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Geometry of Lagrangean structures. 3

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GEOMETRY OF LAGRANGEAN STRUCTURES. 3. *) **)

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Abstract. *The concept of a Lepagean form, and its meaning for the geometrization of the higher order calculus of variations, is discussed. It is shown that in the first order case this concept leads to a unification of the theory of fundamental forms of the Hamilton-Poincaré-Cartan type; a Lepagean form of a different type is also considered. The global infinitesimal (higher order) first variation formula expressed by means of a Lepagean form is derived in terms of differential-geometric operations, and the Euler-Lagrange form is defined. All the differential forms used are odd base forms; this extends the variational theory to fibered manifolds with arbitrary (not necessarily orientable) bases.*

Key words. *Lepagean form, lagrangian, Euler-Lagrange form, jet prolongation of a vector field, variation, variational function, extremal, first variation formula.*

MS classification. *58 E 99, 49 F 05.*

3. LEPAGEAN FORMS AND THE FIRST VARIATION

In Parts 1 and 2 of this work we have developed the basic theory of odd base differential forms, and the theory of horizontal and contact forms on a fibered manifold; some other adequate references to these topics are e.g. [4], [7], and [9]. In this part of the work we begin to study the calculus of variations of differential odd base forms on a fibered manifold. Our goals in the next sections are the following:

(1) To explain the theory of Lepagean differential odd base forms which constitute, in our opinion, an adequate fundamental concept for geometrization of the classical calculus of variations. These forms allow us to investigate the

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invariant, coordinate-free structure of the calculus of variations, to generalize the main concepts, methods, and results of the classical theory to higher order variational problems in many independent variables, and to extend them to a wider class of underlying spaces - smooth manifolds and fibered spaces.

(2) To describe geometrically variations of sections of a fibered manifold, and their prolongations to higher order jet prolongations of this fibered manifold; to derive, with the help of a Lepagean form and the exterior derivative operator, the global first variation formula. These considerations lead naturally to a global notion characterizing the extremals - the Euler-Lagrange differential odd base form.

Our exposition is based on a non-standard understanding of the so called *second Lepage's congruence* for the first order variational problems on Euclidean topological spaces [43], the "congruence mystériense de Lepage" [20, p. 152], and differs from the other authors in many respects. We see the meaning of the second Lepage's congruence in its relation to the first variation formula. Following [9] (see also [7] and [8]) we introduce a *Lepagean form* as a differential n-form on the s-jet prolongation of a fibered manifold with n-dimensional base, whose exterior derivative defines, in a sense to be precised below, the correct "infinitesimal first variation formula". Then using an analogue of the *first Lepagean congruence* we define a Lepagean equivalent of a lagrangian λ of order r as a Lepagean form ρ "variationally equivalent" with λ, i.e. such that the integral variational functionals defined by ρ and λ, coincide.

From now on, Y denotes a fixed fibered manifold with base X and projection π, and we set $n = \dim X$, $m = \dim Y - n$. Notations of Secs. 2.1 - 2.4 concerning jet prolongations of a fibered manifold, horizontal and contact forms, are used. In order to simplify the language, we mean by a *(differential) form*, when there is no danger of confusion, a differential odd form, a differential odd base form, or an (ordinary) differential form; the exact meaning will usually be clear from the context. We note that in general, there is no need to suppose that X is orientable; if X is orientable and its orientation has been chosen, differential odd forms, and differential odd base forms are canonically identified with (ordinary) differential forms (see Sec. 1.1). The reader who wishes to work with oriented X, may simply omit the factor "$\tilde{\varphi}$" in all the local formulas below, and think of the "forms" as "ordinary forms".

3.1. Lepagean forms. We start by a simple lemma.

Lemma 3.1. *Let P be a fibered manifold with base S and projection τ, let ξ be a vector field. There exists a τ-projectable vector field Ξ on P whose projection is ξ, i.e.* $T\tau \,.\, \Xi = \xi \circ \tau$.

Proof. Let (V_ι, ψ_ι), $\psi_\iota = (x^i_\iota, y^\sigma_\iota)$, $\iota \in I$, be fiber charts on P, defining an atlas, (U_ι, φ_ι), $\varphi_\iota = (x^i_\iota)$, the associated charts on S. Let (χ_ι), $\iota \in I$, be a locally finite partition of unity, subordinate to the covering (V_ι) of P. Let $\xi = \xi^i_\iota(\partial/\partial x^i_\iota)$ be the expression of ξ with respect to (U_ι, φ_ι). We put $\Xi = \Sigma \chi_\iota \xi^i_\iota(\partial/\partial x^i_\iota)$ (summation over ι and i); Ξ is a vector field on P. Expressing Ξ with respect to a fiber chart (V,ψ) such that only finitely many functions χ_ι do not vanish on V, we easily obtain that Ξ is τ-projectable and its τ-projection is ξ.

Let $s \geq 0$ be an integer, and let $W \subset J^{s+1}Y$ be an open set. Recall that each element $\eta \in \tilde{\Omega}^p_{J^s Y}(W)$ can be uniquely expressed in the form $\eta = \Sigma \eta_q$ (summation over $q = 0,1,\ldots,p$), where $\eta_q = p_q(\eta) \in \tilde{\Omega}^{p-q,q}(W)$, $p_0 = h$, and p_q $(q \geq 1)$ is the q-th contact projection (see 2.3).

Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y. We denote as before

$$\omega_0 = dx^1 \wedge \ldots \wedge dx^n, \tag{3.1.1}$$
$$\omega_i = (-1)^{i-1} dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n,$$
$$\omega^\sigma_{j_1 \ldots j_k} = dy^\sigma_{j_1 \ldots j_k} - y^\sigma_{j_1 \ldots j_k l} dx^l,$$

where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, $0 \leq k \leq s$, $1 \leq j_1, \ldots, j_k \leq n$. These forms are usually considered as defined on V_{s+1}. Denote by $N(j_1 \ldots j_k)$ the number defined by (2.2.10).

Theorem 3.1. *Let $W \subset J^sY$ be an open set, and let $\rho \in \tilde{\Omega}^n(W)$. The following five conditions are equivalent:*

(1) $\pi^*_{s+1,s} d\rho$ *has the form*

$$\pi^*_{s+1,s} d\rho = E + F, \tag{3.1.2}$$

where E is a 1-contact, $\pi_{s+1,0}$-horizontal form, and F is a form of order of constant ≥ 2.

(2) $p_1(\pi^*_{s+1,s} d\rho)$ *is a $\pi_{s+1,0}$-horizontal form.*

(3) *For each $\pi_{s,0}$-projectable vector field Ξ on W the form $h(i_\Xi d\rho)$ depends on the $\pi_{s,0}$-projection only.*

(4) *For each $\pi_{s,0}$-vertical vector field Ξ on W, $h(i_\Xi d\rho) = 0$.*

(5) *For any fiber chart (V,ψ), $\psi = (x^i, y^\sigma)$, whose associated chart on X is (U,φ), $\pi^*_{s+1,s}\rho$ has an expression*

$$\pi^*_{s+1,s}\rho = \tilde{\varphi} \otimes \rho_0 + \nu, \tag{3.1.3}$$

where the order of contact of ρ_0 (resp. ν) is ≤ 1 (resp. ≥ 2), and

$$\rho_0 = f_0\omega_0 + \sum_{k=0}^{s} \Sigma f^{i,j_1 \ldots j_k}_\sigma \omega^\sigma_{j_1 \ldots j_k} \wedge \omega_i, \tag{3.1.4}$$

where the components of ρ_0 satisfy

$$(3.1.5)\qquad \frac{\partial f_0}{\partial y^\sigma_{j_1}} - d_i f_\sigma^{i,j_1} - f_\sigma^{j_1} = 0,$$

$$\frac{\partial f_0}{\partial y^\sigma_{j_1\ldots j_k}} - d_i f_\sigma^{i,j_1\ldots j_k} - \frac{1}{k} N(j_1\ldots j_k)\Big(\frac{1}{N(j_1\ldots j_{k-1})} f_\sigma^{j_k,j_1\ldots j_{k-1}} + \ldots + \frac{1}{N(j_2\ldots j_k)} f_\sigma^{j_1,j_2\ldots j_k}\Big) = 0, \quad 2 \le k \le s,$$

$$\frac{\partial f_0}{\partial y^\sigma_{j_1\ldots j_{s+1}}} - \frac{1}{s+1} N(j_1\ldots j_{s+1})\Big(\frac{1}{N(j_1\ldots j_s)} f_\sigma^{j_{s+1},j_1\ldots j_s} + \ldots + \frac{1}{N(j_2\ldots j_{s+1})} f_\sigma^{j_1,j_2\ldots j_{s+1}}\Big) = 0.$$

Proof. Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart on Y. The form $\pi^*_{s+1,s}\rho$ has a unique expression (3.1.3) (Theorem 2.6), and ρ_0 has a unique expression (3.1.4). By a direct computation

$$(3.1.6)\qquad \pi^*_{s+1,s} d\rho_0 = \Bigg[\left(\frac{\partial f_0}{\partial y^\sigma} - d_i f_\sigma^i\right)\omega^\sigma + \left(\frac{\partial f_0}{\partial y^\sigma_{j_1}} - d_i f_\sigma^{i,j_1} - f_\sigma^{j_1}\right)\omega^\sigma_{j_1}$$
$$+ \sum_{k=2}^{s}\sum\Bigg(\frac{\partial f_0}{\partial y^\sigma_{j_1\ldots j_k}} - d_i f_\sigma^{i,j_1\ldots j_k}$$
$$- \frac{1}{k} N(j_1\ldots j_k)\left(\frac{1}{N(j_1\ldots j_{k-1})} f_\sigma^{j_k,j_1\ldots j_{k-1}} + \ldots\right.$$
$$\left.\left. + \frac{1}{N(j_2\ldots j_k)} f_\sigma^{j_1,j_2\ldots j_k}\right)\Bigg)\omega^\sigma_{j_1\ldots j_k} + \sum\Bigg(\frac{\partial f_0}{\partial y^\sigma_{j_1\ldots j_{s+1}}}$$
$$- \frac{1}{s+1} N(j_1\ldots j_{s+1})\left(\frac{1}{N(j_1\ldots j_s)} f_\sigma^{j_{s+1},j_1\ldots j_s} + \ldots\right.$$
$$\left.\left. + \frac{1}{N(j_2\ldots j_{s+1})} f_\sigma^{j_1,j_2\ldots j_{s+1}}\right)\Bigg) dy^\sigma_{j_1\ldots j_{s+1}}\Bigg] \wedge \omega_0$$
$$+ \sum_{k=0}^{s}\sum p(df_\sigma^{i,j_1\ldots j_k}) \wedge \omega^\sigma_{j_1\ldots j_k} \wedge \omega_i,$$

where d_i stands for the formal derivative with respect to x^i (Sec. 2.2), and we have used (2.3.20) and (2.3.31).

By definition, the 1-contact form $p_1(\pi^*_{s+1,s} d\rho)$ is equal to E in (3.1.2), and is precisely, up to the factor $\tilde{\varphi}$, the 1-contact term in (3.1.6). Thus conditions (1), (2), and (5) are equivalent.

Let Ξ be a $\pi_{s,0}$-projectable vector field on W, and let θ be any $\pi_{s+1,s}$-

-projectable vector field on $W' = \pi^{-1}_{s+1,s}(W)$ whose $\pi_{s+1,s}$-projection is Ξ (Lemma 3.1). We have $h(i_{\Xi}d\rho) = h(\pi^*_{s+1,s}i_{\Xi}d\rho) = h(i_{\theta}\pi^*_{s+1,s}d\rho)$ (1.3.22). Applying (1.3.6) we get at once that (3) is equivalent with (5). If Ξ is $\pi_{s,0}$-vertical we proceed in the same way, and we get that (4) is equivalent with (5). This proves Theorem 3.1.

A form $\rho \in \widetilde{\Omega}^n(W)$, satisfying one of the equivalent conditions (1) - (5) of Theorem 3.1, is called a *Lepagean form*.

Corollary 1. *Let* $\rho \in \widetilde{\Omega}^n(W)$ *be a Lepagean form.*
(*a*) *If* $\nu \in \widetilde{\Omega}^n(W)$ *has the order of contact* ≥ 2, *then* $\rho + \nu$ *is Lepagean.*
(*b*) *If* $\eta \in \widetilde{\Omega}^n(W)$ *is a closed form, then* $\rho + \eta$ *is Lepagean.*

Corollary 2. (*a*) *Let* $\rho \in \widetilde{\Omega}^n(W)$ *be a Lepagean form. Suppose that* $p_1(\pi^*_{s+1,s}d\rho) = 0$. *Then to each point* $J^{s+1}_x\gamma \in \pi^{-1}_{s+1,s}(W)$ *there exists a Lepagean form* ρ', *defined on a neighborhood of* $J^{s+1}_x\gamma$, *such that*

$$h(\rho) = h(\rho'), \quad d\rho' = 0. \tag{3.1.7}$$

(*b*) *Let* $r \geq s$, $W' = \pi^{-1}_{r,s}(W)$, *and let* $\rho \in \widetilde{\Omega}^n(W)$, $\rho' \in \widetilde{\Omega}^n(W')$ *be Lepagean forms. Suppose that* $p_1(\pi^*_{r+1,s}d\rho) = p_1(\pi^*_{r+1,r}d\rho')$. *Then to each point* $J^{r+1}_x\gamma \in \pi^{-1}_{r+1,r}(W')$ *there exists an* $(n-1)$*-form* η, *defined on a neighborhood of* $J^{r+1}_x\gamma$, *such that*

$$h(\rho) - h(\rho') = h(d\eta). \tag{3.1.8}$$

Proof. (a) If $p_1(\pi^*_{s+1,s}d\rho) = 0$, then $\pi^*_{s+1,s}d\rho = F = p_2(\pi^*_{s+1,s}d\rho) + \dots + p_{n+1}(\pi^*_{s+1,s}d\rho)$. Let $J^{s+1}_x\gamma \in \pi^{-1}_{s+1,s}(W)$ be a point. Since $dF = 0$, there exists a form η, defined on a neighborhood of $J^{s+1}_x\gamma$, whose order of contact is ≥ 1, such that $F = d\eta$ (Theorem 2.7). We set $\rho' = \pi^*_{s+1,s}\rho - \eta$. ρ' is defined on a neighborhood of $J^{s+1}_x\gamma$, and we have $d\rho' = E + F - F = 0$ (3.1.2); in particular, ρ' is Lepagean. Moreover, $h(\rho') = h(\pi^*_{s+1,s}\rho)$ since η is a contact form. This proves (3.1.7).

(b) By hypothesis, $p_1(\pi^*_{r+1,r}d(\rho' - \pi^*_{r,s}\rho)) = 0$. Thus, locally, $h(\rho' - \pi^*_{r,s}\rho) = h(\rho'')$, where $d\rho'' = 0$ (this Corollary, (a)). Writing $\rho'' = d\eta$ we get (3.1.8).

We note that relations (3.1.5) contain formal derivatives of functions, defined on an open set in $J^{s+1}Y$; since these formal derivatives are, by definition, functions on an open set in $J^{s+2}Y$, depending polynomially on certain variables, conditions (3.1.5) split into a rather complicated system of equations for the components of ρ_0 and their derivatives. One could write down these equations explicitly; we shall study them however, in the next section in a simpler way.

Convention 3.1. *Symmetrization in the indices* $j_1,\dots,j_k$ *is denoted by placing parentheses around these indices, that is, by writing* $(j_1,\dots,j_k)$. *Later, in non standard situations when this notation is not possible, we shall use another one.*

Conditions (3.1.5) can be expressed in a recurrent way as follows:

$$(3.1.9)\quad \frac{1}{N(j_1\dots j_{k-1})} f_\sigma^{j_k,j_1\dots j_{k-1}} =$$

$$= \sum_{l=0}^{s+1-k} \sum_{i_1,\dots,i_l} (-1)^l \frac{1}{N(j_1\dots j_k i_1\dots i_l)} d_{i_1}\dots d_{i_l} \frac{\partial f_0}{\partial y^\sigma_{j_1\dots j_k i_1\dots i_l}}$$

$$+ g_\sigma^{j_k,j_1\dots j_{k-1}} + \sum_{l=1}^{s+1-k} \sum_{i_1,\dots,i_l} (-1)^l d_{i_1}\dots d_{i_l} g_\sigma^{i_l,j_1\dots j_k i_1\dots i_{l-1}},$$

$$2 \le k \le s+1,$$

$$f_\sigma^{j_1} = \sum_{l=0}^{s} \sum_{i_1,\dots,i_l} (-1)^l \frac{1}{N(j_1 i_1\dots i_l)} d_{i_1}\dots d_{i_l} \frac{\partial f_0}{\partial y^\sigma_{j_1 i_1\dots i_l}}$$

$$+ \sum_{l=1}^{s} \sum_{i_1,\dots,i_l} (-1)^l d_{i_1}\dots d_{i_l} g_\sigma^{i_l,j_1 i_1\dots i_{l-1}},$$

where $g_\sigma^{j_k,j_1\dots j_{k-1}}$ are some functions such that

$$(3.1.10)\quad g_\sigma^{(j_k,j_1\dots j_{k-1})} = 0.$$

Corollary 3. *If $\rho \in \tilde{\Omega}^n(W)$ is Lepagean, then the $(n+1)$-form E, defined by (3.1.2) is expressed, in terms of the fiber chart (V,ψ), by*

$$(3.1.11)\quad E = E_\sigma \tilde{\varphi} \otimes (\omega^\sigma \wedge \omega_0),$$

where

$$(3.1.12)\quad E_\sigma = \frac{\partial f_0}{\partial y^\sigma} + \sum_{l=1}^{s+1} \sum_{i_1,\dots,i_l} (-1)^l \frac{1}{N(i_1\dots i_l)} d_{i_1}\dots d_{i_l} \frac{\partial f_0}{\partial y^\sigma_{i_1\dots i_l}}$$

$$= \frac{\partial f_0}{\partial y^\sigma} + \sum_{l=1}^{s+1} (-1)^l \sum d_{i_1}\dots d_{i_l} \frac{\partial f_0}{\partial y^\sigma_{i_1\dots i_l}}$$

(summation over $i_1 \le \dots \le i_l$). In particular, the $(n+1)$-form E depends on $h(\rho) = f_0 \cdot \tilde{\varphi} \otimes \omega_0$ only.

Proof. This follows from (3.1.9) and (3.1.6).

The $(n+1)$-form (3.1.11) is called the *Euler-Lagrange form* of the Lepagean form ρ. The components E_σ (3.1.12) of the Euler-Lagrange form are called the *Euler--Lagrange expressions* relative to the fiber chart (V,ψ).

3.2. Lagrangians and their Lepagean equivalents. Let $W \subset J^rY$ be an open set. An element $\lambda \in \tilde{\Omega}^n_X(W)$ is called a *lagrangian* (*of order* r) for Y. Let $s \ge 0$ be an integer, and put $W' = \pi^{-1}_{s,r}(W)$ if $s \ge r$, or $W' = \pi_{r,s}(W)$ if $s < r$. A Lepagean form

$\rho \in \widetilde{\Omega}^n(W')$ such that $\lambda = h(\rho)$, is called a *Lepagean equivalent* (*of order* s) *of* λ.

In this definition of a Lepagean equivalent we have applied our notational convention on projectable forms (Sec. 2.2).

Each Lepagean form ρ is a Lepagean equivalent of a unique lagrangian λ; obviously, $\lambda = h(\rho)$.

If a lagrangian λ of order r is *not* $\pi_{r,r-1}$-projectable, then the order of any Lepagean equivalent ρ of λ is $\geq r - 1$; this follows from the condition $\lambda = h(\rho)$. On the other hand, if ρ is a Lepagean equivalent of λ of order s and η is any n-form on $J^{s+1}Y$ whose order of contact is ≥ 2, then $\pi^*_{s+1,s}\rho + \eta$ is a Lepagean equivalent of λ of order $s + 1$ (Corollary 1 to Theorem 3.1).

Our main problem to be considered in this section is the problem of *existence* of Lepagean equivalents. According to Theorem 3.1 we may restrict ourselves to searching them among forms whose order of contact is ≤ 1.

Convention 3.2. *Let* (V,ψ), $\psi = (x^i, y^\sigma)$, *be a fiber chart on* Y, $f : V_s \to R$ *a function. The partial derivatives* $\partial f/\partial y^\sigma_{j_1\dots j_k}$, *where* $2 \leq k \leq s$, $j_1 \leq \dots \leq j_k$, *will also be denoted by* $\partial f/\partial y^\sigma_{p_1\dots p_k}$, *where* $(p_1,\dots,p_k)$ *is any permutation of* $(j_1,\dots,j_k)$.

In the following theorem we consider Lepagean equivalents of order $s \geq 2r - 1$ of a lagrangian of order r.

Theorem 3.2. *Let* W *be an open set, let* $\lambda \in \widetilde{\Omega}^n_X(W)$ *be a lagrangian. Let* $\rho \in \widetilde{\Omega}^n(W')$, *where* $s \geq 2r - 1$ *and* $W' = \pi^{-1}_{s,r}(W)$, *be a form whose order of contact is* ≤ 1. *The following two conditions are equivalent:*

(1) ρ *is a Lepagean equivalent of* λ.

(2) *For any fiber chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *with associated chart* (U,φ) *on* X, $\rho = \tilde{\varphi} \otimes \rho_0$, *where*

$$(3.2.1)\qquad \pi^*_{s+1,s}\rho_0 = L\omega_0 + \Big(\sum_{k=0}^{s} \sum f_\sigma^{i,j_1\dots j_k}\,\omega^\sigma_{j_1\dots j_k}\Big) \wedge \omega_i,$$

L is defined by the chart expression

$$(3.2.2)\qquad \lambda = L\tilde{\varphi} \otimes \omega_0,$$

and there exist functions $g_\sigma^{i,j_1\dots j_k} : V_s \to R$ *such that*

$$(3.2.3)\qquad f_\sigma^{j_1} = \frac{\partial L}{\partial y^\sigma_{j_1}} - d_i f_\sigma^{i,j_1},$$

$$\frac{1}{N(j_1\dots j_{k-1})} f_\sigma^{j_k,j_1\dots j_{k-1}} = \frac{1}{N(j_1\dots j_k)}\left(\frac{\partial L}{\partial y^\sigma_{j_1\dots j_k}} - d_i f_\sigma^{i,j_1\dots j_k}\right) + g_\sigma^{j_k,j_1\dots j_{k-1}},\qquad 2 \leq k \leq r,$$

$$\frac{1}{N(j_1\ldots j_{k-1})}f_\sigma^{j_k,j_1\ldots j_{k-1}} = -\frac{1}{N(j_1\ldots j_k)}d_i f_\sigma^{i,j_1\ldots j_k} + g_\sigma^{j_k,j_1\ldots j_{k-1}},$$

$$r+1 \le k \le s,$$

$$\frac{1}{N(j_1\ldots j_s)}f_\sigma^{j_{s+1},j_1\ldots j_s} = g_\sigma^{j_{s+1},j_1\ldots j_s},$$

and

$$g_\sigma^{(i,j_1\ldots j_k)} = 0. \tag{3.2.4}$$

Proof. 1. Suppose that ρ is Lepagean. Since the order of contact of ρ is ≤ 1, $\pi^*_{2r,2r-1}\rho = \tilde{\varphi} \otimes \rho_0$, where (3.1.5) holds. The condition $h(\rho) = \lambda$ gives $f_0 = L$, and we get $\pi^*_{s+1,s}\rho$ in the form (3.2.1). Decomposing now the functions $(1/N(j_1\ldots j_{k-1}))f_\sigma^{i,j_1\ldots j_k}$ into the symmetric and complementary parts, we obtain (3.2.3) and (3.2.4) from (3.1.5).

2. If (2) holds, we obtain (1) from Theorem 3.1,(5).

Remark 3.1. Conditions (3.2.3) are a particular case of (3.1.9). This implies (see $f_\sigma^{j_1}$ in (3.1.9)) that they cannot be satisfied, for a *general* lagrangian λ of order r, unless $s \ge 2r-1$. For $s = 2r-1$ conditions (3.2.3) can be expressed in a recurrent way as follows:

$$\frac{1}{N(j_1\ldots j_{k-1})}f_\sigma^{j_k,j_1\ldots j_{k-1}} = \tag{3.2.5}$$

$$= \sum_{l=0}^{r-k} \sum_{i_1,\ldots,i_l} (-1)^l \frac{1}{N(j_1\ldots j_k i_1\ldots i_l)} d_{i_1}\ldots d_{i_l} \frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k i_1\ldots i_l}}$$

$$+ g_\sigma^{j_k,j_1\ldots j_{k-1}} + \sum_{l=1}^{r-k} \sum_{i_1,\ldots,i_l} (-1)^l d_{i_1}\ldots d_{i_l} g_\sigma^{i_l,j_1\ldots j_k i_1\ldots i_{l-1}},$$

$$2 \le k \le r,$$

$$f_\sigma^{j_1} = \sum_{l=0}^{r-1} \sum_{i_1,\ldots,i_l} (-1)^l \frac{1}{N(j_1\ldots j_k i_1\ldots i_l)} d_{i_1}\ldots d_{i_l} \frac{\partial L}{\partial y^\sigma_{j_1 i_1\ldots i_l}}$$

$$+ \sum_{l=1}^{r-1} \sum_{i_1,\ldots,i_l} (-1)^l d_{i_1}\ldots d_{i_l} g_\sigma^{i_l,j_1 i_1\ldots i_{l-1}}.$$

We shall now discuss some particular cases.

Let us suppose that $\dim X = n = 1$, and introduce the following notation. Let (V,ψ), $\psi = (t,q^\sigma)$, be a fiber chart on Y, (V_s,ψ_s), $\psi_s = (t,q^\sigma,q^\sigma_{(1)},\ldots,q^\sigma_{(s)})$ the associated chart on J^sY. We denote $q^\sigma_{(0)} = q^\sigma$ and set for each $0 \le k \le s-1$,

$$\omega^\sigma_{(k)} = dq^\sigma_{(k)} - q^\sigma_{(k+1)}dt. \tag{3.2.6}$$

The formal derivative with respect to t will be denoted by d/dt. Since every 1-dimensional manifold is orientable, we may restrict ourselves to (ordinary) forms.

Corollary 1. *Suppose that* dim $X = 1$. *Then each lagrangian* $\lambda \in \Omega^1_X(J^rY)$ *has a unique Lepagean equivalent* ρ. *This Lepagean equivalent belongs to* $\Omega^1_{J^{r-1}Y}(J^{2r-1}Y)$, *and for any fiber chart* (V,ψ), $\psi = (t,q^\sigma)$, *on* Y,

$$(3.2.7)\qquad \rho = Ldt + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l \frac{d^l}{dt^l}\frac{\partial L}{\partial q^\sigma_{(k+l+1)}}\right)\omega^\sigma_{(k)},$$

where $\lambda = Ldt$.

Proof. Let $\rho \in \Omega^1(J^sY)$ be a Lepagean equivalent of λ. Then locally,

$$(3.2.8)\qquad \pi^*_{s+1,s}\rho = Ldt + \sum_{k=0}^{s} f^{(k+1)}_\sigma \omega^\sigma_{(k)}.$$

By Theorem 3.1

$$(3.2.9)\qquad f^{(s+1)}_\sigma = \frac{\partial L}{\partial q^\sigma_{(s+1)}},\quad f^{(j)}_\sigma = \frac{\partial L}{\partial q^\sigma_{(j)}} - \frac{d}{dt} f^{(j+1)}_\sigma,\qquad 1 \le j \le s.$$

Hence

$$(3.2.10)\qquad f^{(j)}_\sigma = \sum_{l=0}^{s+1-j}(-1)^l \frac{d^l}{dt^l}\frac{\partial L}{\partial q^\sigma_{(j+l)}},\qquad 1 \le j \le s+1.$$

Since $\partial L/\partial q^\sigma_{(s)} = 0$ for $s > r$, substituting this expression in (3.2.8) we get (3.2.7). Since the chart expression of ρ is determined uniquely, ρ is globally well-defined.

Remark 3.2. If $r = 1$, (3.2.7) is usually called the *Cartan form*, with reference to [17]; the same form had been used earlier in analytical mechanics (see Whittaker [53]). For general r, (3.2.7) appears in the papers by Gelfand and Dikii [32] and Sternberg [50] (see also Dedecker [23] and the references therein).

The following two corollaries are concerned with the case when $r = 1$ and n,m re arbitrary.

Corollary 2. *Each lagrangian of order* 1 $\lambda \in \tilde{\Omega}^n_X(W)$ *has a unique Lepagean equivalent* ρ *of order* 1 *whose order of contact is* ≤ 1. *For any fiber chart* (V,ψ), $\psi = (x^i,y^\sigma)$, *with* (U,φ) *the associated chart on* X, ρ *is expressed by* $\rho = \tilde{\varphi} \otimes \rho_0$, *where*

$$(3.2.11)\qquad \rho_0 = L\omega_0 + \frac{\partial L}{\partial y^\sigma_i}\omega^\sigma \wedge \omega_i,$$

and $\lambda = L.\tilde{\varphi} \otimes \omega_0$.

Proof. This follows from Theorem 3.2.

Notice that for $\eta \in \widetilde{\Omega}^n(Y)$, the lagrangian $h(\eta) \in \widetilde{\Omega}^n_X(J^1Y)$ is a multilinear expression in y^σ_j for any fiber chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y, and the form $\pi^*_{1,0}\eta$ is a Lepagean equivalent of $h(\eta)$. Thus the components of $\pi^*_{1,0}\eta$ can be obtained from $h(\eta)$ by means of a differentation procedure with respect to y^σ_j, and the Lepagean equivalent $\pi^*_{1,0}\eta$ can be reconstructed from $h(\eta)$. Generalizing this procedure to any lagrangian $\lambda \in \widetilde{\Omega}^n_X(J^1Y)$ we obtain a new example of a Lepagean equivalent, differing from (3.2.11).

In what follows, $\varepsilon^{i_1\ldots i_k}$ and $\varepsilon^{i_1\ldots i_k}_{j_1\ldots j_k}$ are totally antisymmetric symbols.

Lemma 3.2. *Let* ρ *be an* n*-form on* Y. *For any fiber chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *on* Y $\pi^*_{1,0}\rho$ *is expressed by*

$$(3.2.12)\qquad \pi^*_{1,0}\rho = f_0 dx^1 \wedge\ldots\wedge dx^n + \sum_{k=1}^{n} \sum_{i_1<\ldots<i_k} \sum_{\sigma_1,\ldots,\sigma_k} \frac{\partial^k f_0}{\partial y^{\sigma_1}_{i_1}\ldots\partial y^{\sigma_k}_{i_k}} \cdot$$

$$\cdot dx^1\wedge\ldots\wedge dx^{r_1-1}\wedge\omega^{\sigma_1}\wedge dx^{r_1+1}\wedge\ldots\wedge dx^{i_k-1}\wedge\omega^{\sigma_k}\wedge dx^{i_k+1}\wedge\ldots\wedge dx^n,$$

where $f_0 : V_1 \to R$ *is a function.*

Proof. 1. Let ρ be an n-form on Y, (V,ψ), $\psi = (x^i, y^\sigma)$, a fiber chart on Y. ρ has a unique expression

$$(3.2.13)\qquad \rho = h_0 dx^1 \wedge\ldots\wedge dx^n + \sum_{k=1}^{n} \frac{1}{k!(n-k)}h_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k} dx^{i_1}\wedge\ldots$$
$$\ldots\wedge dx^{i_{n-k}} \wedge dy^{\sigma_1} \wedge\ldots\wedge dy^{\sigma_k},$$

where $h_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k}$ are functions on V, antisymmetric with respect to $i_1,\ldots,i_{n-k}$, and with respect to $\sigma_1,\ldots,\sigma_k$. The n-form $\pi^*_{1,0}\rho$ has a unique expression

$$(3.2.14)\qquad \pi^*_{1,0}\rho = g_0 dx^1 \wedge\ldots\wedge dx^n + \sum_{k=1}^{n} \frac{1}{k!(n-k)}g_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k} dx^{i_1}\wedge\ldots$$
$$\ldots\wedge dx^{i_{n-k}} \wedge \omega^{\sigma_1} \wedge\ldots\wedge \omega^{\sigma_k},$$

where $g_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k}$ are functions on V_1, antisymmetric with respect to $i_1,\ldots,i_{n-k}$, and with respect to $\sigma_1,\ldots,\sigma_k$. We shall find the relations between the systems of functions h_0, $h_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k}$ and g_0, $g_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k}$.

We substitute ω^σ for dy^σ in (3.2.13) and substract the added terms. Let us compute the coefficient at $dx^{i_1}\wedge\ldots\wedge dx^{i_{n-k}}\wedge\omega^{\sigma_1}\wedge\ldots\wedge\omega^{\sigma_k}$. This coefficient is defined by the summand

$$(3.2.15)\qquad \frac{1}{k!(n-k)!}h_{i_1\ldots i_{n-k}\sigma_1\ldots\sigma_k} dx^{i_1}\wedge\ldots\wedge dx^{i_{n-k}}\wedge\omega^{\sigma_1}\wedge\ldots\wedge\omega^{\sigma_k}$$

and the contributions from the summands

(3.2.16) $$\frac{1}{(k+l)!(n-k-l)!}h_{i_1\dots i_{n-k-l}\sigma_1\dots\sigma_{k+l}}dx^{i_1}\wedge\dots\wedge dx^{i_{n-k-l}}\wedge\omega^{\sigma_1}\wedge\dots\wedge\omega^{\sigma_{k+l}},\quad l=1,2,\dots,n-k.$$

Using the antisymmetry properties of the functions $h_{i_1\dots i_{n-k-l}\sigma_1\dots\sigma_{k+l}}$ we get for these contributions

(3.2.17) $$\frac{1}{(k+l)!(n-k-l)!}\binom{k+l}{l}h_{i_1\dots i_{n-k}\sigma_1\dots\sigma_k\nu_1\dots\nu_l}y^{\nu_1}_{i_{n-k-l+1}}\dots y^{\nu_l}_{i_{n-k}}\cdot dx^{i_1}\wedge\dots\wedge dx^{i_{n-k-l}}\wedge\omega^{\sigma_1}\wedge\dots\wedge\omega^{\sigma_k}\wedge dx^{i_{n-k-l+1}}\wedge\dots\wedge dx^{i_{n-k}}$$

$$=\frac{(-1)^{kl}}{k!l!(n-k-l)!}h_{i_1\dots i_{n-k-l}\sigma_1\dots\sigma_k\nu_1\dots\nu_l}y^{\nu_1}_{i_{n-k-l+1}}\dots y^{\nu_l}_{i_{n-k}}\cdot dx^{i_1}\wedge\dots\wedge dx^{i_{n-k}}\wedge\omega^{\sigma_1}\wedge\dots\wedge\omega^{\sigma_k},\quad l=1,2,\dots,n-k.$$

Thus the coefficient antisymmetric in $i_1,\dots,i_{n-k}$ and in $\sigma_1,\dots,\sigma_k$ equals

(3.2.18) $$\frac{1}{k!(n-k)!}\frac{(-1)^{kl}}{l!(n-k-l)!}h_{j_1\dots j_{n-k-l}\sigma_1\dots\sigma_k\nu_1\dots\nu_l}y^{\nu_1}_{j_{n-k-l+1}}\dots y^{\nu_l}_{j_{n-k}}\varepsilon^{j_1\dots j_{n-k}}_{i_1\dots i_{n-k}}.$$

Notice that for $l = 0$ this expression reduces to the coefficient in (3.2.15). Hence

(3.2.19) $$g_{i_1\dots i_{n-k}\sigma_1\dots\sigma_k}=\sum_{l=0}^{n-k}\frac{(-1)^{kl}}{l!(n-k-l)!}h_{j_1\dots j_{n-k-l}\sigma_1\dots\sigma_k\nu_1\dots\nu_l}\cdot y^{\nu_1}_{j_{n-k-l+1}}\dots y^{\nu_l}_{j_{n-k}}\varepsilon^{j_1\dots j_{n-k}}_{i_1\dots i_{n-k}}.$$

For g_0 we get from (3.2.13)

(3.2.20) $$g_0=h_0+\sum_{k=1}^{n}\frac{1}{k!(n-k)!}h_{j_1\dots j_{n-k}\sigma_1\dots\sigma_k}y^{\sigma_1}_{j_{n-k+1}}\dots y^{\sigma_k}_{j_n}\cdot\varepsilon^{j_1\dots j_n}.$$

These are the desired relations.

2. Let $(p_1,\dots,p_s)$, $(\nu_1,\dots,\nu_s)$ be any sequences such that $1\le p_1<\dots<p_s\le n$, $1\le\nu_1<\dots<\nu_s\le m$, and let $(q_1,\dots,q_{n-s})$ be the unique sequence such that $1\le q_1<\dots<q_{n-s}\le n$ and $q_i\neq p_j$ for all i and j; thus $(q_1,\dots,q_{n-s})$ is the complementary increasing sequence to $(p_1,\dots,p_s)$ in the sequence $(1,2,\dots,n)$. We shall show that

$$(3.2.21)\qquad \frac{\partial^k g_0}{\partial y^{\nu_1}_{p_1}\ldots\partial y^{\nu_s}_{p_s}} = \frac{1}{s!}\, c^{q_1\ldots q_{n-s}}\, g_{q_1\ldots q_{n-s}\nu_1\ldots\nu_s},$$

where

$$(3.2.22)\qquad c^{q_1\ldots q_{n-s}} = (-1)^{ns-\frac{1}{2}n(n+1)+q_1+\ldots+q_{n-s}-\frac{1}{2}s(s+1)}$$

(no summation in (3.2.21)).

Write (3.2.20) in the form

$$(3.2.23)\qquad g_0 = h_0 + \sum_{k=1}^{n} \frac{1}{k!(n-k)!} h_{j_1\ldots j_{n-k}\sigma_1\ldots\sigma_k} y^{\sigma_1}_{l_1}\ldots y^{\sigma_k}_{l_k} \varepsilon^{j_1\ldots j_{n-k}l_1\ldots l_k}.$$

Since the coefficient at $y^{\sigma_1}_{l_1}\ldots y^{\sigma_k}_{l_k}$ is symmetric in $\binom{\sigma_1}{l_1},\ldots,\binom{\sigma_k}{l_k}$, we have

$$(3.2.24)\qquad \frac{\partial^k g_0}{\partial y^{\nu_1}_{p_1}\ldots\partial y^{\nu_s}_{p_s}} = \sum_{k=s}^{n} \frac{1}{k!(n-k)!} h_{j_1\ldots j_{n-k}\sigma_1\ldots\sigma_k} \varepsilon^{j_1\ldots j_{n-k}l_1\ldots l_k} \cdot$$

$$\cdot \frac{\partial^s}{\partial y^{\nu_1}_{p_1}\ldots\partial y^{\nu_s}_{p_s}}(y^{\sigma_1}_{l_1}\ldots y^{\sigma_k}_{l_k}) = \sum_{k=s}^{n} \frac{1}{k!(n-k)!}\binom{k}{s} \cdot$$

$$\cdot h_{j_1\ldots j_{n-k}\sigma_1\ldots\sigma_k} \varepsilon^{j_1\ldots j_{n-k}l_1\ldots l_k} \delta^{\sigma_1}_{\nu_1}\ldots\delta^{\sigma_s}_{\nu_s}\delta^{p_1}_{l_1}\ldots\delta^{p_s}_{l_s} \cdot$$

$$\cdot y^{\sigma_{s+1}}_{l_{s+1}}\ldots y^{\sigma_k}_{l_k} = \sum_{l=0}^{n-s} \frac{1}{s!\,l!\,(n-s-l)!} \cdot$$

$$\cdot h_{j_1\ldots j_{n-s-l}\nu_1\ldots\nu_s\sigma_1\ldots\sigma_l} \varepsilon^{j_1\ldots j_{n-s-l}p_1\ldots p_s i_1\ldots i_l} y^{\sigma_1}_{i_1}\ldots y^{\sigma_l}_{i_l} .$$

But

$$(3.2.25)\qquad \varepsilon^{j_1\ldots j_{n-s-l}p_1\ldots p_s i_1\ldots i_l} = (-1)^{sl}\cdot\varepsilon^{j_1\ldots j_{n-s-l}i_1\ldots i_l p_1\ldots p_s} =$$

$$= (-1)^{sl}\cdot\varepsilon^{j_1\ldots j_{n-s-l}i_1\ldots i_l}_{q_1\ldots\quad q_{n-s}}\, \varepsilon^{q_1\ldots q_{n-s}p_1\ldots p_s}$$

(no summation), and

$$(3.2.26)\qquad \varepsilon^{q_1\ldots q_{n-s}p_1\ldots p_s} = (-1)^{n-s-p_1}(-1)^{n-s+1-p_2}\ldots(-1)^{n-1-p_s} =$$

$$= (-1)^{ns-p_1-p_2-\ldots-p_s-1-2-\ldots-s} = (-1)^{ns-p_1-\ldots-p_s-(1/2)s(s+1)}$$

so that

$$(3.2.27)\qquad \varepsilon^{j_1\ldots j_{n-s-l}p_1\ldots p_s i_1\ldots i_l} = (-1)^{sl}\cdot c^{q_1\ldots q_{n-s}}\cdot\varepsilon^{j_1\ldots j_{n-s-l}i_1\ldots i_l}_{q_1\ldots\qquad q_{n-s}}.$$

Substituting these expressions in (3.2.24) we obtain

$$(3.2.28)\qquad \frac{\partial^k g_0}{\partial y^{\nu_1}_{p_1}\ldots\partial y^{\nu_s}_{p_s}} = \sum_{l=0}^{n-s}\frac{1}{s!\,l!\,(n-s-l)!}(-1)^{sl}c^{q_1\ldots q_{n-s}}$$

$$\cdot\, h_{j_1\ldots j_{n-s-l}\nu_1\ldots\nu_s\sigma_1\ldots\sigma_l}\varepsilon^{j_1\ldots j_{n-s-l}i_1\ldots i_l}_{q_1\ldots\qquad q_{n-s}}y^{\sigma_1}_{i_1}\ldots y^{\sigma_l}_{i_l}$$

(no summation over $q_1,\ldots,q_{n-s}$). Comparing this result with (3.2.19) we get the formula

$$(3.2.29)\qquad \frac{\partial^k g_0}{\partial y^{\nu_1}_{p_1}\ldots\partial y^{\nu_s}_{p_s}} = \frac{1}{s!}c^{q_1\ldots q_{n-s}}g_{q_1\ldots q_{n-s}\nu_1\ldots\nu_s}$$

(no summation).

3. Let $(p_1,\ldots,p_k),(\nu_1,\ldots,\nu_k)$, and $(q_1,\ldots,q_{n-k})$ be as above. Using (3.2.22) we easily get

$$(3.2.30)\qquad dx^1\wedge\ldots\wedge dx^{p_1-1}\wedge\omega^{\nu_1}\wedge dx^{p_1+1}\wedge\ldots\wedge dx^{p_k-1}\wedge\omega^{\nu_k}\wedge dx^{p_k+1}\wedge\ldots\wedge dx^n$$

$$= c^{q_1\ldots q_{n-k}}dx^1\wedge\ldots\wedge dx^{p_1-1}\wedge dx^{p_1+1}\wedge\ldots\wedge dx^{p_k-1}\wedge dx^{p_k+1}\wedge$$

$$\wedge\,\omega^{\nu_1}\wedge\ldots\wedge\omega^{\nu_k} = c^{q_1\ldots q_{n-k}}dx^{q_1}\wedge\ldots\wedge dx^{q_{n-k}}\wedge\omega^{\nu_1}\wedge\ldots\wedge\omega^{\nu_k}$$

(no summation). Let us consider the form ρ (3.2.13). $\pi^*_{1,0}\rho$ (3.2.14) can be uniquely expressed by

$$(3.2.31)\qquad \pi^*_{1,0}\rho = f_0dx^1\wedge\ldots\wedge dx^n + \sum_{k=1}^{n}\sum_{j_1<\ldots<j_k}\sum_{\sigma_1<\ldots<\sigma_k}f^{j_1\ \ j_k}_{\sigma_1\ldots\sigma_k}dx^1\wedge\ldots$$

$$\ldots\wedge dx^{j_1-1}\wedge\omega^{\sigma_1}\wedge dx^{j_1+1}\wedge\ldots\wedge dx^{j_k-1}\wedge\omega^{\sigma_k}\wedge dx^{j_k+1}\wedge\ldots\wedge dx^n.$$

Obviously $f_0 = g_0$. Thus, using (3.2.30) and (3.2.29) we get

$$(3.2.32)\qquad \pi^*_{1,0}\rho = f_0dx^1\wedge\ldots\wedge dx^n + \sum_{k=1}^{n}\sum_{j_1<\ldots<j_k}\sum_{\sigma_1<\ldots<\sigma_k}f^{j_1\ \ j_k}_{\sigma_1\ldots\sigma_k}\cdot$$

$$\cdot\, c^{i_1\ldots i_{n-k}}dx^{i_1}\wedge\ldots\wedge dx^{i_{n-k}}\wedge\omega^{\sigma_1}\wedge\ldots\wedge\omega^{\sigma_k},$$

where $(i_1,\ldots,i_{n-k})$ is the unique increasing sequence, complementary to $(j_1,\ldots,j_k)$ in the sequence $(1,2,\ldots,n)$. Obviously

$$g_{i_1\dots i_{n-k}\sigma_1\dots\sigma_k} = c^{i_1\dots i_{n-k}j_1\dots j_k} f^{j_1 \dots j_k}_{\sigma_1\dots\sigma_k} \tag{3.2.33}$$

so that

$$\frac{1}{k!}c^{i_1\dots i_{n-k}} g_{i_1\dots i_{n-k}\sigma_1\dots\sigma_k} = \frac{1}{k!}f^{j_1\dots j_k}_{\sigma_1\dots\sigma_k} = \frac{\partial^k f_0}{\partial y^{\sigma_1}_{j_1}\dots\partial y^{\sigma_k}_{j_k}} . \tag{3.2.34}$$

Now (3.2.31) is written in the form

$$\begin{aligned}\pi^*_{1,0}\rho = {} & f_0 dx^1\wedge\dots\wedge dx^n + \sum_{k=1}^{n} \sum_{j_1<\dots<j_k} \sum_{\sigma_1,\dots,\sigma_k} \frac{1}{k!}f^{j_1\dots j_k}_{\sigma_1\dots\sigma_k} dx^1 \wedge \\ & \dots\wedge dx^{j_1-1}\wedge\omega^{\sigma_1}\wedge dx^{j_1+1}\wedge\dots\wedge dx^{j_k}\wedge\omega^{\sigma_k}\wedge dx^{j_k+1}\wedge\dots\wedge dx^n \\ = {} & f_0 dx^1\wedge\dots\wedge dx^n + \sum_{k=1} \sum_{j_1<\dots<j_k} \sum_{\sigma_1,\dots,\sigma_k} \frac{\partial^k f_0}{\partial y^{\sigma_1}_{j_1}\dots\partial y^{\sigma_k}_{j_k}} dx^1 \wedge \\ & \dots\wedge dx^{j_1-1}\wedge\omega^{\sigma_1}\wedge dx^{j_1+1}\wedge\dots\wedge dx^{j_k-1}\wedge\omega^{\sigma_k}\wedge dx^{j_k+1}\wedge\dots\wedge dx^n .\end{aligned} \tag{3.2.35}$$

This proves our assertion.

Corollary 3. *Let* $\lambda \in \tilde{\Omega}^n_X(J^1Y)$ *be a lagrangian. There exists one and only one Lepagean equivalent* ρ *of* λ *such that for any fiber chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *with the associated chart* (U,φ) *on* X, $\rho = \tilde{\varphi}\otimes\rho_0$, *where*

$$\begin{aligned}\rho_0 = L\omega_0 + & \sum_{k=1}^{n} \sum_{i_1<\dots<i_k} \sum_{\sigma_1,\dots,\sigma_k} \frac{\partial^k L}{\partial y^{\sigma_1}_{i_1}\dots\partial y^{\sigma_k}_{i_k}} . \\ & . \, dx^1\wedge\dots\wedge dx^{i_1-1}\wedge\omega^{\sigma_1}\wedge dx^{i_1+1}\wedge\dots\wedge dx^{i_k-1}\wedge\omega^{\sigma_k}\wedge dx^{i_k+1}\wedge\dots\wedge dx^n ,\end{aligned} \tag{3.2.36}$$

and L *is defined by the chart expression* $\lambda = L.\tilde{\varphi}\otimes\omega_0$.

Proof. By Theorem 3.1 and Theorem 3.2 it is enough to show that the form $\tilde{\varphi}\otimes\rho_0$ (3.2.36) is globally well-defined. The proof involves essentially the Laplace's theorem on expansion of a determinant, and is in fact elementary; we shall present it here for the sake of completeness.

Let (V,ψ) be as above, and let $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, be another fiber chart, with the associated chart $(\bar{U},\bar{\varphi})$ on X. Using (1.2.5), (2.1.3), (2.3.8) and (3.2.30) we obtain, with the obvious notation

$$\begin{aligned}\sum_{j_1<\dots<j_k} \sum_{\nu_1,\dots,\nu_k} & \frac{\partial^k \bar{L}}{\partial \bar{y}^{\nu_1}_{j_1}\dots\partial\bar{y}^{\nu_k}_{j_k}} d\bar{x}^1\wedge\dots\wedge d\bar{x}^{j_1-1}\wedge\bar{\omega}^{\nu_1}\wedge d\bar{x}^{j_1+1} \\ & \wedge\dots\wedge d\bar{x}^{j_k-1}\wedge\bar{\omega}^{\nu_k}\wedge d\bar{x}^{j_k+1}\wedge\dots\wedge d\bar{x}^n =\end{aligned} \tag{3.2.37}$$

$$= \det D\varphi\bar{\varphi}^{-1} \cdot \sum_{j_1<\dots<j_k} \sum_{\sigma_1,\dots,\sigma_k} c^{1\dots j_1-1\, j_1+1\dots j_k-1\, j_k+1\dots n} \cdot$$

$$\cdot \frac{\partial^k L}{\partial y_{i_1}^{\sigma_1}\dots\partial y_{i_k}^{\sigma_k}} \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}}\dots\frac{\partial \bar{x}^{j_k}}{\partial x^{i_k}} \cdot \varepsilon_{1\dots s_1-1\, s_1+1\dots s_k-1\, s_k+1\dots n}^{p_1\dots p_{j_1-1}p_{j_1+1}\dots p_{j_k-1}p_{j_k+1}\dots p_n}$$

$$\cdot \frac{\partial \bar{x}^1}{\partial x^{p_1}}\dots\frac{\partial \bar{x}^{j_1-1}}{\partial x^{p_{j_1-1}}}\frac{\partial \bar{x}^{j_1+1}}{\partial x^{p_{j_1+1}}}\dots\frac{\partial \bar{x}^{j_k-1}}{\partial x^{p_{j_k-1}}}\frac{\partial \bar{x}^{j_k+1}}{\partial x^{p_{j_k+1}}}\dots\frac{\partial \bar{x}^{n}}{\partial x^{p_n}} \cdot$$

$$\cdot\, dx^1\wedge\dots\wedge dx^{s_1-1}\wedge dx^{s_1+1}\wedge\dots\wedge dx^{s_k-1}\wedge dx^{s_k+1}\wedge\dots\wedge dx^n\wedge$$

$$\cdot\, \omega^{\sigma_1}\wedge\dots\wedge\omega^{\sigma_k},$$

where $(1,\dots,s_1-1,s_1+1,\dots,s_k-1,s_k+1,\dots,n)$ is the unique increasing sequence defined as a permutation of $(p_1,\dots,p_{j_1-1},p_{j_1+1},\dots,p_{j_k-1},p_{j_k+1},\dots,p_n)$. This expression takes the form

$$\text{(3.2.38)}\quad \det D\varphi\bar{\varphi}^{-1} \cdot \sum_{s_1<\dots<s_k} \sum_{\sigma_1,\dots,\sigma_k} \Bigg(\sum_{j_1<\dots<j_k} c^{1\dots j_1-1\, j_1+1\dots j_k-1\, j_k+1\dots n} \cdot$$

$$\cdot\, c^{1\dots s_1-1\, s_1+1\dots s_k-1\, s_k+1\dots n} \cdot \frac{\partial^k L}{\partial y_{q_1}^{\sigma_1}\dots\partial y_{q_k}^{\sigma_k}} \cdot$$

$$\cdot \frac{1}{k!}\varepsilon_{q_1\dots q_k}^{i_1\dots i_k} \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}}\dots\frac{\partial \bar{x}^{j_k}}{\partial \bar{x}^{i_k}} \cdot$$

$$\cdot\, \varepsilon_{1\quad s_1-1\, s_1+1\dots s_k-1\quad s_k+1\dots n}^{p_1\dots p_{j_1-1}p_{j_1+1}\dots p_{j_k-1}p_{j_k+1}\dots p_n} \frac{\partial \bar{x}^1}{\partial x^{p_1}}\dots\frac{\partial \bar{x}^{j_1-1}}{\partial x^{p_{j_1-1}}}\frac{\partial \bar{x}^{j_1+1}}{\partial x^{p_{j_1+1}}}\dots$$

$$\dots\frac{\partial \bar{x}^{j_k-1}}{\partial x^{p_{j_k-1}}}\frac{\partial \bar{x}^{j_k+1}}{\partial x^{p_{j_k+1}}}\dots\frac{\partial \bar{x}^{n}}{\partial x^{p_n}}\Bigg) dx^1\wedge\dots\wedge dx^{s_1-1}\wedge\omega^{\sigma_1}\wedge dx^{s_1+1}$$

$$\wedge\dots\wedge dx^{s_k-1}\wedge\omega^{\sigma_k}\wedge dx^{s_k+1}\wedge\dots\wedge dx^n = \det D\varphi\bar{\varphi}^{-1}.$$

$$\cdot \sum_{s_1<\dots<s_k} \sum_{\sigma_1,\dots,\sigma_k} \Bigg(\sum_{j_1<\dots<j_k} c^{1\dots j_1-1\, j_1+1\dots j_k-1\, j_k+1\dots n} \cdot$$

$$\cdot c^{1\dots s_1-1\; s_1+1\dots s_k-1\; s_k+1\dots n} \cdot \sum_{q_1<\dots<q_k} \frac{\partial^k L}{\partial y^{\sigma_1}_{q_1}\dots\partial y^{\sigma_k}_{q_k}} \cdot$$

$$\cdot\left(\varepsilon^{i_1\dots i_k}_{q_1\dots q_k}\frac{\partial\bar{x}^{j_1}}{\partial x^{i_1}}\dots\frac{\partial\bar{x}^{j_k}}{\partial x^{i_k}}\right)\cdot\varepsilon^{p_1\dots p_{j_1-1}p_{j_1+1}\dots p_{j_k-1}p_{j_k+1}\dots p_n}_{1\dots s_1-1\; s_1+1\dots s_k-1\; s_k+1\dots n} \cdot$$

$$\cdot\left.\frac{\partial\bar{x}^1}{\partial x^{p_1}}\dots\frac{\partial\bar{x}^{j_1-1}}{\partial x^{p_{j_1-1}}}\frac{\partial\bar{x}^{j_1+1}}{\partial x^{p_{j_1+1}}}\dots\frac{\partial\bar{x}^{j_k-1}}{\partial x^{p_{j_k-1}}}\frac{\partial\bar{x}^{j_k+1}}{\partial x^{p_{j_k+1}}}\dots\frac{\partial\bar{x}^n}{\partial x^{p_n}}\right)\cdot$$

$$\cdot\; dx^1\wedge\dots\wedge dx^{s_1-1}\wedge\omega^{\sigma_1}\wedge dx^{s_1+1}\wedge\dots\wedge dx^{s_k-1}\wedge\omega^{\sigma_k}\wedge dx^{s_k+1}\wedge\dots\wedge dx^n.$$

Let us consider the determinants

$$\varepsilon^{i_1\dots i_k}_{q_1\dots q_k}\frac{\partial\bar{x}^{j_1}}{\partial x^{i_1}}\dots\frac{\partial\bar{x}^{j_k}}{\partial x^{i_k}} = \frac{\partial(\bar{x}^{j_1},\dots,\bar{x}^{j_k})}{\partial(x^{q_1},\dots,x^{q_k})}, \tag{3.2.39}$$

$$\varepsilon^{p_1\dots p_{j_1-1}p_{j_1+1}\dots p_{j_k-1}p_{j_k+1}\dots p_n}_{1\dots s_1-1\; s_1+1\dots s_k-1\; s_k+1\dots n}\frac{\partial\bar{x}^1}{\partial x^{p_1}}\dots\frac{\partial\bar{x}^{j_1-1}}{\partial x^{p_{j_1-1}}}\frac{\partial\bar{x}^{j_1+1}}{\partial x^{p_{j_1+1}}}\dots$$

$$\dots\frac{\partial x^{j_k-1}}{\partial x^{p_{j_k-1}}}\frac{\partial x^{j_{k+1}}}{\partial x^{p_{j_k+1}}}\dots\frac{\partial\bar{x}^n}{\partial x^{p_n}} =$$

$$= \frac{\partial(\bar{x}^1,\dots,\bar{x}^{j_1-1},\bar{x}^{j_1+1},\dots,\bar{x}^{j_k-1},\bar{x}^{j_k+1},\dots,\bar{x}^n)}{\partial(x^1,\dots,x^{s_1-1},x^{s_1+1},\dots,x^{s_k-1},x^{s_k+1},\dots,x^n)}.$$

Considering the first one as the algebraic complement of the second we get

$$\sum_{j_1<\dots<j_k}(-1)^{j_1+\dots+j_k+s_1+\dots+s_n}\frac{\partial(\bar{x}^{j_1},\dots,\bar{x}^{j_k})}{\partial(x^{q_1},\dots,x^{q_k})}\cdot \tag{3.2.40}$$

$$\cdot\frac{\partial(\bar{x}^1,\dots,\bar{x}^{j_1-1},\bar{x}^{j_1+1},\dots,\bar{x}^{j_k-1},\bar{x}^{j_k+1},\dots,\bar{x}^n)}{\partial(x^1,\dots,x^{s_1-1},x^{s_1+1},\dots,x^{s_k-1},x^{s_k+1},\dots,x^n)} =$$

$$= \frac{\partial(\bar{x}^1,\dots,\bar{x}^{j_1-1},\bar{x}^{j_1},\bar{x}^{j_1+1},\dots,\bar{x}^{j_k-1},\bar{x}^{j_k},x^{j_k+1},\dots,x^n)}{\partial(x^1,\dots,x^{s_1-1},x^{q_1},x^{s_1+1},\dots,x^{s_k-1},x^{q_k},x^{s_k+1},\dots,x^n)};$$

this expression vanishes whenever $(q_1,\dots,q_n) \neq (s_1,\dots,s_k)$, and is equal to det $D\bar{\varphi}\varphi^{-1}$ if $(q_1,\dots,q_k) = (s_1,\dots,s_k)$. Since by (3.2.22)

$$(3.2.41)\quad c^{1\ldots j_1-1\, j_1+1\ldots j_k-1\, j_k+1\ldots n}\, c_{1\ldots s_1-1\, s_1+1\ldots s_k-1\, s_k+1\ldots n} = $$
$$= (-1)^{j_1+\ldots+j_k+s_1+\ldots+s_k},$$

(3.2.38) reduces to

$$(3.2.42)\quad \sum_{s_1<\ldots<s_k} \sum_{\sigma_1,\ldots,\sigma_k} \frac{\partial^k L}{\partial y^{\sigma_1}_{s_1}\ldots\partial y^{\sigma_k}_{s_k}} dx^1\wedge\ldots\wedge dx^{s_1-1}\wedge\omega^{\sigma_1}\wedge dx^{s_1+1}\wedge\ldots$$
$$\ldots\wedge dx^{s_k-1}\wedge\omega^{\sigma_k}\wedge dx^{s_k+1}\wedge\ldots\wedge dx^n$$

as desired.

Remark 3.3. The form (3.2.11) first appeared as one element of the family of forms introduced by Lepage [43] in his theory of geodesic fields. This family was defined by means of two congruences which have become known as the *first* and the *second Lepage's congruneces*. Later, the congruences of Lepage were interpreted and generalized by Dedecker [25], [21], [24]; he extended his approach also to higher order variational problems [20]. The class of problems he considered substantially differs from that one discussed in this work, which produces serious difficulties in comparing of the methods and results. For some special fibered manifolds, the form (3.2.11) was used by Sniatycki [49], García and Pérez-Rendón [31] (see also [29]), and Nôno and Mimura [47]; Goldschmidt and Sternberg [33] introduced it formally by means of some geometric axioms.

Lepagean equivalent (3.2.36) was introduced independently by Krupka [39] and Betounes [16], [15]. In our exposition in this paper we have followed [39], with the basis of forms (dx^i,ω^σ) instead of (dx^i,dy^σ); this enables us a direct comparison of the results of the above mentioned papers.

Our definition of a Lepagean form is taken from [9] (see also [7]); an equivalent definition has been given by Mangiarotti and Modugno [44].

Remark 3.4. Restricting ourselves in (3.2.36) to terms of order of contact $\leq k$ we obtain another Lepagean equivalent of λ.

We shall need the transformation formulas for the components of 1-contact n-forms. In the following, obvious notation related to fiber charts is used.

Lemma 3.3. *Let* (V,ψ), $\psi=(x^i,y^\sigma)$, *and* $(\bar V,\bar\psi)$, $\bar\psi=(\bar x^i,\bar y^\sigma)$, *be two fiber charts on* Y, *and let* $\eta\in\Omega^{n-1,1}(V_s)$ *and* $\bar\eta\in\Omega^{n-1,1}(\bar V_s)$ *be two* 1*-contact forms expressed by*

$$(3.2.43)\quad \eta = \sum_{k=0}^{s-1}\sum \eta_\sigma^{i,j_1\ldots j_k}\,\omega^\sigma_{j_1\ldots j_k}\wedge\omega_i,$$

$$\bar{\eta} = \sum_{l=0}^{s-1} \sum \bar{\eta}_{\nu}^{q,p_1\dots p_l} \cdot \omega'_{p_1\dots p_l} \wedge \bar{\omega}_q .$$

Then $\eta = \bar{\eta}$ *on* $V_s \cap \bar{V}_s$ *if and only if*

$$\text{(3.2.44)} \qquad \eta_{\sigma}^{i,j_1\dots j_k} = \det\left(\frac{\partial\bar{x}}{\partial x}\right) \cdot \sum_{l=k}^{s-1} \sum_{p_1\leq\dots\leq p_l} \frac{\partial\bar{y}^{\nu}_{p_1\dots p_l}}{\partial y^{\sigma}_{j_1\dots j_k}} \frac{\partial x^i}{\partial\bar{x}^q} \bar{\eta}_{\nu}^{q,p_1\dots p_l} .$$

Proof. We have

$$\text{(3.2.45)} \qquad \bar{\omega}_q = i_{\partial/\partial\bar{x}^q}\bar{\omega}_0 = \det\left(\frac{\partial\bar{x}}{\partial x}\right) \frac{\partial x^i}{\partial\bar{x}^q} i_{\partial/\partial x^i}\omega_0 = \det\left(\frac{\partial\bar{x}}{\partial x}\right) \frac{\partial x^i}{\partial\bar{x}^q} \omega_i .$$

Applying this formula and Theorem 2.4, (2.38) we easily get this assertion.

Now we take $r = 2$ and let n and m be arbitrary.

Corollary 4. *Each lagrangian of order 2* $\lambda \in \tilde{\Omega}^n_X(W)$ *has a Lepagean equivalent whose order of contact is* ≤ 1. *If* (V,ψ), $\psi = (x^i, y^{\sigma})$, *is a fiber chart on* Y *with associated chart* (U,φ) *on* X *and* λ *is expressed by* $\lambda = L.\tilde{\varphi} \otimes \omega_0$, *then* $\rho = \tilde{\varphi} \otimes \rho_0$, *where*

$$\text{(3.2.46)} \qquad \rho_0 = L\omega_0 + \sum_i \left[\left(\frac{\partial L}{\partial y^{\sigma}_i} - \sum_j \frac{1}{N(ij)} d_j \frac{\partial L}{\partial y^{\sigma}_{ij}}\right)\omega^{\sigma} + \sum_j \frac{1}{N(ij)} \frac{\partial L}{\partial y^{\sigma}_{ij}} \omega^{\sigma}_j\right] \wedge \omega_i ,$$

is such a Lepagean equivalent.

Proof. By Theorem 3.2 and Lemma 3.3 there exists one and only one Lepagean equivalent ρ of λ whose order of contact is ≤ 1, such that $\rho = \tilde{\varphi} \otimes \rho_0$, $\rho_0 = L\omega_0 + (f^i_{\sigma}\omega^{\sigma} + f^{i,j}_{\sigma}\omega^{\sigma}_j) \wedge \omega_i$, and the components $f^{i,j}_{\sigma}$ satisfy the invariant conditions $f^{i,j}_{\sigma} = f^{j,i}_{\sigma}$; this is precisely $\tilde{\varphi} \otimes \rho_0$ (3.2.46).

Remark 3.5. Lepagean equivalent (3.2.46) was derived by Krupka [9, p. 26]. An analogous form has been discussed by Dedecker [22].

Conditions (3.2.3) of Theorem 3.2 can be always satisfied locally, which ensures the existence of *local* Lepagean equivalents in the following sense.

Corollary 5. *Let* $\lambda \in \tilde{\Omega}^n_X(J^rY)$ *be a lagrangian of order* r, (V,ψ), $\psi = (x^i, y^{\sigma})$, *a fiber chart with associated chart* (U,φ) *on* X, *let* $\lambda = L.\tilde{\varphi} \otimes \omega_0$ *be the expression of* λ *for this fiber chart. We put*

$$\text{(3.2.47)} \qquad \rho_{\lambda,V} = \tilde{\varphi} \otimes \rho_0,$$

$$\rho_0 = L\omega_0 + \left(\sum_{k=0}^{r-1} \sum f_{\sigma}^{i,j_1\dots j_k} \omega^{\sigma}_{j_1\dots j_k}\right) \wedge \omega_i ,$$

$$f_\sigma^{j_r,j_1\ldots j_{r-1}} = \frac{N(j_1\ldots j_{r-1})}{N(j_1\ldots j_r)} \frac{\partial L}{\partial y^\sigma_{j_1\ldots j_r}},$$

$$f_\sigma^{j_k,j_1\ldots j_{k-1}} = \frac{N(j_1\ldots j_{k-1})}{N(j_1\ldots j_k)}\left(\frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k}} - d_i f_\sigma^{i,j_1\ldots j_k}\right), \quad 2 \le k \le r-1,$$

$$f_\sigma^j = \frac{\partial L}{\partial y^\sigma_j} - d_i f_\sigma^{i,j}.$$

Then $\rho_{\lambda,V}$ *is a Lepagean equivalent of the restriction of* λ *to* V_r*.*

Proof. Substituting (3.2.47) in (3.2.1) we obviously obtain a Lepagean equivalent of $L.\tilde{\varphi} \otimes \omega_0$.

Relations (3.2.18) can be written in the form

$$(3.2.48) \quad \frac{1}{N(j_1\ldots j_{k-1})} f_\sigma^{j_k,j_1\ldots j_{k-1}} =$$

$$= \sum_{l=0}^{r-k} \sum_{i_1,\ldots,i_l} \frac{(-1)^l}{N(j_1\ldots j_k i_1\ldots i_l)} d_{i_1}\ldots d_{i_l} \frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k i_1\ldots i_l}},$$

$2 \le k \le r$,

$$f_\sigma^j = \sum_{l=0}^{r-1} \sum_{i_1,\ldots,i_l} \frac{(-1)^l}{N(j i_1\ldots i_l)} d_{i_1}\ldots d_{i_l} \frac{\partial L}{\partial y^\sigma_{j i_1\ldots i_l}}.$$

If we denote

$$(3.2.49) \quad P_\sigma^{j_k j_1\ldots j_{k-1}} = \frac{1}{N(j_1\ldots j_{k-1})} f_\sigma^{j_k,j_1\ldots j_{k-1}}, \quad P_\sigma^j = f_\sigma^j,$$

then the local Lepagean equivalent $\rho_{\lambda,V} = \tilde{\varphi} \otimes \rho_0$ of λ gets the form

$$(3.2.50) \quad \rho_0 = L\omega_0 + \sum_{k=0}^{r-1} P_\sigma^{ij_1\ldots j_k} \omega^\sigma_{j_1\ldots j_k} \wedge \omega_i.$$

If ρ is any Lepagean equivalent of λ whose order of contact is ≤ 1, then by (3.2.5) and Theorem 3.2, (2)

$$(3.2.51) \quad \rho = \rho_{\lambda,V} + \tilde{\varphi} \otimes \nu_V,$$

and ν_V is given by

$$(3.2.52) \quad \nu_V = \sum_{k=0}^{r-1} Q_\sigma^{ij_1\ldots j_k} \omega^\sigma_{j_1\ldots j_k} \wedge \omega_i,$$

$$Q_\sigma^{j_k j_1 \dots j_{k-1}} = g_\sigma^{j_k, j_1 \dots j_{k-1}} +$$

$$+ \sum_{l=1}^{r-k} \sum_{i_1, \dots, i_l} (-1)^l d_{i_1} \dots d_{i_l} g_\sigma^{j_l, j_1 \dots j_k i_1 \dots i_{l-1}} ,$$

$$Q_\sigma^j = \sum_{l=1}^{r-1} \sum_{i_1, \dots, i_l} (-1)^l d_{i_1} \dots d_{i_l} g_\sigma^{i_l, j_1 \dots j_k i_1 \dots i_{l-1}} ,$$

where $g_\sigma^{(j_k, j_1 \dots j_{k-1})} = 0$.

We shall now show that if $n \neq 1$ and $r > 2$, then the local Lepagean equivalents (3.2.47) of a lagrangian do not necessarily define a global differential form on $J^{2r-1}Y$. Following Kolář (private communication) we shall analyze transformation properties of the functions (3.2.49). For this purpose it is sufficient to consider the case $r = 3$ and ordinary differential forms, although the general case does not require additional, other than typographical, efford.

Let (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\overline{V}, \overline{\psi})$, $\overline{\psi} = (\overline{x}^i, \overline{y}^\sigma)$, be two fiber charts. Denote by $\overline{\omega}_0$, $\overline{\omega}_q$, $\overline{\omega}^\nu$, $\overline{\omega}_p^\nu$, and $\overline{\omega}_{pq}^\nu$ the forms (3.1.1) related to the fiber chart $(\overline{V}, \overline{\psi})$. Let $\lambda \in \Omega_X^n(J^3Y)$ be a lagrangian. Consider the local Lepagean equivalents $\rho_{\lambda,V} = \tilde{\varphi} \otimes \rho_0$ (3.2.50) and $\rho_{\lambda,\overline{V}} = \tilde{\varphi} \otimes \overline{\rho}_0$, $\overline{\rho}_0 = \overline{L}\overline{\omega}_0 + (\overline{P}_\nu^q \overline{\omega}^\nu + \overline{P}_\nu^{qp} \overline{\omega}_p^\nu + \overline{P}_\nu^{qpr} \overline{\omega}_{pr}^\nu) \wedge \overline{\omega}_q$. Using (2.3.8) and (3.2.45) we get on $V_{2r-1} \cap \overline{V}_{2r-1}$

$$\rho_0 - \overline{\rho}_0 = \left[\left(P_\sigma^i - \det\left(\frac{\partial \overline{x}}{\partial x}\right) \frac{\partial x^i}{\partial \overline{x}^q} \left(\overline{P}_\nu^q \frac{\partial \overline{y}^\nu}{\partial y^\sigma} + \overline{P}_\nu^{qp} \frac{\partial \overline{y}_p^\nu}{\partial y^\sigma} + \overline{P}_\nu^{qpr} \frac{\partial \overline{y}_{pr}^\nu}{\partial y^\sigma} \right) \right) \omega^\sigma + \left(P_\sigma^{ij} - \det\left(\frac{\partial \overline{x}}{\partial x}\right) \frac{\partial x^i}{\partial \overline{x}^q} \left(\overline{P}_\nu^{qp} \frac{\partial \overline{y}_p^\nu}{\partial y_j^\sigma} + \overline{P}_\nu^{qpr} \frac{\partial \overline{y}_{pr}^\nu}{\partial y_j^\sigma} \right) \right) \omega_j^\sigma + \left(P_\sigma^{ijk} - \det\left(\frac{\partial \overline{x}}{\partial x}\right) \frac{\partial x^i}{\partial \overline{x}^q} \overline{P}_\nu^{qpr} \cdot \frac{1}{N(jk)} \frac{\partial \overline{y}_{pr}^\nu}{\partial y_{jk}^\sigma} \right) \cdot \omega_{jk}^\sigma \right] \wedge \omega_i . \tag{3.2.53}$$

By definition, $\tilde{\varphi} \otimes (\rho_0 - \overline{\rho}_0)$ is a Lepagean equivalent of the zero lagrangian. Hence by Theorem 3.2 there exist functions $g_\sigma^{i,jk}$ and $g_\sigma^{i,j}$ such that $g_\sigma^{(i,jk)} = 0$, $g_\sigma^{(i,j)} = 0$, and

$$P_\sigma^{ijk} - \det\left(\frac{\partial \overline{x}}{\partial x}\right) \frac{\partial x^i}{\partial \overline{x}^q} \overline{P}_\nu^{qpr} \frac{1}{N(jk)} \frac{\partial \overline{y}_{pr}^\nu}{\partial y_{jk}^\sigma} = g_\sigma^{i,jk}, \tag{3.2.54}$$

$$P_\sigma^{ij} - \det\left(\frac{\partial \overline{x}}{\partial x}\right) \frac{\partial x^i}{\partial \overline{x}^q} \left(\overline{P}_\nu^{qp} \frac{\partial \overline{y}_p^\nu}{\partial y_j^\sigma} + \overline{P}_\nu^{qpr} \frac{\partial \overline{y}_{pr}^\nu}{\partial y_j^\sigma} \right) = g_\sigma^{i,j} - d_s g_\sigma^{s,ij}$$

$$p^i_\sigma - \det\left(\frac{\partial\bar{x}}{\partial x}\right)\frac{\partial x^i}{\partial\bar{x}^q}\left(\bar{p}^q_\nu\frac{\partial\bar{y}^\nu}{\partial y^\sigma} + \bar{p}^{qp}_\nu\frac{\partial\bar{y}^\nu_p}{\partial y^\sigma} + \bar{p}^{qpr}_\nu\frac{\partial\bar{y}^\nu_{pr}}{\partial y^\sigma}\right)$$

$$= - d_s g^{s,i}_\sigma + d_s d_k g^{s,ik}_\sigma .$$

Since

$$(3.2.55)\qquad \frac{\partial\bar{y}^\nu_{pr}}{\partial y^\sigma_{jk}} = N(jk)\frac{\partial\bar{y}^\nu}{\partial y^\sigma}\frac{\partial x^{(j}}{\partial\bar{x}^p}\frac{\partial x^{k)}}{\partial\bar{x}^r}$$

(symmetrization in j,k) and the first expression on the left is symmetric in i,j,k, we have $g^{i,jk}_\sigma = 0$. The second equality gives by a simple calculation

$$(3.2.56)\qquad g^{i,j}_\sigma = \frac{1}{2}\det\left(\frac{\partial\bar{x}}{\partial x}\right)\left(\frac{\partial x^j}{\partial\bar{x}^q}\frac{\partial\bar{y}^\nu_{pr}}{\partial y^\sigma_i} - \frac{\partial x^i}{\partial\bar{x}^q}\frac{\partial\bar{y}^\nu_{pr}}{\partial y^\sigma_j}\right)\bar{p}^{qpr}_\nu =$$

$$= \frac{1}{2}\det\left(\frac{\partial\bar{x}}{\partial x}\right)\frac{\partial\bar{y}^\nu}{\partial y^\sigma}\left(\frac{\partial^2 x^i}{\partial\bar{x}^p\partial\bar{x}^r}\frac{\partial x^j}{\partial\bar{x}^q} - \frac{\partial^2 x^j}{\partial\bar{x}^p\partial\bar{x}^r}\frac{\partial x^i}{\partial\bar{x}^q}\right)\bar{p}^{qpr}_\nu .$$

This expression is seen not to vanish unless additional assumptions are imposed on the lagrangian λ, or on the topological structure of the manifold X, or both. Thus in general, $\tilde{\varphi}\otimes(\rho_0 - \bar{\rho}_0) \neq 0$, and the local Lepagean equivalents do not agree on intersections of coordinate neighborhoods.

These remarks show that so far the problem of *existence* of a Lepagean equivalent of a lagrangian remains open; positive answer has only been given for a few particular cases. To obtain the fundamental existence theorem, notice that if for a lagrangian $\lambda \in \tilde{\Omega}^n_X(J^rY)$ there exists a fiber chart (V,ψ) such that supp $\lambda \subset V_r$, then by Corollary 5, or by (3.2.50), $\rho_{\lambda,V}$ can be extended to a *global* Lepagean equivalent of λ by putting $\rho_{\lambda,V} = 0$ outside V_{2r-1}. We can now prove the following result.

Theorem 3.3. *Each lagrangian of order r has a Lepagean equivalent.*

Proof. Let (V_ι,ψ_ι), $\iota \in I$, be fiber charts on Y, defining an atlas, let $((V_\iota)_s,(\psi_\iota)_s)$ be the associated charts on J^sY (see 2.1). We may suppose that (V_ι) is a locally finite covering of Y. Let (χ_ι), $\iota \in I$, be a partition of unity, subordinate to this covering. Consider any lagrangian $\lambda \in \tilde{\Omega}^n_X(J^rY)$. For each $\iota \in I$, $\chi_\iota\lambda$ is a lagrangian for which supp $\chi_\iota\lambda \subset (V_\iota)_r$, where supp $\chi_\iota\lambda$ is the support of $\chi_\iota\lambda$. Define a Lepagean equivalent ρ_ι of $\chi_\iota\lambda$ as the local Lepagean equivalent (3.2.47) on $(V_\iota)_{2r-1}$, and the zero form outside $(V_\iota)_{2r-1}$, and set

$$(3.2.57)\qquad \rho = \sum_\iota \rho_\iota .$$

We have $\pi^*_{2r,2r-1}d\rho = \sum_\iota \pi^*_{2r,2r-1}d\rho_\iota$. Since $p_1(\pi^*_{2r,2r-1}d\rho_\iota)$ is $\pi_{2r,0}$-horizontal for each ι, so is $p_1(\pi^*_{2r,2r-1}d\rho)$, and by Theorem 3.1, ρ is a Lepagean form. Moreover, $h(\rho) = \Sigma h(\rho_\iota) = \Sigma\chi_\iota\lambda = (\Sigma\chi_\iota)\lambda = \lambda$, and ρ is a Lepagean equivalent of λ.

The Lepagean equivalent constructed in the proof of Theorem 3.3 can be characterized more precisely.

Theorem 3.4. *Each lagrangian* $\lambda \in \tilde{\Omega}^n_X(J^rY)$ *has a Lepagean equivalent* $\rho \in \tilde{\Omega}^n(J^{2r-1}Y)$ *such that the following condition holds: To each point* $y \in Y$ *there exists a fiber chart* (V,ψ), $\psi = (x^i, y^\sigma)$, *at* y *for which* ρ *is expressed by (3.2.54), (3.2.55), where* $Q^{ij_1\dots j_{r-1}}_\sigma = 0$ *and the functions* $Q^{ij_1\dots j_{k-1}}_\nu$ *depend on* $x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1\dots j_s}$, $s = 2r - k - 1$, *only.*

Proof. The proof consists in finding the expression of ρ (3.2.57) for a proper fiber chart.

1. Let (V_ι,ψ_ι), (χ_ι), λ, and ρ_ι be as in the proof of Theorem 3.3. Expressing ρ_ι with respect to (V_ι,ψ_ι) we obtain, with the obvious notation, $\rho_\iota = \tilde{\varphi}_\iota \otimes \rho_{\iota,0}$, where

$$(3.2.58)\qquad \rho_{\iota,0} = \chi_\iota L_\iota \omega_{0(\iota)} + \sum_{s=0}^{r-1} P^{qq_1\dots q_s}_{\nu(\iota)} \omega^\nu_{q_1\dots q_s(\iota)} \wedge \omega_{q(\iota)} ,$$

L_ι is defined by the chart expression $\lambda = L_\iota . \tilde{\varphi}_\iota \otimes \omega_{0(\iota)}$, and the functions $P^{qq_1\dots q_s}_{\nu(\iota)}$ are obtained from $\chi_\iota L_\iota$ by means of (3.2.49) and (3.2.48). Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart such that $V \cap V_\iota \neq \phi$ for only finitely many $\iota \in I$, let (U,φ) be the associated chart on X. Writting $\lambda = L.\tilde{\varphi} \otimes \omega_0$ with respect to (V,ψ) and using the transformation formulas between L_ι and L, $\omega_{q(\iota)}$ and ω_i (3.2.45), and $\bar{\omega}^\nu_{q_1\dots q_s(\iota)}$ and $\omega^\sigma_{j_1\dots j_k}$ (2.3.8) we obtain

$$(3.2.59)\qquad \tilde{\varphi}_\iota \otimes \rho_{\iota,0} = \tilde{\varphi} \otimes \Big(\chi_\iota L\omega_0 + |\det D\varphi_\iota\varphi^{-1}| . \sum_{s=0}^{r-1} \sum_{k=0}^{s} \sum_{j_1\leq\dots\leq j_k} \frac{\partial y^\nu_{q_1\dots q_s(\iota)}}{\partial y^\sigma_{j_1\dots j_k}} . \frac{\partial x^i}{\partial x^q_{(\iota)}} P^{qq_1\dots q_s}_{\nu(\iota)} . \omega^\sigma_{j_1\dots j_k} \wedge \omega_i\Big) .$$

Let us subtract from this expression the local Lepagean equivalent $\rho_{\chi,\lambda,V}$ of $\chi_\iota\lambda$, expressed by

$$(3.2.60)\qquad \rho_{\chi_\iota\lambda,V} = \tilde{\varphi} \otimes \Big(\chi_\iota L.\omega_0 + \sum_{k=0}^{r-1} S^{ij_1\dots j_k}_{\sigma(\iota)} \omega^\sigma_{j_1\dots j_k} \wedge \omega_i\Big).$$

We get on V_{2r-1} $\rho_\iota - \rho_{\chi_\iota\lambda,V} = \tilde{\varphi} \otimes \nu_{\iota,V}$, where

$$(3.2.61)\qquad \nu_{\iota,V} = \sum_{k=0}^{r-1} Q^{ij_1\dots j_k}_{\sigma(\iota)} \omega^\sigma_{j_1\dots j_k} \wedge \omega_i ,$$

and

$$(3.2.62)\quad Q_{\sigma(\iota)}^{ij_1\dots j_k} = \frac{1}{N(j_1\dots j_k)}\det\left(\frac{\partial x_{(\iota)}}{\partial x}\right)\sum_{s=k}^{r-1}\frac{\partial y^{\nu}_{q_1\dots q_s(\iota)}}{\partial y^{\sigma}_{j_1\dots j_k}}\frac{\partial x^i}{\partial x^p_{(\iota)}}P^{pq_1\dots q_s}_{\nu(\iota)} - S_{\sigma(\iota)}^{ij_1\dots j_k}, \quad 1 \le k \le r-1,$$

$$Q^i_{\sigma(\iota)} = \det\left(\frac{\partial x_{(\iota)}}{\partial x}\right)\sum_{s=0}^{r-1}\frac{\partial y^{\nu}_{q_1\dots q_s(\iota)}}{\partial y^{\sigma}}\frac{\partial x^i}{\partial x^p_{(\iota)}}P^{pq_1\dots q_s}_{\nu(\iota)} - S^i_{\sigma(\iota)} .$$

Since $\tilde{\varphi}\otimes\nu_{\iota,V}$ is a Lepagean equivalent of the zero lagrangian, it is completely determined, via (3.2.52), by the non-symmetric parts of the components (3.2.62), i.e., by the functions

$$(3.2.63)\quad g_{\sigma(\iota)}^{i,j_1\dots j_k} = Q_{\sigma(\iota)}^{ij_1\dots j_k} - Q_{\sigma(\iota)}^{(ij_1\dots j_k)} .$$

It is easily seen that $g_{\sigma(\iota)}^{ij_1\dots j_{r-1}} = Q_{\sigma}^{ij_1\dots j_{r-1}} = 0$ and that (3.2.63) implies **that** the functions $g_{\sigma(\iota)}^{ij_1\dots j_k}$ are independent of $y^{\nu}_{p_1\dots p_{2r-k-1}},\dots,y^{\nu}_{p_1\dots p_{2r-1}}$.

2. Let us put $\rho = \Sigma\rho_\iota$ and consider this form on V_{2r-1}. We have $\rho = \rho_{\lambda,V} + \tilde{\varphi}\otimes\nu_V$ (3.2.51) so that $\Sigma(\rho_{\chi_\iota\lambda,V} + \tilde{\varphi}\otimes\nu_{\iota,V}) = \rho_{\lambda,V} + \tilde{\varphi}\otimes\nu_V$. Using (3.2.52) and comparing the non-symmetric parts of the coefficients on both sides we get

$$(3.2.64)\quad g_{\sigma}^{i,j_1\dots j_k} = \sum_{\iota} g_{\sigma(\iota)}^{i,j_1\dots j_k} = Q_{\sigma}^{ij_1\dots j_k} - Q_{\sigma}^{(ij_1\dots j_k)} .$$

These functions determine uniquely ν_V, by (3.2.52). Thus, using the first part of the proof, we obtain that ρ has all the required properties.

Remark 3.6. Higher order Lepagean forms have been introduced by Krupka [9], [8] in full analogy with [7] (see also [3], [41]). The local Lepagean equivalents (3.2.50) have been considered by Aldaya and Azcárraga [13] (for vector bundles), Krupka (see the above - mentioned papers), and Shadwick [48]; in fact, the quantities (3.2.49) appear in the classical first variation formula, and are called the *variational derivatives* of L (see de Donder [26, (27) - (30)]). Following the approach and terminology of Garcia [29], some authors have been developing the theory of "Poincaré-Cartan forms" (i.e., Lepagean forms whose order of contact is ≤ 1) with the help of auxiliary connections, or fibered connections (Ferraris and Francaviglia [28], Garcia and Muñoz [30], Ferraris [27], Kolář [37]), or in the language of morphisms (Horák and Kolář [35], Kolář [38]; see also [27 §2.3]). In [27], the third order case has been analyzed in detail. The same axioms and motivation for the fundamental variational form (Lepagean form) as those of our, (see [9], [7]) were re-discovered by Kupershmidt [10, (1.7)]. Marvan [45] has proved the existence of a Lepagean equivalent by means of elementary sheaf theory. The proof given in this paper is taken from [42].

We shall give an example of a second order lagrangian which has a first order Lepagean equivalent. Our discussion is based on [9, p. 26], and on the papers by Szczyrba [51], Kijowski [36], and Novotný [46].

Let X be an n-dimensional manifold, $T^*X \odot T^*X$ the vector bundle of symmetric covariant tensors of degree 2 over X. Let $x \in X$ be a point, (U,φ), $\varphi = (x^i)$, a chart at x. Each tensor $h \in T^*X \odot T^*X$ at x has a unique expression $h = g_{ij}(h)dx^i \odot dx^j$, where $g_{ij}(h) = g_{ji}(h)$; h is called *regular*, if $\det(g_{ij}(h)) \neq 0$. Denote by $T_{\text{met}}X \subset T^*X \odot T^*X$ the open subset of regular tensors. $T_{\text{met}}X$, viewed as a fibered manifold with base X and projection τ, defined as the restriction of the vector bundle projection of $T^*X \odot T^*X$, is called the *fibered manifold of metrics* over X. For any chart (U,φ), $\varphi = (x^i)$, on X, the pair (V,ψ), where $V = \tau^{-1}(U)$, $\psi = (x^i, g_{jk})$, $j \leq k$, is a fiber chart on $T_{\text{met}}X$, called *associated* with (U,φ). The corresponding associated chart on $J^rT_{\text{met}}X$ is denoted by (V_r,ψ_r), where $V_r = \tau_r^{-1}(U)$, $\psi_r = (x^i, g_{jk}, g_{jk,l_1}, \ldots, g_{jk,l_1\ldots l_r})$, $j \leq k$, $l_1 \leq \ldots \leq l_r$. We define g^{ij} as the elements of the inverse matrix of (g_{ij}), and denote $g = |\det(g_{ij})|$; g_{ij}, g^{ij}, and g may be considered as functions on any of the sets U, V, V_s, $1 \leq s \leq r$. For a function f on V_r we denote $d_i f = f_{,i}$.

The *Hilbert lagrangian* for $T_{\text{met}}X$ is the odd base n-form $\lambda \in \tilde{\Omega}^n(J^2T_{\text{met}}X)$ such that for any chart (U,φ), $\varphi = (x^i)$, $\lambda = R.\tilde{\varphi} \otimes \omega_0$, where $\omega_0 = dx^1 \wedge \ldots \wedge dx^n$, $\mathcal{R} = R.\sqrt{g}$, $R = g^{ik}g^{jl}R_{ijkl}$, and R_{ijkl} are the components of the *formal curvature tensor*,

$$(3.2.65) \qquad R_{ijkl} = \frac{1}{2}(g_{il,jk} - g_{ik,jl} + g_{jk,il} - g_{jl,ik}) + g_{pq}(\Gamma^p_{jk}\Gamma^q_{il} - \Gamma^p_{jl}\Gamma^q_{ik}),$$

where

$$(3.2.66) \qquad \Gamma^p_{jk} = \frac{1}{2} g^{pm}(g_{mj,k} + g_{mk,j} - g_{jk,m})$$

are the *formal Christoffel symbols*, considered as real valued functions on $\tau_1^{-1}(U)$, or $\tau_s^{-1}(U)$, $1 \leq s \leq r$.

Denote $\omega_{kl} = dg_{kl} - g_{kl,p}dx^p$, $\omega_{kl,j} = dg_{kl,j} - g_{kl,jp}dx^p$. According to (3.2.46) the Lepagean equivalent of λ has the form $\rho = \varphi \otimes \rho_0$, where

$$(3.2.67) \qquad \rho_0 = \mathcal{R}\omega_0 + \sum_i \sum_{k\leq l}\left[\left(\frac{\partial\mathcal{R}}{\partial g_{kl,i}} - \sum_j d_j \frac{1}{N(ij)} \frac{\partial\mathcal{R}}{\partial g_{kl,ij}}\right)\omega_{kl} + \sum_j \frac{1}{N(ij)} \frac{\partial\mathcal{R}}{\partial g_{kl,ij}} \omega_{kl,j}\right] \wedge \omega_i .$$

After some calculation we obtain

$$(3.2.68) \qquad \rho_0 = \sqrt{g}\Big[g^{pq}(\Gamma^k_{pq}\Gamma^s_{ks} - \Gamma^k_{sq}\Gamma^s_{kp})\omega_0 + \sum_i(\Gamma^k_{pq}(g^{pl}g^{iq} - g^{pq}g^{il})dg_{kl} +$$

$$+ (g^{kj}g^{il} - g^{kl}g^{ij})dg_{kl,j}\Big) \wedge \omega_i\Big] .$$

Using the relation

$$(3.2.69) \qquad g_{kl,j} = g_{pk}\Gamma^p_{lj} + g_{pl}\Gamma^p_{ki}$$

we can express ρ_0 in forms of the coordinates $(x^i, g_{ij}, \Gamma^i_{jk})$ on $J^1T_{\text{met}}X$. We obtain

$$(3.2.70) \qquad \rho_0 = \sqrt{g}\Big[g^{pq}(\Gamma^k_{pq}\Gamma^s_{ks} - \Gamma^k_{sq}\Gamma^s_{kp})\omega_0 + \sum_i((g^{kj}g^{ip} - g^{kp}g^{ij})\Gamma^l_{pj}dg_{kl} + g^{kj}d\Gamma^i_{kj} - g^{ij}d\Gamma^l_{lj}) \wedge \omega_i\Big] .$$

3.3. Variations of sections and their jet prolongations. Let Ξ be a vector field on Y, α_t its local one-parameter group. It is easily seen that Ξ is π-projectable if and only if each point $y \in Y$ has a neighborhood V such that α_t is defined on V for any sufficiently small t, and is an isomorphism of the fibered manifold V onto $\alpha_t(V)$.

Suppose that Ξ is a π-projectable vector field on Y, and denote by ξ its π-projection. For each t, denote by $J^r\alpha_t$ the r-jet prolongation of the isomorphism of fibered manifold α_t (Sec. 2.1). $J^r\alpha_t$ is an isomorphism of J^rY considered with any of the projections π_r, $\pi_{r,s}$, and these isomorphisms define a local one parameter transformation group of J^rY. We put for each point $J^r_x\gamma$ from the domain of definition of $J^r\alpha_t$

$$(3.3.1) \qquad J^r\Xi(J^r_x\gamma) = \left\{\frac{d}{dt} J^r\alpha_t(J^r_x\gamma)\right\}_0 .$$

This relation defines a vector field $J^r\Xi$ on J^rY, called the *r-jet prolongation* of the π-projectable vector field Ξ. $J^r\Xi$ is π_r-projectable (resp. $\pi_{r,s}$-projectable for any s, $0 \le s \le r$), and its π_r-projection (resp. $\pi_{r,s}$-projection) is ξ (resp. $J^s\Xi$).

The following theorem describes the local structure of $J^r\Xi$. In its formulation we use the notion of formal derivative of a function (2.2.15).

Theorem 3.5. *Let Ξ be a π-projectable vector field on Y, (V,ψ), $\psi = (x^i, y^\sigma)$, a fiber chart, and let Ξ be expressed by*

$$(3.3.2) \qquad \Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} .$$

Then $J^r\Xi$ is expressed with respect to (V_r, ψ_r) by

$$(3.3.3) \qquad J^r\Xi = \xi^i \frac{\partial}{\partial x^i} + \sum_{k=0}^{r} \sum \Xi^\sigma_{j_1\ldots j_k} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}} ,$$

where $\Xi^\sigma_{j_1\ldots j_k}$ are functions on V_r determined by the recurrent formula

$$(3.3.4)\qquad \Xi^{\sigma}_{j_1\ldots j_k} = d_{j_k}\Xi^{\sigma}_{j_1\ldots j_{k-1}} - y^{\sigma}_{j_1\ldots j_{k-1}l}\frac{\partial\xi^l}{\partial x^{j_k}} .$$

Proof. Let $J^r_x\gamma \in V_r$. By definition,

$$(3.3.5)\qquad \Xi^{\sigma}_{j_1\ldots j_k}(J^r_x\gamma) = \left\{\frac{d}{dt}\, y^{\sigma}_{j_1\ldots j_k}\circ J^r\alpha_t(J^r_x\gamma)\right\}_0 ,$$

where α_t is the local one-parameter group of Ξ. Let β_t be the π-projection of α_t, and let (U,φ), $\varphi = (x^i)$, be the associated chart on X. We have

$$(3.3.6)\qquad y^{\sigma}_{j_1\ldots j_k}\circ J^r\alpha_t(J^r_x\gamma) = \left\{\frac{\partial^k}{\partial x^{j_1}\ldots\partial x^{j_k}}\, y^{\sigma}\alpha_t\gamma\beta_t^{-1}\varphi^{-1}\right\}_{\varphi\beta_t(x)} .$$

Since this expression is equal to

$$(3.3.7)\qquad \left\{\frac{\partial}{\partial x^{j_k}}\frac{\partial^{k-1}}{\partial x^{j_1}\ldots\partial x^{j_{k-1}}}\, y^{\sigma}\alpha_t\gamma\beta_t^{-1}\varphi^{-1}\right\}_{\varphi\beta_t(x)}$$

$$= \left\{\frac{\partial}{\partial x^i}\left(\frac{\partial^{k-1}}{\partial x^{j_1}\ldots\partial x^{j_{k-1}}}\, y^{\sigma}\alpha_t\gamma\beta_t^{-1}\varphi^{-1}\right)\circ\varphi\beta_t\varphi^{-1}\right\}_{\varphi(x)} \cdot$$

$$\cdot\left\{\frac{\partial}{\partial x^{j_k}}\, x^i\beta_t^{-1}\varphi^{-1}\right\}_{\varphi\beta_t(x)} ,$$

we obtain

$$(3.3.8)\qquad \Xi^{\sigma}_{j_1\ldots j_k}(J^r_x\gamma) =$$

$$= \left\{\frac{\partial}{\partial x^i}\frac{d}{dt}\left(\left(\frac{\partial^{k-1}}{\partial x^{j_1}\ldots\partial x^{j_{k-1}}}\, y^{\sigma}\alpha_t\gamma\beta_t^{-1}\varphi^{-1}\right)\circ\varphi\beta_t\varphi^{-1}\right)\right\}_{\varphi(x)}\delta^i_{jk} -$$

$$- y^{\sigma}_{j_1\ldots j_{k-1}i}\frac{\partial\xi^i}{\partial x^{j_k}} .$$

In this expression, we have the derivative of the mapping

$$(3.3.9)\qquad (t,x^1,\ldots,x^n) \to \left(\left(\frac{\partial^{k-1}}{\partial x^{j_1}\ldots\partial x^{j_{k-1}}}\, y^{\sigma}\alpha_t\gamma\beta_t^{-1}\varphi^{-1}\right)\circ\varphi\beta_t\varphi^{-1}\right)(x^1,\ldots,x^n).$$

Since the derivative with respect to t at $t = 0$ is the component $\Xi^{\sigma}_{j_1\ldots j_{k-1}}$ of $J^r\xi$, (3.3.8) immediately gives (3.3.4).

In the proof of the next theorem we use the non-holonomic jets (Sec. 2.1).

Theorem 3.6. *For any two π-projectable vector fields ξ,ζ on Y the Lie bracket $[\xi,\zeta]$ is also a π-projectable vector field, and*

(3.3.10) $\quad J^r[\xi,\zeta] = [J^r\xi,\ J^r\zeta]$.

Proof. If $r = 1$, (3.3.10) can be proved in fiber coordinates by a direct calculation.

We shall now suppose that $J^{r-1}[\xi,\zeta] = [J^{r-1}\xi,\ J^{r-1}\zeta]$. Let $\iota : J^rY \to J^1(J^{r-1}Y)$ be the canonical embedding (2.1.9). Let α be an isomorphism of Y, defined on an open set $V \subset Y$, and $\alpha_0 = \mathrm{pr}\ \alpha$. We have for each $J^r_x\gamma \in \pi^{-1}_{r,0}(V)$

(3.3.11) $\quad \iota\circ J^r\alpha(J^r_x\gamma) = J^1_{\alpha_0(x)}(J^{r-1}\alpha\gamma\alpha_0^{-1})$.

Let us consider the 1-jet prolongation $J^1(J^{r-1}\alpha)$. We have

(3.3.12) $\quad J^1(J^{r-1}\alpha)(\iota(J^r_x\gamma)) = J^1_{\alpha_0(x)}(J^{r-1}\alpha\circ J^{r-1}\gamma\circ\alpha_0^{-1})$.

Put $J^{r-1}\alpha\circ J^{r-1}\gamma\circ\alpha_0^{-1} = J^{r-1}(\alpha\gamma\alpha_0^{-1})\circ\alpha_0\circ\alpha_0^{-1} = J^{r-1}(\alpha\gamma\alpha_0^{-1})$ so that

(3.3.13) $\quad J^1(J^{r-1}\alpha)\circ\iota = \iota\circ J^r\alpha$.

Denote by $J^1(J^{r-1}\xi)$ (resp. $J^1(J^{r-1}\zeta)$) the 1-jet prolongation of the π_{r-1}-
- projectable vector field $J^{r-1}\xi$ (resp. $J^{r-1}\zeta$). Applying (3.3.13) to the local one-parameter groups of these vector fields we get

(3.3.14) $\quad J^1(J^{r-1}\xi)\circ\iota = T\iota\ .\ J^r\xi, \quad J^1(J^{r-1}\zeta)\circ\iota = T\iota\ .\ J^r\zeta$.

Let us consider the vector field $J^r[\xi,\zeta]$. By (3.3.14),

$$T\iota\ .\ J^r[\xi,\zeta] = J^1(J^{r-1}[\xi,\zeta])\circ\iota = J^1([J^{r-1}\xi,\ J^{r-1}\zeta])\circ\iota \tag{3.3.15}$$
$$= [J^1(J^{r-1}\xi),\ J^1(J^{r-1}\zeta)]\circ\iota,$$

where we have applied our inductive assumption. On the other hand, (3.3.14) implies

(3.3.16) $\quad [J^1(J^{r-1}\xi), J^1(J^{r-1}\zeta)]\circ\iota = T\iota\ .\ [J^r\xi,\ J^r\zeta]$.

Comparing (3.3.15) and (3.3.16) we get $T\iota.(J^r[\xi,\zeta] - [J^r\xi, J^r\zeta]) = 0$, and (3.3.10) follows from the fact that $T\iota$ is injective.

We note that for any π-projectable vector field Ξ on Y and any form $\rho \in \tilde{\Omega}^p(J^rY)$

(3.3.17) $\quad \partial_{J^{r+1}\Xi} h(\rho) = h(\partial_{J^r\Xi}\rho)$.

This follows from Theorem 2.1 (f).

The concept of the r-jet prolongation of a π-projectable vector field can be generalized to vector fields along sections of Y. Let γ be a section of Y, W its domain of definition. Recall that a *vector field along* γ is a mapping $\zeta : W \to TY$ such that for each $x \in W$, $\zeta(x) \in T_{\gamma(x)}Y$. A vector field along γ is also called a *variation* of γ. If ζ is a variation of γ, then the formula

$$(3.3.18) \qquad \zeta_0 = T\pi.\zeta$$

defines a vector field on W, called the *π-projection* of ζ.

Theorem 3.7. *Let γ be a section of Y defined on W, ζ a variation of γ.*

(a) There exists a π-projectable vector field Ξ, defined on a neighborhood of the set $\gamma(W) \subset Y$, such that for each $x \in W$

$$(3.3.19) \qquad \Xi(\gamma(x)) = \zeta(x) .$$

(b) For any two π- projectable vector fields Ξ_1, Ξ_2, defined on a neighborhood of $\gamma(W)$, such that $\Xi_1(\gamma(x)) = \Xi_2(\gamma(x)) = \zeta(x)$ for all $x \in W$

$$(3.3.20) \qquad J^r\Xi_1(J^r_x\gamma) = J^r\Xi_2(J^r_x\gamma) .$$

Proof. (a) Let (V_ι, ψ_ι), $\psi_\iota = (x^i_\iota, y^\sigma_\iota)$, be some fiber charts on Y such that $\cup V_\iota \supset \gamma(W)$, and let (U_ι, φ_ι) be the associated charts on X. With respect to (U_ι, φ_ι) and (V_ι, ψ_ι), ζ is expressed by $\zeta = \zeta^i_\iota(\partial/\partial x^i_\iota) + Z^\sigma_\iota(\partial/\partial y^\sigma_\iota)$, where ζ^i_ι, Z^σ_ι are some functions of $x^1_\iota, \dots, x^n_\iota$. Let (χ_ι) be a partition of unity, subordinate to the covering (V_ι) of $\cup V_\iota$. We set for each ι $\Xi^\sigma_\iota = Z^\sigma_\iota$, $\xi^i_\iota = \zeta^i_\iota$, and define a vector field Ξ_ι on V_ι by $\Xi_\iota = \xi^i_\iota(\partial/\partial x^i_\iota) + \Xi^\sigma_\iota(\partial/\partial y^\sigma_\iota)$. Now it is easily verified by means of a fiber chart and the transformation properties of the components of ζ, that the vector field $\Xi = \Sigma\chi_\iota\Xi_\iota$ has all the required properties.

(b) Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fiber chart, (U, φ) the associated chart on X. By (2.2.15), for any function $f : V_r \to R$ and any section of Y over U, $d_i f \circ J^{r+1}\gamma = \partial(f \circ J^r\gamma)/\partial x^i$, and $d_i f$ depends only on the restriction of f to $J^r\gamma(U)$. Thus the components of $J^r\Xi_1$ depend only on the components of Ξ_1 resctricted to $\gamma(W)$, i.e., on the components of ζ, by (3.3.4); this implies (3.3.20).

A π-projectable vector field Ξ satisfying condition (a) of Theorem 3.7, is called a *π-projectable extension* of the variation ζ. According to Theorem 3.7 (b) we may put

$$(3.3.21) \qquad J^r\zeta(J^r_x\gamma) = J^r\Xi(J^r_x\gamma) ,$$

where Ξ is any π-projectable extension of ζ and $x \in W$. $J^r\zeta$ is a vector field along the r-jet prolongation $J^r\gamma$ of γ; we call it the *r-jet prolongation* of the variation ζ.

Remark 3.7. The notion of the 1-jet prolongation of a projectable vector field has been introduced, in the case of the product of Euclidean spaces, by Trautman

[52] to obtain a geometric characterization of one-parameter symmetries of a lagrangian, and was easily transferred to r-jet prolongations of arbitrary fibered manifolds; the prolonged vector fields turned out to get the meaning of "prolongations of variations" of sections of fibered manifolds (Krupka [40]).

3.4. The first variation formula. Let $W \subset X$ be a set. From now on, $\Gamma_W(\pi)$ denotes the set of sections γ of Y such that the domain of definition of γ is a neighborhood of W.

Let $\lambda \in \tilde{\Omega}^n_X(J^rY)$ be a lagrangian of order r for Y, $\Omega \subset X$ a piece, i.e., a compact, n-dimensional submanifold of X with boundary. The function

$$(3.4.1)\qquad \Gamma_\Omega(\pi) \ni \gamma \to \lambda_\Omega(\gamma) = {}_\Omega\!\int J^r\gamma^*\lambda \in R$$

is called the *variational function*, or the *action function*, of the lagrangian λ over Ω. Our main purpose in this series of papers is to investigate the family $\{\lambda_\Omega\}$ of variational functions, labeled by Ω.

Let Ξ be a π-projectable vector field on Y, ξ its π-projection, α_t (resp. α_{0t}) the local one-parameter group of Ξ (resp. ξ). Let $\gamma \in \Gamma_\Omega(\pi)$ be a section, $U \supset \Omega$ its domain of definition. To each point $x_0 \in \Omega$ there exist $\varepsilon_0 > 0$ and a neighborhood U_0 of x_0 such that the mapping $(s,x) \to \alpha_{0s}(x)$ (the global flow of ξ) is defined on $(-\varepsilon_0,\varepsilon_0) \times U_0$. Since Ω is compact, we can find finite sequences $x_1,\dots,x_n \in \Omega$, $\varepsilon_1,\dots,\varepsilon_N > 0$, and $U_1,\dots,U_N$ such that for each $i = 1,2,\dots,N$, the mapping $(s,x) \to \alpha_{0s}(x)$ is defined on $(-\varepsilon_i,\varepsilon_i) \times U_i$. Putting $\varepsilon = \min\{\varepsilon_1,\dots,\varepsilon_N\}$ we obtain that this mapping is defined on $(-\varepsilon,\varepsilon) \times W$, where $W = \cup U_i$ is a neighborhood of Ω. Hence the formula

$$(3.4.2)\qquad \gamma_s = \alpha_s\gamma\alpha_{0s}^{-1},$$

where $s \in (-\varepsilon,\varepsilon)$, defines a one-parameter family $\{\gamma_s\}$ of sections of Y; the domain of definition of γ_s contains $\alpha_{0s}(\Omega)$. $\{\gamma_s\}$ is called the *deformation* of the section γ *induced* by Ξ.

Let us consider the real-valued function

$$(3.4.3)\qquad (-\varepsilon,\varepsilon) \ni s \to \lambda_{\alpha_{0t}(\Omega)}(\alpha_s\gamma\alpha_{0s}^{-1}) = \int_{\alpha_{0s}(\Omega)} J^r(\alpha_s\gamma\alpha_{0s}^{-1})^*\lambda \in R.$$

This function is obviously smooth. Differentiating with respect to s at $s = 0$ and using Theorem 1.4 and Theorem 1.5 we obtain

$$(3.4.4)\qquad \left\{\frac{d}{ds}\lambda_{\alpha_{0s}(\Omega)}(\alpha_s\gamma\alpha_{0s}^{-1})\right\}_0 = {}_\Omega\!\int J^r\gamma^*\partial_{J^r\Xi}\lambda,$$

where $\partial_{J^r\Xi}\lambda$ denotes the Lie derivative of λ with respect to the r-jet prolongation $J^r\Xi$ of Ξ. The arising function

$$(3.4.5)\qquad \Gamma_\Omega(\pi) \ni \gamma \to (\partial_{J^r\Xi}\lambda)_\Omega(\gamma) = {}_\Omega\!\int J^r\gamma^*\partial_{J^r\Xi}\lambda \in R$$

is the variational function of the lagrangian $\partial_{J^r\Xi}\lambda$ over Ω. We call it the *first variation* of the variational function λ_Ω, *induced* by Ξ.

Theorem 3.8. *Let* $\lambda \in \widetilde{\Omega}^n_X(J^rY)$ *be a lagrangian,* $\rho \in \widetilde{\Omega}^n(J^sY)$ *a Lepagean equivalent of* λ.

(*a*) *For every* π*-projectable vector field* Ξ *on* Y*,*

$$(3.4.6) \qquad \partial_{J^r\Xi}\lambda = h(i_{J^s\Xi}d\rho) + h(di_{J^s\Xi}\rho) .$$

(*b*) *For every* π*-projectable vector field* Ξ *on* Y *and every section* γ *of* Y*,*

$$(3.4.7) \qquad J^r\gamma^*\partial_{J^r\Xi}\lambda = J^s\gamma^* i_{J^s\Xi}d\rho + dJ^s\gamma^* i_{J^s\Xi}\rho .$$

(*c*) *For every* π*-projectable vector field on* Y*, every piece* Ω *with boundary* $\partial\Omega$*, and every section* $\gamma \in \Gamma_\Omega(\pi)$*,*

$$(3.4.8) \qquad \int_\Omega J^r\gamma^*\partial_{J^r\Xi}\lambda = \int_\Omega J^s\gamma^* i_{J^s\Xi}d\rho + \int_{\partial\Omega} J^s\gamma^* i_{J^s\Xi}\rho .$$

Proof. (a) Since $h(\rho) = \lambda$ (see Convention 2.1) (3.4.6) follows from (3.3.17) and (1.3.29).

(b) This follows from (3.4.6) and from the definition of h (see Sec. 2.2).

(c) This follows from (3.4.7) and from the Stokes' formula (Theorem 1.6).

We call either of the relations (3.4.6), (3.4.7) the *infinitesimal first variation formula;* (3.4.8) is called the *integral first variation formula.*

Remark 3.8. Infinitesimal first variation formula (3.4.6), or (3.4.7), explains the axioms defining a Lepagean equivalent ρ of a lagrangian λ: The first axiom, stating that $p_1(d\rho)$ should be $\pi_{s,0}$-horizontal, guaranties that the first term in (3.4.7) depends only on Ξ, not on the prolongations of Ξ (i.e., on the derivatives of the components of Ξ), while the complementary term (the second term) is precisely the "boundary term"; the second axiom, stating that $h(\rho) = \lambda$, implies $\int_\Omega J^r\gamma^*\lambda = \int_\Omega J^s\gamma^*\rho$, which means that ρ should define the same variational problem as λ.

Our definition of the variational function (3.4.1) is analogous as that one of Hermann [34]. His first variation formula is, however, not correct, since it does not lead to the required decomposition of the variation into "Euler-Lagrange" and the "boundary" terms.

The following corollary says that the (global) first variation formula (3.4.6) can also be obtained by means of the local Lepagean equivalents (3.2.47).

Corollary 1. *For every* π*-projectable vector field* Ξ *on* Y*,*

$$(3.4.9) \qquad \partial_{J^r\Xi}\lambda = h(i_{J^{2r-1}\Xi}d\rho_{\lambda,V}) + h(di_{J^{2r-1}\Xi}\rho_{\lambda,V}) ,$$

where (V,ψ) *is any fiber chart on* Y *and* $\rho_{\lambda,V}$ *is the local Lepagean equivalent* (3.2.47) *of* λ.

Proof. Let $\rho \in \tilde{\Omega}^n(J^{2r-1}Y)$ be any Lepagean equivalent of λ (Theorem 3.4). For each fiber chart (V,ψ) on Y ρ is expressible in the form $\rho = \rho_{\lambda,V} + \nu_V$, where ν_V is a local Lepagean equivalent of the zero lagrangian (compare with (3.2.51) and (3.1.3)). If Ξ is a π-projectable vector field on Y, we have

$$h(\partial_{J^{2r-1}\Xi}\nu_V) = 0 \tag{3.4.10}$$

since h and $\partial_{J^{2r-1}\Xi}$ commute (in the sense of (3.3.17)). Hence

$$h(\partial_{J^{2r-1}\Xi}\rho) = h(\partial_{J^{2r-1}\Xi}\rho_{\lambda,V}) \tag{3.4.11}$$

on V_{2r}. Applying (3.3.17) again and using (1.3.29) and Theorem 2.1, (a) we obtain (3.4.9).

Let $\lambda \in \tilde{\Omega}^n_X(J^rY)$ be a Lagrangian, ρ a Lepagean equivalent of λ; we may suppose that $\rho \in \tilde{\Omega}^n_{J^{r-1}Y}(J^{2r-1}Y)$. We know that the Euler-Lagrange form E of ρ (Sec. 3.1) depends only on λ (Corollary 3 to Theorem 3.1). We denote $E = E_\lambda$ and call E_λ the *Euler-Lagrange form* of the lagrangian λ. The mapping $\lambda \to E_\lambda$ of $\tilde{\Omega}^n_X(J^rY)$ into $\tilde{\Omega}^{n,1}(J^{2r}Y)$ is called the *Euler-Lagrange mapping*. The chart expression of E_λ is given by (3.1.11) and (3.1.12) with $f_0 = L$, where $\lambda = L.\tilde{\varphi} \otimes \omega_0$ (3.2.2), and $s = 2r - 1$. Writing $E_\lambda = E_\sigma(L).\tilde{\varphi} \otimes (\omega^\sigma \wedge \omega_0)$ we obtain the *Euler-Lagrange expressions* $E_\sigma(L)$ (3.1.12) of the lagrangian λ with respect to (V,ψ).

Corollary 2. *For every π-vertical vector field Ξ on Y,*

$$\partial_{J^r\Xi}\lambda = i_{J^{2r}\Xi}E_\lambda + h(di_{J^{2r-1}\Xi}\rho). \tag{3.4.12}$$

Proof. This follows from (3.4.6), the identity $h(i_{J^{2r}\Xi}E_\lambda) = i_{J^{2r}\Xi}E_\lambda$, and from Theorem 3.1, (1).

Remark 3.9. The Euler-Lagrange form as considered above, is due to Krupka (see e.g. [9, (4.19), [7], and [3]) and Anderson and Duchamp [14, (3.4)]. Goldschmidt and Sternberg [33] interpreted the Euler-Lagrange expressions as the vector-valued differential n-form ($n = \dim X$). Garcia [29] (see also [30]) introduced the Euler-Lagrange form as a vector-valued n-form, depending on an auxiliary connection.

3.5. Extremals. Let $W \subset X$ be a set. From now on, $\Gamma_W(\pi)$ denotes the set of sections γ of Y such that the domain of definition of γ is a neighborhood of W. Let $\lambda \in \tilde{\Omega}^n_X(J^rY)$ be a lagrangian of order r for Y, $\Omega \subset X$ a piece, i.e., a compact, n-dimensional submanifold of X with boundary. The function

$$\Gamma_\Omega(\pi) \ni \gamma \to \lambda_\Omega(\gamma) = \int_\Omega J^r\gamma^*\lambda \in R \tag{3.5.1}$$

is called the *variational function*, or the *action function* of the lagrangian λ over Ω. Our main purpose in this series of papers is to investigate the family of functions λ_Ω, labeled by Ω.

By a *one-parameter family of sections* of Y we shall mean a mapping $(-\varepsilon,\varepsilon) \times W \ni (s,x) \to \gamma(s,x) \in Y$, where $\varepsilon > 0$ and $W \subset X$ is an open set, such that for each $s \in (-\varepsilon,\varepsilon)$ the mapping $\gamma_s : W \to Y$, defined by the relation $\gamma_s(x) = \gamma(s,x)$, is a section of Y over W. When there is no need of further specification, we denote a one-parameter family of sections by $\{\gamma_s\}$. A *deformation* of a section γ we mean a one-parameter family of sections $\{\gamma_s\}$ such that $\gamma_0 = \gamma$.

If $\{\gamma_s\}$ is a deformation of a section $\gamma \in \Gamma_W(\pi)$, then the relation

$$(3.5.2) \qquad \zeta(x) = \left\{\frac{d}{ds}\,\gamma_s(x)\right\}_0$$

defines a vector field along γ. Since $T\pi.\zeta(x) = 0$ for every x, this vector field is formed by π-vertical vectors. By the *support* of the deformation $\{\gamma_s\}$ we mean the set $\mathrm{cl}\{x \in W \mid \zeta(x) \neq 0\}$, where cl means the closure. We say that $\{\gamma_s\}$ is of *compact support*, if its support is a compact set in W.

If Ξ is a π-vertical vector field on Y and (α_s) is its local one-parameter group, then for any $\gamma \in \Gamma_\Omega(\pi)$, $\{\alpha_s\gamma\}$ is a deformation of γ; this deformation is said to be *induced* by Ξ.

A section $\gamma \in \Gamma_\Omega(\pi)$ is called an *extremal*, or a *critical section*, of the lagrangian λ *on* Ω, if

$$(3.5.3) \qquad \left\{\frac{d}{ds}\,\lambda_\Omega(\gamma_s)\right\}_0 = 0$$

for every deformation $\{\gamma_s\}$ of γ whose support is contained in Ω. A section $\gamma \in \Gamma_W(\pi)$ is called an *extremal*, or a *critical point*, of λ, if the restriction of γ to any piece $\Omega \subset W$ is an extremal of λ on Ω. If $W = X$ we suppose of *global* extremals.

Thus the extremals of λ are those sections γ, for which the value $\lambda_\Omega(\gamma)$ of the variational function (3.5.1) is "stable" with respect to "small compact deformations" of γ, for every $\Omega \subset X$.

Lemma 3.4. *A section $\gamma \in \Gamma_\Omega(\pi)$ is an extremal of λ on Ω if and only if*

$$(3.5.4) \qquad \int_\Omega J^r\gamma^* \partial_{J^r\Xi}\lambda = 0$$

for every π-vertical vector field Ξ, defined on a neighborhood of $\gamma(\Omega) \subset Y$, such that $\mathrm{supp}\,\Xi \subset \pi^{-1}(\Omega)$.

Proof. 1. Let γ be an extremal of λ on Ω, let Ξ be a π-vertical vector field on a neighborhood of $\gamma(\Omega) \subset Y$, α_t its local one-parameter group. To each point $y_0 \in \gamma(\Omega)$ there exist $\varepsilon_0 > 0$ and a neighborhood V_0 of Y_0 such that the mapping $(s,y) \to \alpha_s(y)$ is defined on $(-\varepsilon_0,\varepsilon_0) \times V_0$. Since γ is an embedding, $\gamma(\Omega)$ is compact, and we can find a finite sequences $y_1,\ldots,y_N$, $\varepsilon_1,\ldots,\varepsilon_N$, and $V_1,\ldots,V_N$ such that for each i, $1 \le i \le N$, the mapping $(s,y) \to \alpha_s(y)$ is defined on $(-\varepsilon_i,\varepsilon_i)\times \times V_i$. Putting $\varepsilon = \min(\varepsilon_1,\ldots,\varepsilon_N)$ we obtain that this mapping is defined on $(-\varepsilon,\varepsilon)\times V$, where $V = \cup V_i$ is a neighborhood of $\gamma(\Omega)$. Thus $\gamma_s = \alpha_s\gamma$ is a deformation of γ. Let U be the domain of definition of γ; we may suppose that $U \subset \pi(V)$. If $x \in U$ is a point such that $\Xi(\gamma(x)) \neq 0$, then $\gamma(x) \in \operatorname{supp}\Xi \subset \pi^{-1}(\Omega)$, and $x \in \Omega$ which implies that the support of γ_s $\operatorname{cl}\{x \in U \mid \Xi(\gamma(x)) \neq 0\} \subset \Omega$. Now by definition (3.5.3)

$$(3.5.5)\qquad 0 = \left\{\frac{d}{ds}\lambda_\Omega(\gamma_s)\right\}_0 = {}_\Omega\!\int \left\{\frac{d}{ds} J^r(\alpha_s\gamma)^*\lambda\right\}_0 = {}_\Omega\!\int \left\{\frac{d}{ds}(J^r\alpha_s\circ J^r\gamma)^*\lambda\right\}_0$$

$$= {}_\Omega\!\int \left\{\frac{d}{ds} J^r\gamma^* J^r\alpha_s^*\lambda\right\}_0 = {}_\Omega\!\int J^r\gamma^*\partial_{J^r\Xi}\lambda ,$$

where we have used Theorem 1.5, (2.1.5), and (1.3.17).

2. Suppose that (3.4.4) holds for every Ξ. Let $\{\gamma_s\}$ be a deformation of γ whose support is contained in Ω. Take for Ξ the π-projectable extension of the vector field ζ (3.5.2) (Theorem 3.7 (a)); obviously, Ξ must be π-vertical. Using Theorem 1.5 we obtain

$$(3.5.6)\qquad \left\{\frac{d}{ds}\lambda_\Omega(\gamma_s)\right\}_0 = {}_\Omega\!\int \left\{\frac{d}{ds} J^r\gamma_s^*\lambda\right\}_0 .$$

Express the integrand by means of a fiber chart (V,ψ), $\psi = (x^i, y^\sigma)$. If $\lambda = L.\tilde\varphi \otimes \omega_0$, then $J^r\gamma_s^*\lambda = (L_0 J^r\gamma_s).\tilde\varphi \otimes \omega_0$, and

$$(3.5.7)\qquad \left\{\frac{d}{ds} J^r\gamma_s^*\lambda\right\}_0 = \left(\sum_{k=0}^{r}\sum \frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k}}\left\{\frac{d(y^\sigma_{j_1\ldots j_k}J^r\gamma_s)}{ds}\right\}_0\right).\tilde\varphi\otimes\omega_0$$

$$= \left(\sum_{k=0}^{r}\sum \frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k}} \frac{\partial^k}{\partial x^{j_1}\ldots\partial x^{j_k}} \left\{\frac{d}{ds}.Y^\sigma\gamma_s\right\}_0\right)\tilde\varphi\otimes\omega_0 ,$$

where these expressions are considered along $J^r\gamma$, and the definition of the associated coordinates on J^rY is used. But $\zeta(x) = \Xi(\gamma(x))$, and if we write $\Xi = \Xi^\sigma(\partial/\partial y^\sigma)$ we obtain $\Xi^\sigma = \{d(y^\sigma\gamma_s)/ds\}_0$ which implies, using Theorem 3.5, (3.3.3), and (1.3.28) that this expression equals $J^r\gamma^*\partial_{J^r\Xi}\lambda$. Now (3.5.3) follows from the assumption.

Lemma 3.4 says that for the study of extremals γ of λ one can use π-vertical vector fields in place of variations of γ. This immediately leads to the following result.

Theorem 3.9. *Let $\lambda \in \tilde\Omega^n_X(J^rY)$ be a lagrangian, E_λ the Euler-Lagrange form of λ, γ a section of Y, and ρ a Lepagean equivalent of λ of order $2r-1$. The following conditions are equivalent:*

(1) γ is an extremal of λ.

(2) *The Euler-Lagrange form* E_λ *vanishes along* $J^{2r}\gamma$,

(3.5.8) $$E_\lambda \circ J^{2r}\gamma = 0 .$$

(3) *For every* π*-projectable vector field* Ξ *on* Y

(3.5.9) $$J^{2r-1}\gamma^* i_{J^{2r-1}\Xi} d\rho = 0 .$$

(4) *For any fiber chart* (V,ψ) *the restriction of* γ *to the set* $\pi(V)$ *satisfies the system of partial differential equations*

(3.5.10) $$E_\sigma(L)\circ J^{2r}\gamma = 0, \quad 1 \le \sigma \le m.$$

Proof. 1. If γ is an extremal then by Lemma 3.4 and Corollary 2, $\int_\Omega J^{2r}\gamma^* i_{J^{2r}\Xi}E_\lambda = 0$ for each Ξ π-vertical, such that $\operatorname{supp}\Xi \subset \pi^{-1}(\Omega)$. Thus (1) implies (2). 2. Suppose that condition (2) holds. Since for any π-projectable vector field Ξ on Y

(3.5.11) $$i_{J^{2r}\Xi}\pi^*_{2r,2r-1}d\rho = \pi^*_{2r,2r-1}i_{J^{2r-1}\Xi}d\rho = i_{J^{2r-1}\Xi}d\rho = i_{J^{2r}\Xi}E_\lambda + i_{J^{2r}\Xi}F$$

(see (1.3.22) and (3.1.2)) and the order of contact of F is ≥ 2, we have $J^{2r-1}\gamma^* i_{J^{2r-1}\Xi}d\rho = J^{2r}\gamma^* i_{J^{2r}\Xi}E_\lambda = 0$. 3. Writing (3.5.9) locally we see at once that (3) implies (4). 4. Finally, condition (4) together with (3.4.12) and Lemma 3.4 imply (1).

Any of the equivalent equations (3.5.8) - (3.5.10) are a global system of m partial differential equations of order $2r$ on $J^{2r}Y$; they are called the *Euler-Lagrange equations* of the lagrangian λ. Thus the extremals of λ are precisely solutions of the Euler-Lagrange equations.

Remark 3.10. A variational function containing, in addition to the integral over Ω, an integral over the boundary $\partial\Omega$, has been studied by Chrastina [18].

Remark 3.11. Some authors use formula (3.5.4) for the definition of an extremal, with the vector fields $J^r\Xi$ replaced by a wider class of the so called *infinitesimal contact transformations* ([19], [29]). It should be pointed out, however, that deformations of sections induced by the infinitesimal contact transformations are no more sections, in general, and are not adequate to the considered type of variational problems in fibered spaces.

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