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ALGEBRAIC PROPERTIES OF THE 3-STATE VECTOR POTTS MODEL

A.K. Kwaśniewski

Abstract

The purpose of this note is to report on further progress in algebraic approach to Potts models as to compare with [1] and [2].

The generalized hyperbolic functions are shown again to be a structural ingredient of these models.

The case of arbitrary n-state Potts model was treated in [2]. Here, in order to compare our considerations with [1], we restrict ourselves to n=3 vector Potts model on the p×q torus lattice.

Its transfer matrix M can be expressed as M = AB where

$$A = \left[\det \hat{a}(a)\right]^{p/3} \exp\left\{a* \sum_{k=1}^{p} (X_k + X_k^2)\right\} \tag{1}$$

with $\hat{a}(a)$ - an interaction matrix, and

$$B = \prod_{k=1}^{p} \exp\left[b(Z_k^2 Z_{k+1} + Z_{k+1}^2 Z_k)\right] \tag{2}$$

with $Z_{p+1} \equiv Z_1$.
The $3^p \times 3^p$ matrices X_k & Z_k; k=1,...,p, are of the familiar form:

$$\begin{aligned} X_k &= I\times \dots \times I \times \sigma_1 \times I \times \dots \times I \qquad \text{(p - terms)} \\ Z_k &= I\times \dots \times I \times \sigma_3 \times I \times \dots \times I \qquad \text{(p - terms)} \end{aligned} \tag{3}$$

where I, σ_1, σ_3 are 3×3 matrices belonging to the so called generalized Pauli algebra i.e. to say

$\sigma_1\sigma_2 = \omega\sigma_2\sigma_1$ and $\sigma_3 = \sigma_1^2\sigma_2$ where $\omega = \exp\left\{\frac{2\pi i}{3}\right\}$. Explicitly the matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{4}$$

Another useful (and familiar from the Ising case) representation of A&B matrices is by means of generaliæd γ's (generators of generalized Clifford algebras) which might be define as follows:

$$\begin{aligned}
\gamma_1 &= \sigma_3 \times I \times I \times \ldots \times I \times I \\
\gamma_2 &= \sigma_1 \times \sigma_3 \times I \times \ldots \times I \times I \\
&\vdots \\
\gamma_p &= \sigma_1 \times \sigma_1 \times \sigma_1 \times \ldots \times \sigma_1 \times \sigma_3 \\
\gamma_{p+1} &= \bar{\gamma}_1 = \sigma_2 \times I \times I \times \ldots \times I \times I \\
\gamma_{p+2} &= \bar{\gamma}_2 = \sigma_1 \times \sigma_2 \times I \times \ldots \times I \times I \\
&\vdots \\
\gamma_{2p} &= \bar{\gamma}_p = \sigma_1 \times \sigma_1 \times \sigma_1 \times \ldots \times \sigma_1 \times \sigma_2
\end{aligned} \tag{5}$$

It is easily seen that $\gamma_i\gamma_j = \omega\gamma_j\gamma_i$; $i<j$ and $\gamma_i^3 = 1$ where $i,j=1,\ldots,2p$.

In terms of generalized γ's the A & B matrices read:

$$B = \prod_{k=1}^{p} \exp\left\{b\left[\gamma_{k+1}^2 \bar{\gamma}_k + \bar{\gamma}_k^{-2} \gamma_{k+1}\right]\right\} \tag{6}$$

$$A = \left[\det \hat{a}(a)\right]^{p/3} \exp\left\{a^* \sum_{k=1}^{p} (\omega^2 \gamma_k^2 \bar{\gamma}_k + \bar{\gamma}_k^{-2} \gamma_k)\right\} \tag{7}$$

where $\hat{a}(a) = \sum_{k=0}^{2} e^{aRe\omega^k} \sigma_1^k$ and a^* is the dual of a , to be found from [3] the equation:

$$\det \hat{a}(a) = 3^3[\det \hat{a}(a)]^{-1} \tag{8}$$

The expressions (6) & (7) are the starting point for the two major observations of

this note.
At first we show how the diagonalization of the $3^p \times 3^p$ transfer matrix is reduced to the independent diagonalizations of three matrices T_o, T_1', T_2'' to be defined.

Then we show how the matrices A & B induce inner automorphisms in the three-dimensional subspaces of generalized Clifford algebra, in a similar way as it was done in [1] for the standard Potts model. This time however, the matrices representing the very automorphisms are expressed in terms of hyperbolic functions of order 3 [4] , which is of great advantage as the system of their properties is well established by now, and also the hyperbolic functions of order n were recently shown to be crucial in the Onsager problem for Potts models [2] .
For completeness, we recall [4] the definition of these functions:

$$h_i(x) = \frac{1}{3} \sum_{k=0}^{2} \omega^{-ki} \exp\left\{\omega^k x\right\} , \qquad x \in R, \quad i=0,1,2 . \tag{9}$$

The first observation:
Define the following operators V_k, $V_k V_l = \delta_{kl} V_l$ [2]

$$V_k = \frac{1}{3} \sum_{i=0}^{2} \omega^{-ki} U^i , \qquad \text{where } k,l = 0,1,2 \quad \text{and} \tag{10}$$

$$U = \omega^2 \otimes^p \sigma_1 . \tag{11}$$

Denote by B_k (k=0,1,2) the following matrices

$$B_k = \prod_{\alpha=1}^{p-1} \exp\left\{b\left[\bar{\gamma}_\alpha^2 \gamma_{\alpha+1} + \gamma_{\alpha+1}^2 \bar{\gamma}_\alpha\right]\right\} \exp\left\{b\left[\omega^k \bar{\gamma}_p^2 \gamma_1 + \omega^{-k} \gamma_1^2 \bar{\gamma}_p\right]\right\}. \tag{12}$$

Then one has:

$$M = \sum_{k=0}^{2} V_k (AB_k) \equiv V_o T_o + V_1 T_1 + V_2 T_2 \tag{13}$$

However, due to

$$[U, T_k] = 0 \qquad \text{and } U^3 = 1 , \tag{14}$$

we conclude that there exists such similarity transformation that U becomes U of the form

$$\tilde{U} = \begin{pmatrix} I & & \\ & \omega I & \\ & & \omega^2 I \end{pmatrix} , \qquad I = (3^{p-1} \times 3^{p-1}) . \tag{15}$$

Therefore $T_j \to \tilde{T}_j$ (j=0,1,2), where

$$\tilde{T}_o = \begin{pmatrix} T_o & & \\ & T_o' & \\ & & T_o'' \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} T_1 & & \\ & T_1' & \\ & & T_1'' \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} T_2 & & \\ & T_2' & \\ & & T_2'' \end{pmatrix}. \tag{16}$$

In this representation V_k matrices are particularly simple. Namely:

$$V_o = \begin{pmatrix} I & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & & \\ & I & \\ & & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & I \end{pmatrix}. \tag{17}$$

Therefore

$$\tilde{M} = \begin{pmatrix} T_o & & \\ & T_1' & \\ & & T_2'' \end{pmatrix}, \tag{18}$$

which proves our first observation.
The same observation in the case of Ising model just enables to solve it.
For Potts model however, it is no more "that easy", as in our case $[A,B] \neq 0$!

<u>The second observation:</u>

For the sake of second observation we come back to formulas (6) & (7) and we see that the factors to be studied are the following:

$$\hat{b}_k = \exp\{b\ \gamma_{k+1}^2 \gamma_{p+k}\}\ ; \qquad [\hat{b}_k, \hat{b}_1] = 0 \tag{19}$$

and

$$\hat{a}_k = \exp\{a^*\ \omega^2 \gamma_k^2 \gamma_{p+k}\}; \qquad [\hat{a}_k, \hat{a}_1] = 0 \tag{20}$$

where now $k,1=1,2,\ldots,p$.

Therefore both kinds of factors are of the type:

$$\exp\{x\gamma_\nu^2\gamma_\mu\} \qquad \text{where} \quad \nu < \mu\ .$$

One then proves:

Lemma 1.

$$\exp\{x\gamma_\nu^2\gamma_\mu\} \begin{pmatrix} \gamma_\nu \\ \gamma_\mu \\ \gamma_\nu^2\gamma_\mu^2 \end{pmatrix} \exp\{-x\gamma_\nu^2\gamma_\mu\} = V(x) \begin{pmatrix} \gamma_\nu \\ \gamma_\mu \\ \gamma_\nu^2\gamma_\mu^2 \end{pmatrix}$$

where $V(x)$ is 3×3 matrix.

Proof:

It is enough to notice that for $\hat{C}^3 = 1 \neq \hat{C}^2$

$$\exp\{xC\} = h_o(x)I + h_1(x)C + h_2(x)C^2 \quad \text{and that } \hat{C} = \gamma_\nu^2\gamma_\mu$$

acting on the "spinor" subspaces spanned by γ_ν, γ_μ, $\gamma_\nu^2\ \gamma_\mu^2$, both from the left and right, becomes an auto morphism just of these subspaces. To give an example: $\hat{C} = \gamma_k^2\ \bar{\gamma}_k$ in the basis γ_k, $\bar{\gamma}_k$, $\gamma_k^2\ \gamma_k^{-2}$ acting from the left is represented by the following matrix

$$C_{\rightarrow} = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} = \sigma_2 \qquad \blacksquare \quad (!)$$

From the proof of Lemma 1 we also conclude:

Lemma 2.

Matrix elements of $V(x)$ are bilinear in hyperbolic functions h_o, h_1, h_2. ∎
It is not difficult to derive $V(x)$ in explicit form. This we leave however - altogether with study of its further properties based on properties of h_i functions - for the forthcoming paper.

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