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CONVEXITY IN COMBINATORIAL STRUCTURES (*)

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SUMMARY. The recent (since 1968) combinatorial developments of abstract convexity are surveyed. The combinatorial properties of convex sets (Helly property, Eckhoff's partition problem ...) are considered in the general setting of finitary closure systems ("Convexity Spaces" or "Alignements"). In ordered sets, in tree-like structures and in combinatorial structures inspired by Geometry (e.g. "oriented matroids") there are natural definitions of convex sets : an axiomatic common background is the theory of "convex geometries" (or anti-exchange convexities) of Edelman and Jamison or dually the theory of "shelling structures" (or "APS-greedoids") of Korte and Lovász. Convexity in graphs recently appeared of independent interest (contraction into complete graphs, universal properties of geodesic convexity ...)

I - INTRODUCTION

Properly speaking, convexity is not a mathematical theory, but rather a notional domain where five basic concepts operate : *betweenness* (+ medians, convex dependance), *algebraicness* (+ convex hull operator, dimension), *separation* (+ hemispaces, copoints), *connectedness* and *optimization* (+ extremal points, face-lattices, duality). Thus, convexity is present in almost all constituted combinatorial theories: Finite Geometry (namely oriented matroids) of course, but also Graphs, Set Systems, Ordered Sets, Extremal Set Theory, Enumeration Problems, Designs and Combinatorial Optimization.

The first aspect of the interplay between Combinatorics and Convexity is the use of combinatorial methods when studying ordinary convexity (i.e. standard convexity in Euclidean spaces \mathbb{R}^d) : combinatorial properties of families of convex sets, facial structure of polytopes.

(*) This paper is in final form and no version of it will be submitted for publication elsewhere.

The second aspect is the use of convexity concepts in combinatorial research. The title of the present article refers to a definite part of the second approach : we restrict our attention to those combinatorial structures where the notions relative to convexity are introduced by *intrinsic* means, or to put it differently, where the definition of the convex sets only depends on the combinatorial structure itself. Nevertheless, since most convex approaches of combinatorial structures are motivated and inspired by abstract generalization of Euclidean space situations, we briefly recall in Section 2 the major *combinatorial themes* of convexity theory in real vector spaces. But convexities that arise in an extrinsic way in Combinatorics are not considered in this survey. For instance the reader who is interested in polyhedral combinatorics (discrete optimization, linear programming, packing and covering problems) is referred to the classical literature of this important domain : Rockefellar [1970], Stoer and Witzgall [1970], Lawler [1976], Hammer et al. [1979], Schrijver [1979], Grötschel et al. [1981], Lovász [1984] Yemelichev et al. [1984], Karmarkar [1985].

The general context, a sort of "*Prototheory of Convexity*" is introduced in Section 3. The interest of an independent development of an abstract and axiomatic convexity is underlined by recent results that concern *combinatorial parameters* attached to convexity spaces (Section 4) and by Eckhoff's partition problem. A typically interesting direction is the theory of *Convex Geometries* (or anti-exchange convexities) where every convex set is the convex hull of its extreme points : this theory, described in Section 5, can be viewed as a part of the Theory of Greedoids (a relaxation of Matroids). In Section 6, we deal with *convex sets in graphs*. Graphical convexities appear as a very important example of finite convexity spaces with possible applications both in Graph Theory (e.g. Hadwiger's conjecture) and in Abstract Convexity (universality of geodesic convexity). The last three sections present more specialized works on combinatorial structures where convexity appears in a very natural way : *oriented matroids* (Section 7) with facial structure and separation problems, *ordered sets* (Section 8) with betweenness or medians, *tree-like structures* (Section 9) with coherent convexities.

Throughout the paper, strict inclusion is denoted by \subset and set difference by \setminus . Square brackets indicate the year of a reference issue. Open problems are numbered separately. A notation of the form (3.7) refer to the seventh paragraph of Section 3.

2. FROM EUCLIDEAN CONVEXITY ...

(2.1) Combinatorial properties of ordinary convex sets were progressively discovered when mathematicians tries to find geometrical proofs of properties of convex sets. So, in a parallel direction with geometric approaches by Brünn [1887], Bonnesen [1929], Alexandroff [1950], the combinatorial works follow one another : Carathéodory [1907], Radon [1921], Helly [1923], Kakutani [1937]. Further ideas and references may be found in usual books on Convexity and Combinatorial Geometry : Bonnesen and Fenchel [1934], Yaglom and Boltianski [1956, 1961], Hadwiger and Debrunner [1964] (wonderful !), Valentine [1964] Boltianski and Soltan [1978]. Interesting survey papers are : Dantzer et al. [1963], Eckhoff [1979, 1986].

(2.2) The second major theme in Combinatorial Convexity is the following problem, attributed to Baker.

PROBLEM 1. (Baker) Characterize the nerves of finite families of convex sets in \mathbb{R}^d .

The nerve of a finite family $(C_i)_{i \in I}$ of convex sets in \mathbb{R}^d is the abstract simplicial complex (*) whose vertex set is I and simplices (= faces) are those subsets $J \subseteq I$ such that $\bigcap_{i \in J} (C_i; i \in J) \neq \emptyset$. An important stage towards a solution of Baker's problem was recently reached : Kalai [1984], using technics of exterior algebra, characterized the f-vectors (**) associated to the nerves of families of convex sets — thus solving a conjecture due to Eckhoff —

A special unsolved part of Baker's problem deserves mention :

PROBLEM 2 (Wegner) Characterize the intersection graphs of families of convex sets in \mathbb{R}^2 .

The characterization problem for lattices of Euclidean convex sets is closely related to Baker's problem : see Bennett [1974] and its references.

(2.3) At last, combinatorial Convexity includes the systematic examination of face lattices of convex bodies, espacially of polytopes : McMullen [1971] solved the Upper Bound Conjecture (Motzkin) when

(*) An (*abstract*) *simplicial complex* is a collection K of subsets — the *faces* of K — such that the properties $F \in K$ and $F' \subseteq F$ imply $F' \in K$. In Berge's terminology [1973, 1986] a simplicial complex is a *hereditary hypergraph*.

(**) The *f-vector* (*face-vector*) of a cell complex K is $(f_0, f_1, \dots, f_k, \dots)$, where f_k is the number of k -dimensionnal cells of K . In case of simplicial complexes, f_k is the number of $k+1$ -elements simplices.

determining the maximum number of k -dimensional faces of real d -dimensional polytope with a given number of vertices. Stanley [1975] generalizes the result to shellable complexes. More, the characterizations of f -vectors of simplicial (*) polytopes was obtained by Stanley [1980] (necessity of McMullen's conditions) and Billera and Lee [1981] (sufficiency). Stanley intensively uses homological and commutative algebra.

(2.4) Evident similarities exist between Eckhoff's conditions for nerves (2.2) and McMullen's conditions for f -simplicial boundary complexes of polytopes (2.3). Attempts to find a common general setting (**) to both properties can also help for a solution to a nice conjecture due to Chvátal [1974a,b].

PROBLEM 3 (Chvátal) Let \mathcal{K} be a simplicial complex with at least two faces. Let Δ denote the maximum number of \mathcal{K} -faces that have a common vertex; Is it true that every family of $\Delta+1$ faces contains two disjoint faces ?

Berge (see[1986]) conjectures a stronger property : the line-graph (****) of the faces has chromatic number Δ .

3. ... TO ABSTRACT CONVEXITY.

(3.1) How to do geometry with convexity ? This was the original motivation of pioneers of axiomatic convexity. The scheme was : convexity \rightarrow convex hull \rightarrow segment \rightarrow line \rightarrow dimension \rightarrow geometry. The best achievements in this domain were obtained by Busemann [1955], Prenowitz [1969], Bryant, Webster [1972,73,77], Cantwell [1974,76], Cantwell and Kay [1978], Prenowitz and Jantosciak [1979]. An axiomatic characterization of \mathbb{R}^d by the means of axioms that only involve abstract convexities, "linearization problem" is possible : see Doignon [1976] and Whitfield and Yong [1981], where further references can be found.

(3.2) Motzkin [1951] was seemingly the first to advice an independent

(*) i.e. in which every face is a simplex.

(**) I share Eckhoff's [1985] point of view : an homological interpretation of the f -vectors (or more precisely of the associated " h -vectors") of a wider class of simplicial complexes is highly desirable.

(****) The line-graph $L(\mathcal{F})$ (or intersection graph) of a family of sets $\mathcal{F} = (F_i)_{i \in I}$ has vertex set I : the vertices $i, j \in I$ are joined in $L(\mathcal{F})$ when $F_i \cap F_j \neq \emptyset$.

development of an axiomatic convexity in a very general abstract setting, namely in the theory of *algebraic closure systems* Cohn [1965], Birkhoff [1967], Graetzer [1968]. As a matter of fact, several approaches of what is now called *Abstract Convexity* pre-existed without any reference to Convexity : Moore [1910], Schmidt [1952, 1953] Tarski [1930], Hammer [1963a]. The purely abstract point of view is now sustained by convergent problems and results. Recent general references on Abstract Convexity are Jamison [1972, 1982] and Soltan [1984].

(3.3) CONVEXITY SPACES There are essentially two ways of defining a convex set : by intersection of "large basic convex sets" (for instance half-spaces in vector spaces), or by the property of being closed with respect to a certain family of finitary operators (for instance n -ary operators of the form

$$x_1, \dots, x_n \in \mathbb{R}^d \rightarrow \sum_{i=1}^n \lambda_i x_i$$

where the λ_i 's are non negative and sum to 1). This remark leads to the following definition : a set X , endowed with a set \mathcal{C} of X -subsets forms a *convexity space* (X, \mathcal{C}) if the following axioms are satisfied :

$$(C_1) \quad \emptyset \in \mathcal{C}, \quad X \in \mathcal{C}$$

(C_2) \mathcal{C} is preserved under intersections

(C_3) \mathcal{C} is preserved by nested union.

Elements of X are points, members of \mathcal{C} are convex sets or convexes. Set families that satisfy (C_1) and (C_2) are known as *Moore families* or *closure systems*. Families satisfying (C_3) are known as *inductive systems*. An alternative terminology for convexity spaces is *algebraic closure systems* (cf [1.3]), *alignements* [Jamison 1972], *geometries* [Wille 1970], *domain finite convexity structures* [Hammer 1963; Sierksma 1976].

(3.4) BASIS When infinite convexity spaces are considered, Choice Axiom is assumed. By Zorn's lemma, Axiom (C_3) implies the existence, for all $x \in X$, of a maximal convex set that does not contain x . Such a convex set is called a *copoint* (relative to x). The set X and copoints form a basis $\mathcal{B} \subseteq \mathcal{C}$, i.e. every convex set is an intersection of members of \mathcal{B} . As easily seen, this set constitutes the unique inclusion-minimal basis of \mathcal{C} . (cf. Soltan [1984])

(3.5) CONVEX HULL, OPERATORS. Let \mathcal{C} be a Moore family (resp. a convexity) on a set X . The *closure* (resp. the *convex hull*) of a X -subset A , denoted by $\langle A \rangle_{\mathcal{C}}$ — or simply by $\langle A \rangle$ when no confusion can arise — is the intersection of all \mathcal{C} -members containing A .

Axiom (C_3) expresses the algebraic nature of the convex hull operator : for Moore families, this axiom is equivalent (Schmidt[1952], see Cohn[1981] p.95) (*), compare Hammer [1963a]) to the following property :

(CF) If $x \in \langle A \rangle$ then $x \in \langle F \rangle$ for some finite A -subset F .

Thus, every convexity \mathcal{C} can be defined by a family of algebraic operators "generating rules" $\omega_i : X^n \rightarrow X$. A set $A \subseteq X$ is convex if and only if it is closed for all operators. This point of view allows recursive proofs.

(3.6) INTERVAL CONVEXITIES. A convexity is n -ary (or n -generated) if it can be defined by a family of n_i -ary operators with $n_i \leq n$ for all i . 2 -generated convexities are known as *Interval convexities*. Most usual convexities are interval convexities : Euclidean convexity in \mathbb{R}^d , geodesic convexity in metric spaces (graphs, Riemann manifolds ...). Although, the oriented matroid convexities (see section 7) are not interval convexities.

Intersection of convexities (see (3.8)) preserves n -arity. Hence there exists a smallest n -ary convexity that contains a given convexity \mathcal{C} on a set X (Burris,[1968]) It may be defined as

$$\mathcal{C}^{(n)} = \{C \subseteq X ; A \subseteq C, |A| \leq n \Rightarrow \langle A \rangle_{\mathcal{C}} \subseteq C\}$$

Burris [1972] has shown that every convexity space is isomorphic to an induced subspace (see (3.8)) for a definition) of an interval convexity space.

(3.7) METRIC CONVEXITY. In every metric space (X, d) one may consider an operator $D(x, y) = \{z \in X ; d(x, y) = d(x, z) + d(z, y)\}$. Any interval convexity which can be defined by a single operator of this form is called a metric convexity. To give an intrinsic characterization of metric convexities should not be difficult. See Busemann [1955].

(3.8) MORPHISMS. Morphisms for convexity spaces are defined as for topological spaces : a *convex morphism* from a convexity space (X, \mathcal{C}) in another (Y, \mathcal{D}) is a mapping $\varphi : X \rightarrow Y$ such that $\varphi^{-1}(D) \in \mathcal{C}$ for every $D \in \mathcal{D}$. Endowed with these morphisms, the class of convexity spaces forms a category (or an "espèce de structure" in Bourbaki's terminology [1966]). Initial and final objects in this category (\sim "Structures dérivées") are easily defined. For instance the *product convexity space* of a family (X_i, \mathcal{C}_i) ($i \in I$) of convexity spaces is the convexity space (X, \mathcal{C}) where $X = \prod_i (X_i; i \in I)$ and $\mathcal{C} = \{\prod_i (C_i; i \in I) ; C_i \in \mathcal{C}_i\}$. The *intersection convexity space* is

(*)Cohn's definition for inductive system is uncorrect; replace by ours in the proof.

$(\bigcap(X_i; i \in I), \bigcap(\mathcal{C}_i; i \in I))$. The convexity subspace (Y, \mathcal{C}_Y) of a convexity space (X, \mathcal{C}) induced by a subset $Y \subseteq X$ is defined by : $\mathcal{C}_Y = \{C \cap Y ; C \in \mathcal{C}\}$. Other derived convexities are considered below : (3.9) and (3.10) and in Degreef [1979,82], Sierksma [1981], Soltan [1984].

(3.9) MINORS. As in Matroid Theory (see Welsh [1976]), a very natural notion of minor space exists for convexity spaces (see Jamison [1982], compare Korte, Lovász [1984b]). Let $S = (X, \mathcal{C})$ be a convexity space and let $Y \subseteq X$. The contracted space of S by A , denoted by S/A is the convexity space $(X \setminus A, \mathcal{C}/A)$ with :

$$\mathcal{C}/A = \{C \setminus A ; A \subseteq C \text{ and } C \in \mathcal{C}\}$$

The subspace $S \setminus A$ is the convexity subspace of S induced by $X \setminus A$. We have $S \setminus A = (X \setminus A, \mathcal{C} \setminus A)$ with :

$$\mathcal{C} \setminus A = \{C \setminus A ; C \in \mathcal{C}\}.$$

When A, B are disjoint subsets, we have $(S/A) \setminus B = (S \setminus B)/A$. A minor of S is every convexity space which is isomorphic to some induced subspace of a contracted space of S . If the flats of a matroids are viewed as convexes of a finite convexity structure, the notions "matroidal minor" and "convex minor" coincide. With this first point of view, matroids are exactly the finite convexity spaces with no minor isomorphic to the space $Q_1^{(2)} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Convex geometries (section 5) are another interesting class which is stable under taking minors.

(3.10) VARIETES. If we order by inclusion the set of all convexities on a given set X we obtain a complete lattice (Jamison, [1982]), in which the upper bound of a family of convexities is called the joint of the family. Most interesting classes of convexity spaces are preserve under the formation of joints. Jamison proposer to call variety a class V of convexity spaces that possesses the following properties :

(V₁) Convexity spaces isomorphic to a number of V are in V .

(V₂) Subspaces of members of V are in V .

(V₃) If every finite subspace of a space (X, \mathcal{C}) is in V then (X, \mathcal{C}) is in V .

When convexities $\mathcal{C}_i (i \in I)$ are taken in a variety V , the convex hull in their joint J has a simple expression :

$$\langle^A\rangle_J = \bigcap_{i \in I} \langle^A\rangle_{\mathcal{C}_i}$$

By (V₂), varieties can be characterized by excluded subspaces. A variety V is finitely based when V can be defined by a finite list of forbidden subspaces.

PROBLEM 4 (Jamison) Let V, W be two finitely based varieties. Let $V \vee W$ be the class of all convexity spaces (X, \mathcal{C}) where \mathcal{C} is a joint of a member of V with a member of W (*). Is $V \vee W$ finitely based?

4. COMBINATORIAL INVARIANTS

(4.1) Essential results in Axiomatic Convexity concern various parameters associated to combinatorial properties. Each of these parameters expresses a certain notion of dimension. All definition listed below are relative to a convexity space (X, \mathcal{C}) .

(4.2) CONVEX DEPENDANCE : a set $A \subseteq X$ is said to be free if $a \notin \langle A \setminus a \rangle$, for every $a \in A$. The rank of \mathcal{C} is the maximum cardinality (if finite) of a free set. The rank, denoted by $\text{rank}(\mathcal{C})$, equals the least integer k that fulfills the following property :

(F_k) If C_1, \dots, C_{k+1} are convex sets of \mathcal{C} , then one of them contains the intersection of the others.

Real intervals of the real line form a convexity of rank 2 (see section 8). Convexities of rank n are obviously n -generated.

(4.3) HELLY PROPERTY :

(H_k) If a finite family of convexes has an empty intersection, then this family contains at most k members with an empty intersection. Notice that (H_k) is a relaxation of (F_k) . The Helly number of denoted by $h(\mathcal{C})$ is the smallest integer k (if it exists) such that (H_k) holds. Ordinary convexity in \mathbb{R}^d has Helly number $d+1$ (Helly, [1923]). Arithmetical progressions in \mathbb{Z} satisfy (H_2) (Chinese Remainder Theorem). For further examples see Dantzer et al. [1963], Jamison [1982], Duchet, Quilliot [1986], Eckhoff [1986].

Berge, Duchet [1975] : $h(\mathcal{C}) = \max(|A| ; A \subseteq X \text{ and } \bigcap_{a \in A} \langle A \setminus a \rangle = \emptyset)$

We have also : $\text{rank}(\mathcal{C}) = \max(h(\mathcal{C}_A) ; A \subseteq X)$. Particular results on Helly number are given in other sections.

Other results may be found in Sierksma [1975/76] and Soltan [1984].

(4.4) PARTITION PROPERTY

$(P_{k,n})$ If $(p_i)_{i \in I}$ is a family of $n = |I|$ points, there exists a partition of I into k parts $I_1 \dots I_k$ such that :

$$\bigcap_{1 \leq \lambda \leq k} \{p_i ; i \in I_\lambda\} \neq \emptyset$$

Ordinary convexity in \mathbb{R}^d has property $(P_{k, (k-1)(d+1)+1})$ for $k, d \geq 1$ (Radon, [1921] for $k=2$, Birch [1960] for $d=2$, Tverberg

(*) It can be shown that $V \vee W$ is a variety.

[1966/82] for the general case). The k -th partition number denoted by $p_k(\mathcal{C})$ is the smallest integer n such that $\{P_{k,n}\}$ holds. Number $p_2(\mathcal{C})$ is usually known as Radon number and is denoted by $r(\mathcal{C})$. An important inequality is $h(\mathcal{C}) \leq r(\mathcal{C}) - 1$ (Levi, [1951]): see (4.7). Eckhoff [1979] asked for a purely combinatorial proof of Tverberg's theorem.:

PROBLEM 5 (Eckhoff) Suppose a convexity space \mathcal{C} has Radon number r . Is the following "partition inequality": $p_k(\mathcal{C}) \leq (k-1)(r-1)+1$ always true?

A first step towards a solution to this surprising problem is taken with the following results [Jamison-Waldner, 1981]:

$$p_{kk} \leq p_k, \quad \text{hence } p_k \leq p_2 k^{\log_2 p_2}$$

$$p_{(k-i)k+1} \leq (p_{k-i})p_{k+1} \text{ for } 1 \leq i \leq k.$$

Jamison proved the partition inequality for order convexities (section 8), tree-convexities (section 9) and more generally for convexities that have the following property:

CIP(3,2): for every point x , among three copoints relative to x two of them are disjoint.

Roudneff [1986] proved the partition inequality for convexities of oriented matroids of rank ≤ 3 (section 7).

(4.5) CARATHEODORY PROPERTY

(C_k) If $x \in \langle A \rangle$, then $x \in \langle F \rangle$ for some $F \subseteq A$ with $|F| \leq k$.

Ordinary convexity in \mathbb{R}^d has property (C_{d+1}) (Caratheodory, [1907]). For related results see Barany [1981]. The Caratheodory number denoted by $c(\mathcal{C})$ is the smallest integer k (if it exists) such that (C_k) holds. We have:

$$c(\mathcal{C}) = \max(|A|; A \subseteq X \text{ and } \langle A \rangle \notin \bigcup_{a \in A} \langle A \setminus a \rangle)$$

For interval convexities (3.6) more can be said [Duchet, 1986a]: the Caratheodory number equals the least integer k such that every $k+1$ -point set A satisfies the following property:

$$\langle A \rangle = \bigcup_{a \in A} \langle A \setminus a \rangle.$$

(4.6) EXCHANGE NUMBER [Reay, 1965; Siersksma, 1976]

(E_k) If $x \in X$, $A \subseteq X$, $|A| \geq k$ then:

$$\langle A \rangle \subseteq \bigcup_{a \in A} \langle (A \cup x) \setminus a \rangle.$$

Ordinary convexity in \mathbb{R}^d satisfies (F_{d+1}) Reay, 1965. The exchange number [Siersksma, 1976], denoted by $e(\mathcal{C})$ is the smallest integer k (if it exists) such that (E_k) holds.

(4.7) INEQUALITIES

- Levi [1951] : $h \leq r-1$
 Sierksma [1976] : $e \leq c+1$
 Sierksma [1977] : $r \leq (h-1) \max(h, e-1)+2$
 If $e \leq c$, $r \leq (c-1)(h-1)+3$
 Jamison-Waldner [1981] : $P_k \leq (k-1)$ rank +1
 Ibid. (Kay Womble [1977] for k=2) : $P_k \leq (k-1)c h - c + 2$
- For $i = 1, 2, \dots, n$ let (X_i, \mathcal{C}_i) be a convexity space. Let (X, \mathcal{C}) denote their product space (3.8). Set $h_i = h(\mathcal{C}_i)$, $h = h(\mathcal{C})$; numbers r_i, c_i, e_i, r, c, e are defined analogously.
- Sierksma [1975] : $h = \max_i h_i$
 Eckhoff [1978, 1979] : $\max_i r_i \leq r$
 for $n = 2$: $r \leq r_1 + r_2 - 1$
 Sierksma [1976] : $r \leq \sum_i r_i - 2n + 2$
- Soltan [1981] (cf. Sierksma [1976]) : $e = 1 + \sum_i (c_i + \text{sign}(e_i - e_{i-1}))$

Soltan [1981], Sierksma [1975] ($n=2$) : $c = e-1+\varepsilon$ (where $\varepsilon = 0$
 (also Reay [1970] if \mathcal{C}_i are standard convexities in \mathbb{R}^{d_i})
 if $e_i = c_i+1$ for every i or if $c_i \geq 2$ for every i ;
 if $e_i = 1$ in other cases).

PROBLEM 6 (Kay, Womble) Characterize possible triples of the form $(h(\mathcal{C}), r(\mathcal{C}), c(\mathcal{C}))$ for some convexity space (X, \mathcal{C}) .

5. CONVEX GEOMETRIES

(5.1) Convex Geometries were introduced independently by Edelman [1980a] and by Jamison [1980]. Convex Geometries are finite convexity spaces in which the (finite) Krein-Milman property holds :

(KM) Every convex set is the convex hull of its extreme (*) points.

Theory of Convex Geometries is interesting at least for two reasons. On the one hand, it is general enough to unify areas that were previously ill-matched : Convex Geometries were independently discovered under dual form by Korte and Lovász [1981, 1984a] (shelling structures) and Crapo [1984] (selectors) in a completely different context; a similar structure appears also in Discrete Optimization [Hoffman 1979]. On the other hand the concept of Convex Geometry is rich enough to allow the growth of a theory.

If all singletons of a convex geometry are convex, then the

(*) A point x of a convex C is extreme in C iff $C \setminus x$ is convex.

convex geometry is completely determined by the lattice of its convexes. Edelman [1980] showed that a finite convexity space is a convex geometry if and only if the lattice of its convexes is meet-distributive (*). Hence one can say that the concept of Convex Geometry is implicit in Dilworth [1940] where meet-distributive lattices are characterized as lattices in which every element has a unique expression as a joint of joint-irreducible elements. These lattices were frequently rediscovered : see Monjardet [1985].

Although Edelman and Jamison restrict their attention to finite spaces, I suspect that no major difficulty arises about extending their results to infinite spaces. (compare Soltan [1984]).

PROBLEM 7 (Duchet) Extend the theory of convex geometries to infinite spaces.

(5.2) EXAMPLES : Convexities of acyclic oriented matroids (section 7); partial order convexities and some related convexities (see section 8); minimal path convexity in triangulated graphs (6.4); geodesic convexity in ptolemaic graphs (6.7); tree convexities (see section 9).

(5.3) CHARACTERIZATIONS AS CONVEXITY SPACES (Edelman, Jamison) A finite convexity space (X, \mathcal{C}) is a convex geometry if and only if each of the following equivalent axioms is satisfied.

(i) (Anti-exchange axiom)

$x, y \in \langle A \rangle$, $x \neq y$ and $x \in \langle A \cup y \rangle$ imply $y \in \langle A \cup x \rangle$

(ii) (Augmentation axiom)

If $C \in \mathcal{C}$ then $(C \cup x) \in \mathcal{C}$ for some $x \in X \setminus C$

(iii) (Generation axiom)

If $C = \langle A \rangle = \langle B \rangle$ then $C = \langle A \cap B \rangle$

(iv) (X, \mathcal{C}) is a joint of monotone convexity spaces (**)

(v) (X, \mathcal{C}) has no minor isomorphic to $\mathbb{Q}_0(2)$ where

$\mathbb{Q}_0(2) = (\{1,2\}; \{\emptyset, \{1,2\}\})$.

Antieexchange axiom (i) is to be compared with exchange axiom for closed sets (flats) of matroids. The equivalence between (i), (iv) and (v) holds for infinite spaces [Jamison, 1982]. A variant of anti-exchange axiom is :

(vi) For every $C \in \mathcal{C}$, the relation $x \in \langle C \cup y \rangle$ between elements x and y of $X \setminus C$ is a partial order on $X \setminus C$.

(*) A lattice is meet-distributive if and only if all intervals $[x, y]$ such that x is the meet of elements that y covers, are Boolean algebras.

(**) See (3.10) and (8.4) for definitions. See (5.8) for details

For other variants see Edelman, Jamison [1985], Thron [1985]. An equivalent form of (ii) is :

- (vii) Every maximal chain $\emptyset = C_1 \subset C_2 \subset \dots \subset X$ of convexes comprises $|X| + 1$ convexes.

Notice that the existence of extreme points for every convex set is not sufficient to imply (ii). The generation axiom (Hoffman, [1979]) is equivalent to the following axiom (cf. Korte, Lovász [1984b]) :

- (viii) Every convex $C \in \mathcal{C}$ contains a unique minimal subset B such that $\langle B \rangle = C$.

(5.4) HELLY NUMBER. In a convexity space (X, \mathcal{C}) , a set $K \subseteq X$ is a clique if K and all K -subsets are convex. Equivalently a clique is a convex set all vertices of which are extremal. In a convex geometry the Helly number equals the maximum number of points in a clique.

(Edelman, Jamison [1985]). Compare with the Helly number of minimal path convexities in graphs (6.4).

(5.5) GREEDOIDS (Korte, Lovász) As a framework for greedy algorithms, Faigle [1979, 1980] proposed an extension of matroidal structure to partially ordered sets. Greedoids (Korte, Lovász, [1981]) are a further generalization of "Faigle-geometries" (see Korte-Lovasz [1982]). A Greedoid (X, \mathcal{F}) is a finite set X endowed with a collection \mathcal{F} of X -subsets such that the following axioms are satisfied.

$$(G_1) \quad \emptyset \in \mathcal{F}$$

- (G₂) If $A, B \in \mathcal{F}$, $|A| < |B|$ then $A \cup \{b\} \in \mathcal{F}$ for some $b \in B \setminus A$
- Members of \mathcal{F} are the feasible sets of the greedoid. The independent sets of a matroid are the feasible sets of a greedoid.

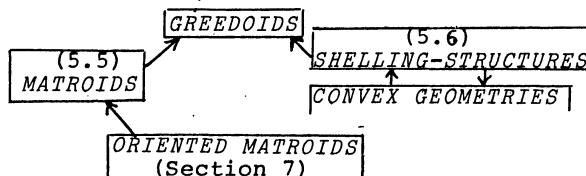
(5.6) SHELLING STRUCTURES (= Selectors, "Alternative precedence structures" or APS-greedoids) [Korte, Lovász, 1984b] are a special class of greedoids. Axioms are :

$$(SH_1) \quad \emptyset, X \in \mathcal{F}$$

- (SH₂) If $A, B \in \mathcal{F}$, $A \not\subseteq B$ then $A \cup \{b\} \in \mathcal{F}$ for some $b \in B \setminus A$.

Axiom (SH₂) is due to Björner [1983] who first pointed out the connection between greedoids and antieexchange axiom (implicit in Crapo [1984]) : let X be a finite set and let $\mathcal{C} \subseteq 2^X$. Set $\overline{\mathcal{C}} = \{X \setminus C ; C \in \mathcal{C}\}$. Then (X, \mathcal{C}) is a convex geometry if and only if $(X, \overline{\mathcal{C}})$ is a shelling structure : see (5.8) below.

(5.7)



(5.8) COMPATIBLE ORDERS. If τ is a total order on a set X and $x \in X$, we denote by τ_x the lower ideal $\{y \in X : y \tau x\}$. Let \mathcal{T} be a family of total orders ("criteria") on X . A total order τ is said to be \mathcal{T} -compatible when every element $x \in X$ is for some criterias in \mathcal{T} , the best element of τ_x . The set of all lower ideals (respectively of all upper ideals) of all \mathcal{T} -compatible orders are the convexes of a convex geometry (respectively the feasible sets of a shelling structure). By (5.3)(iv) every convex geometry (hence every shelling structure) can be generated by this process. The set of all \mathcal{T} -compatible orders induces a connected subgraph of the permutohedron on X (Korte, Lovász, [1984c]; Edelman, Jamison, [1985]; compare Feldman-Högaasen [1969]). Compatible total orderings x_1, \dots, x_n of X are precisely those in which x_i is an extreme point of $\{x_1, x_2, \dots, x_i\}$ for $i = 1, \dots, n$. The sequences $x_n x_{n-1} \dots x_i$ are precisely the words of the shelling structure when considered as a language over X (see Korte, Lovász [1984a]).

PROBLEM 8 (Edelman, Jamison) Determine the convex dimension of a convex geometry, i.e. the minimum number of criteria which are needed for the construction of that convex geometry.

Edelman and Jamison [1985] showed that the convex dimension is not less than the convex rank (4.2) but it may be greater for the construction of a given convex geometry : this number is at least equal to the convex rank (3.); see Edelman, Jamison [1985].

(5.9) CHARACTERIZATIONS AS GREDOIDS (Korte, Lovász, Björner). A greedoid (X, \mathcal{F}) is a shelling structure if and only if $X \in \mathcal{F}$ and each of the following equivalent axiom is satisfied :

- (i) Every union of feasible sets is feasible.
- (ii) (Interval property without upper bound)
If $A, B, A \cup \{x\}$ belongs to \mathcal{F} and $A \subseteq B$, then $(B \cup x) \in \mathcal{F}$
- (iii) If $A, A \cup \{x\}, A \cup \{y\}$ belong to \mathcal{F} then $(A \cup \{x\} \cup \{y\}) \in \mathcal{F}$

When replacing $A \subseteq B$ in (ii) by $B \subseteq A$, we obtain an axiom (Interval property without lower bound) which characterizes matroids among greedoids.

(5.10) CIRCUITS (Korte and Lovász [1984a]) Let (X, \mathcal{C}) be a convex geometry. Free sets of \mathcal{C} are defined as in paragraph (4.2). By analogy with Matroid Theory, a minimal non free set is named a circuit; a circuit R has a unique point r such that $r \in \langle R \setminus r \rangle$, named its root. Finite convexity subspaces of Euclidean convexity have the property ;

(CI) If $A \cup \{x\}$ and $A \cup \{y\}$ are circuits with respective roots x

and y then there exists a unique circuit R with root y such that $\{x,y\} \subseteq R \subseteq A \cup \{x,y\}$.

Say a convex geometry is good if it satisfies (CI) and if every 3-points subset is free.

Korte and Lovász [1984b] proved the following generalization of a famous Erdős-Szekeres theorem [1935] : for some function $f(n)$, every good convex geometry with at least $f(n)$ points contains a free set of n points.

Remark that, in a convex geometry, free sets are precisely the sets of extreme points of convex sets.

(5.11) ENUMERATION Let (X, \mathcal{C}) be a convex geometry. In the meet distributive lattice (\mathcal{C}, \subseteq) , the Möbius function (*) is easy to calculate (Edelman, Jamison [1985]) : for $C \subseteq D$ we have $\mu(C, D) = (-1)^{|D \setminus C|}$ if every point of $D \setminus C$ is extreme in D ; otherwise $\mu(C, D) = 0$. An (unpublished) theorem of Lawrence follows : we have $\sum_k (-1)^k f_k = 0$ where f_k denotes the number of free sets with k points.

In [1985] Edelman and Jamison (**) also give a nice interpretation of the values of the Zeta function (*) of (\mathcal{C}, \subseteq) . Let $Z(n)$ denote the number of nondecreasing sequences of n convexes of \mathcal{C} : $\emptyset \subseteq C_1 \subseteq \dots \subseteq C_n \subseteq X$. Function $Z(n)$ is a polynomial in the variable n . Let us say a function $\varphi : X \rightarrow \{1, \dots, n\}$ is extremal (respectively strictly extremal) if for every convex $C \in \mathcal{C}$, φ achieves its maximum on C at some extreme point of C (respectively φ achieves its maximum only on extreme points). Then :

$Z(n)$ and $(-1)^{|X|} Z(-n)$ are respectively the number of extremal functions $X \rightarrow \{1, \dots, n\}$ and the number of strictly extremal functions $X \rightarrow \{1, \dots, n\}$.

6. CONVEXITY IN GRAPHS AND HYPERGRAPHS

(6.1) As far as I know, the first explicit use of convexity in graphs appears in Feldmann - Högässen [1969] where convexity in the permutohedron is investigated. Most results deal with geodesic convexity (6.5). A more general point of view appeared in Nebesky [1970, 71], Sekanina [1975], Mulder [1978, 1980], Harary, Nieminen [1981]. A systematic approach arises in Duchet, Meyniel [1983] and Farber, Jamison [1983]. All definitions and results below refer to a connected graph G on

(*) See Rota [1964]-or Aigner [1979] for details.

(**) See also Edelman [1980] and Stanley [1974, Prop. 2.1].

vertex set V (finite or infinite). See Berge [1985] for general notions of Graph Theory.

(6.2) A graph convexity (Duchet and Meyniel [1983]) is a pair (G, \mathcal{C}) formed with a (connected) graph G , with vertex set V , and a convexity \mathcal{C} on V such that (V, \mathcal{C}) is a convexity space satisfying the additional axiom :

(GC) Every convex set induces a connected subgraph.

(6.3) HADWIGER'S CONJECTURE is a famous problem that generalizes the four colour problem. Let W be a V -subset that induces a connected subgraph of G . Add a new vertex x_W to the graph $G \setminus W$ and join it to every vertex y , $y \in V \setminus W$ for which G contains a $y - W$ edge. The resulting graph is the graph obtained from G by contraction of W . A contraction of G is every graph that can be obtained by a sequence of contractions of connected subgraphs. Hadwiger [1943] conjecture :

PROBLEM 9 (Hadwiger) If the complete graph with $p+1$ vertices is not a contraction of G then one can color the vertices of G in p colours such that adjacent vertices receive different colours.

A recent survey on this important conjecture is by Duchet, Xuong [1986].

The Hadwiger number $\eta(G)$ of a graph G is the largest integer p such that G has the complete graph with p vertices as a contraction. Helly (4.3) and Radon (4.4) numbers of a graph convexity (G, \mathcal{C}) are intimately related to the Hadwiger number (Duchet, Meyniel [1983]) :

$$\begin{aligned} (i) \quad h(\mathcal{C}) &\leq \eta(G) \\ (ii) \quad r(\mathcal{C}) &\leq 2\eta(G) \end{aligned}$$

Equality is possible in (i).

PROBLEM 10 (Duchet) For every $\epsilon > 0$ there is a graph convexity (G, \mathcal{C}) such that $r(\mathcal{C}) > (2-\epsilon)\eta(G)$.

For further details see Duchet [1984].

(6.4) MINIMAL PATHS. Let $M(x, y)$ denote the set of all vertices of all chordless paths from x to y in a graph G . The convexity generated by the interval function M is called the minimal path-convexity (or M -convexity) on G (Jamison [1982] Duchet [1979, 1986a]). Caratheodory, Helly and Radon numbers (see (4.5) (4.3) and (4.4)) are respectively denoted by $c_M(G)$, $h_M(G)$ and $r_M(G)$; we have (Duchet [1986a]; also Jamison and Nowakowski [1984] for h_M) :

$c_M(G) = 2$ (if G is not complete), $h_M(G) = \omega(G)$, $r_M(G) = \omega(G) + 1$ (if $\omega(G) \geq 3$). If $\omega = 2$, $r_M(G) \leq 4$ and $r_M(G) = 3$ iff G is a "path of blocks". Here above $\omega(G)$ denotes the maximum number of vertices of a complete

subgraph in G .

The M -convexity of a graph G is a convex geometry if and only if G is triangulated — i.e. every cycle of length ≥ 4 has a chord — (Farber, Jamison [1983]). Extremal vertices correspond to simplicial vertices, i.e. vertices whose neighbourhood is complete (see (9.3)).

(6.5) GEODESIC CONVEXITY. Let $D(x,y)$ denote the set of all vertices of all shortest paths between x and y . The convexity generated by the interval function D is called the *geodesic convexity* (or *distance-convexity* or *D-convexity*) on G ; the D -convexity is the metric convexity (3.7) associated to the usual distance function $d(x,y)$ in graphs. First researches on D -convexities were motivated by the following question of Ore [1962] :

PROBLEM 11 (Ore) Characterize the geodetic graphs, i.e. the graphs in which every pair of vertices is joined by a unique shortest path.

Various constructions of geodetic graphs are known. See Stemple and Watkins [1968], Zelinka [1977], Stemple [1979], Parthasarathy and Srinivasan [1984] and Plesnik [1984]. Contrarily to M -convexities of graphs (6.4), the D -convexity is very general (see (6.6)) and became intensively studied since 1981 : see Jamison [1981a], Batten [1983], Soltan [1983], Soltan and Chepoi [1983,84,85], Farber [1985], Farber and Jamison [1986], Sampathkumar [1984].

(6.6) The geodesic convexity is in some sense universal with respect to Caratheodory, Helly and Radon properties : given any finite convexity space (X, \mathcal{C}) , there exists a finite graph G such that the Caratheodory, Helly and Radon numbers of the geodesic convexity in G coincide with those of \mathcal{C} . (Duchet [1986b]).

PROBLEM 12 (Duchet) Does universal property above hold for partition numbers ?

(6.7) METRIC CHARACTERIZATIONS of some classes of graphs appear as a facet of the research on geodesic graph convexity. For instance, certain graphs are completely determined by the list of the isomorphism types of their convex subgraphs ; see Egawa [1986], Van Cruyce [1984a,b]. In Bhaskara Rao and Rao Hebbare [1976] and Rao Hebbare [1979] the authors investigate the graphs having only trivial D -convex sets (\emptyset, V , singletons and edges). Graphs in which D -convexity determines a convex geometry are characterized (Farber, Jamison [1983]) by the *Ptolemaic inequality* :

$$d(x,y)d(z,t) \leq d(x,z)d(y,t) + d(y,z)d(x,t)$$

A related interesting class was introduced in Farber [1985] and Farber, Jamison [1986] : a graph is bridged if each cycle C of

length ≥ 4 contains two vertices whose distance from each other in G is strictly less than in C .

A graph is bridged if and only if every closed neighbourhood of a D -convex set is again D -convex (Farber, Jamison [1986]; Soltan, Chepoi [1983]). In connection with the problem of the metric determination of a graph, the results of Graham and Winkler [1984] on isometric embeddings of graphs are to be considered.

(6.8) MEDIAN GRAPHS (Mulder [1978]) are graphs in which the metric interval function D (see (6.5) has the following property :

$$|D(x,y) \cap D(y,z) \cap D(z,x)| = 1 \text{ for every } x,y,z.$$

These graphs generalise hypercubes and are extensively studied : Mulder [1980a,b], Mulder, Schrijver [1979], Bandelt, Barthelemy [1984], Nieminen [1984].

Mulder [1980a] also considers *interval regular graphs* in which $D(x,y)$ contains exactly $d(x,y)$ neighbours of x , for any two vertices x,y of the graph.

PROBLEM 13 (Mulder) Conjecture : In any interval regular graph the sets $D(x,y)$ are geodesically convex.

Variants on the theme of betweenness or medians are numerous. See for instance Nebesky [1970, 1971] (ternary algebras), Sekanina [1975] Barthelemy [1983] and Batten [1983]. Refer also to Fishburn [1971] and to sections 8, 9 for betweenness in partial orders and in trees.

(6.9) HYPERGRAPHS Natural generalizations of graph convexities may be considered in the context of abstract families of sets (hypergraphs : Berge [1986]); for instance Farber and Jamison [1983] characterized *strongly balanced hypergraphs* — in which every cycle possesses an edge that contains three vertices of the cycle — by the fact that a certain natural convexity associated to the hypergraph determines a convex geometry. One can also consider edges of a hypergraph as copoints of a convexity. Helly property has been studied with this point of view : Mulder, Schrijver [1979], Barthelemy [1985]. In [1979, 1983] Bollobás and Duchet solved the corresponding extremal problem :

Let $h < p \leq n$ be integers: in a n -element set, a convexity with Helly number h has at most $\binom{n-1}{p-1}$ convexes of cardinality p . Moreover, this upper bound is reached only if the convexes of cardinality p are precisely the p -subsets containing a common element.

7. ACYCLIC ORIENTED MATROIDS

All matroids here are supposed to be simple. See Welsh [1976] for general definitions and properties of matroids.

(7.1) Let E be a finite set of points of \mathbb{R}^d . Minimal affinely dependent subsets of E are the circuits of a matroid M on E (*affine matroid*). Every such circuit admits a unique Radon partition (see (4.4), Eckhoff [1975,79]). Oriented matroids constitute an elegant axiomatic setting for a combinatorial treatment of these Radon partitions ; in an oriented matroid M every circuit X of the underlying matroid M admits a unique partition into a positive part X^+ and a negative part X^- . These *signed circuits* satisfy certain axioms. For axioms and fundamental properties, see Bland, Las Vergnas [1978], Folkman, Lawrence [1978]. An important fact is the existence of a canonical way of orienting the orthogonal matroid M : for affine matroids, the positive and the negative part of a cocircuit Y correspond to the separation the hyperplane $H = E \setminus Y$ determines through Y . The structure of Oriented Matroids is thus intimately related to convexity. A fundamental open question is the determination of the set of Radon partitions of finite sets of points in \mathbb{R}^d .

(7.2) *FACES* Affine matroids are *acyclic* : it means that no circuit X is entirely positive ($X = X^+$) or entirely negative ($X = X^-$). Convexity in acyclic oriented matroids was first investigated by Las Vergnas [1980] who proved the following facts : Let M be an acyclic oriented matroid with point set E . A *face* of M is a flat F such that the matroid \overline{F}^M obtained from M by sign interchange of all points in F is again acyclic. *Faces, when ordered by inclusion form a graded lattice* where the meet is the intersection. A point p is extreme if $\{p\}$ is a face. A rank r acyclic matroid admits at least r extreme points.

PROBLEM 14 (Las Vergnas) Characterize face lattices of acyclic oriented matroids.

These lattices are meet-distributive (Edelman [1982]) and satisfy Euler's relations (Cordovil, Mandel, Las Vergnas [1982]). Further results may be found in Munson [1981], Lawrence [1983,84], Billera, Munson [1984].

(7.3) *CONVEX OPERATORS* Let M be an acyclic oriented matroid on a set E . All signed M circuits X whose negative part is a singleton $\{x\}$ may be viewed as operators $X^+ + x$. The convexity on E determined (see (3.5)) by these operators is called the (canonical) *convexity of M* (Las Vergnas [1980]; cf. Goodman, Pollack [1982],

Folkman, Lawrence [1978, p. 204].

PROBLEM 15 Characterize convexity spaces that arise from acyclic oriented matroids.

7.4) SEPARATIONS Let M be an acyclic oriented matroid on a point set E . Two disjoint E -subsets A, B are said to be separable if no signed circuit (X^+, X^-) of M both satisfy $X^+ \subseteq A$ and $X^- \subseteq B$. A hyperplane $H \subseteq E$ separates A and B when the signed parts of the cocircuit $A \setminus B$ contain respectively A and B . A separation of M (or "non-Radon partition", see Brylawski [1976], Cordovil [1985]) is a partition of E into separable sets. Bienia and Cordovil [1985] gave a characterization of the set of separations of an oriented matroid.

(7.5) SEPARATION THEOREMS Let M denote an acyclic oriented matroid on a point set E . For $A \subseteq E$, the restriction of M to A is denoted by $M(A)$, the convex hull of A is denoted by A_M . Let us consider the following properties relative to separation of two subsets A, B of E :

- (i) A and B are separable.
- (ii) $\{A, B\}$ is a separation of $M(A \cup B)$
- (iii) There exists a separation $\{A', B'\}$ of M such that $A \subseteq A'$ and $B \subseteq B'$
- (iv) There exists an extension M' of M where A and B are separated by a hyperplane
- (v) In every extension N of $M(A \cup B)$, we have:

$$A_N \cap B_N = \emptyset$$

- (vi) In every extension M' of M , we have:

$$A_{M'} \cap B_{M'} = \emptyset.$$

Properties (i), (ii), (iii), (iv) are equivalent. Furthermore (iv) \Rightarrow (v) \Rightarrow (vi) (Las Vergnas [1980], Mandel [1982]). The implication (v) \Rightarrow (iv) is conjectured by Mandel who found a rank 4 oriented matroid that satisfies (vi) but not (v).

PROBLEM 16 (Mandel) Conjecture : A pair A, B of sets of points in an oriented matroid M is separable if and only if condition (v) holds.

Equivalence between (iv), (v) and (vi) holds for certain class of oriented matroids : see Bachem, Wanka [1986]. Further results relative to separation may be found in Goodman, Pollack [1982], Cordovil [1982], Cordovil, Duchet [1986].

(7.6) Further results relative to convexity in oriented matroids may be found in Méi [1971], Cheung [1974], Goodman, Pollack [1982], Edelman [1984], Bachem, Kern [1982], Billera, Munson [1984b].

8. PARTIALLY ORDERED SETS

(8.1) Let P denote a finite partially ordered set. Define an interval-function by $I(x, y) = \{z \in P ; x \underset{P}{\leq} z \underset{P}{\leq} y\}$. The interval convexity

(see (3.6)) generated by I is named the *order convexity* of P (Franklin [1962]) and is denoted by \mathcal{C}_P .

For any $x \in P$, at most two copoints relative to x may exist in \mathcal{C}_P , namely the sets $C^+(x) = \{y \in P ; x \leq y\}$ and $C^-(x) = \{y \in P ; x \leq y\}$. Thus, property CIP(3,2) is satisfied (4.4): the partitions inequality holds in every order convexity (Jamison-Waldner [1981a]). Order convexities form a variety (3.10) which is "sum-closed" but not finitely based (see Jamison [1979, 82]).

A convexity space (X, \mathcal{C}) is the order convexity of some partial order P on X if and only if it has the three following properties (Jamison [1979]):

- (i) Every free set is a union of two convex sets.
- (ii) \mathcal{C} has Caratheodory number ≤ 2
- (iii) Every convexity subspace with at most 5 points is an order convexity space.

The first axiomatic of partial orders in terms of betweenness was given by Altwegg [1950]. See Fishburn [1971] for further information.

(8.2) TOTAL ORDER CONVEXITIES: order convexities of totally ordered sets. Every order convexity of a poset P is the joint (3.10) of the total order convexities of linear extensions (*) of P . (Edelman, Jamison [1985]); hence, finite order convexities are convex geometries; remark that linear extensions and their reversals are compatible orders with the meaning of (5.8). Arbitrary joints of total order convexities form an interesting variety we denote by \bigvee_{TO} , indeed they include partial order convexities, ordinary convexities in Euclidean spaces (Jamison [1982]) and convexities of acyclic oriented matroids (Section 7). Variety \bigvee_{TO} is the smallest variety containing all varieties \bigvee^n_{TO} constituted of joint of n total convexities for $n \in \mathbb{N}$.

PROBLEM 17 (Jamison) Characterize the variety \bigvee_{TO} . Is \bigvee_{TO} finitely based for each $n \in \mathbb{N}$?

Variety TO of total order convexities is finitely based since we have: a convexity space (X, \mathcal{C}) is the order convexity space associated to some total order on X if and only if one of the

(*) A linear extension of P is a total order τ such that $x \leq y$ whenever $x \leq y$ $\underset{P}{\leq}$

following equivalent conditions is satisfied.

- (i) Every subconvexity of \mathcal{C} induced by a X -subset of at most 4 points is a total order convexity.
- (ii) \mathcal{C} has rank ≤ 2 , has the antierchange property and has the following separation property : for every pair $a, b \in X$ there exists a partition of X into convex sets A and B so that $a \in A$ and $b \in B$.

Various alternative characterizations of total orders in terms of betweenness are known. See Fischburn [1971] and its references; also (9.2).

(8.3) Finite partially ordered sets may be viewed themselves as convex sets : Feldman-Högaasen [1969] exhibit a Galois connection between partial orders on a finite set X and the geodesically-convex subsets of the permutohedron with basic set X .

(8.4) In the downset convexity determined by a poset P , convexes are lower ideals of P , i.e. P -subsets I such that $y \in I$ whenever $y \leq x$ for some $x \in I$. A finite convexity space (X, \mathcal{C}) is the downset convexity of some poset on X if and only if it is a convex geometry in which every union of convex sets is again convex (*). (Edelman, Jamison [1984], Korte, Lovasz [1984b]). Remark that downset convexities of total orders (monotone convexities in Jamison [1982]) may be considered as bricks in the construction of convex geometries : see (5.2(iv)).

(8.5) A (meet) semi-lattice L is a poset in which each pair of elements has an infimum. The semilattice convexity associated with L consists of all subsemilattices of L . It can be viewed as a "refinement" of the downset convexity of L . Finite semilattice convexities are convex geometries. The Caratheodory number of the semilattice convexity coincides with the "breadth" (see Birkhoff [1967] p. 99, Crawley, Dilworth [1973] p. 38) of the underlying semilattice (Jamison [1982]). Note that the rank of a finite convexity space (X, \mathcal{C}) is the breadth of the semilattice (\mathcal{C}, \subseteq) (Ibid.).

(8.6) Various other examples of convex geometries arise in the context of partially ordered sets. In [1983] Cochand and Duchet investigated the order convexity of a product of total orders. For every positive integer k , Saks defines a closure operator Ω_k (see (3.5)) in a poset P by :

$$\Omega_k(A) = A \cup \{y \in P ; y < a_1 < a_2 < \dots < a_k \text{ for some chain } a_1, a_2, \dots, a_k \text{ of } A\}$$

(*) This result can easily be extended to the infinite case.

Ω_k -closed sets form a convex geometry \mathcal{D}_p^k . Free sets (*) of \mathcal{D}_p^k are named k -families. A k -family A admits a unique partition into disjoint, possibly empty antichains A_1, A_2, \dots, A_k such that $A_k \leq A_{k-1} \leq \dots \leq A_1$ where $A_i \leq A_j$ means that for every $a_i \in A_i$ there exist some $a_j \in A_j$ such that $a_i \leq a_j$. Given two k -families A, B of P , define $A \leq B$ if and only if $A_i \leq B_i$ for all $i = 1, \dots, k$. The set of all k -families ordered in this way is a join-distributive lattice (Greene, Kleitman [1976]), hence may be considered as the dual lattice of convex sets of some convex geometry : for a discussion about these two last examples see Edelman, Jamison [1985].

9. TREE-LIKE STRUCTURES

(9.1) TREE CONVEXITIES. In a tree T the M -convex sets (6.4) coincide with the D -convex sets (6.5) : they form the convexes of the tree-convexity determined by T . Tree-convexities are convex geometries (section 5) and are coherent convexities (**). Results by Duchet [1978], Flament [1978], Slater [1978] imply that a given family \mathcal{F} of subsets of a finite set X is a family of connected subsets of some tree on X if and only if the smallest coherent convexity that contains \mathcal{F} has Helly property (H_2) (4.3). Consequently : finite tree convexities are characterized as the convex geometries with Helly number ≤ 2 in which any two different points are contained in disjoint convexes whose union is the whole space ("hemispaces"; compare with (9.2)). In this characterization the assumption of being a convex geometry can be replaced by the property of being coherent. Compare to Skolander [1952].

Tree convexities satisfy the partitions inequality since they have property CIP(3.2) (see (4.4)).

A metric characterization of trees was provided by Colonius and Schutze [1981] where related references may be found. Further developments may be found in Barthelemy [1983], Duchet [1986] and Duchet, Quilliot [1986].

(9.2) TOTAL ORDER CONVEXITIES (See (8.6)) are tree convexities of a special kind. Using structural characterizations of interval hypergraphs (Tucker [1972], Trotter, Moore [1976], Duchet [1978], Nebesky [1984]) one can obtain : a family \mathcal{F} of subsets of a finite

(*) As defined in (5.10).

(**) A convexity space is coherent when the union of two intersecting convexes is again convex.

set X is a family of intervals of a total order on X if and only if the smallest coherent convexity containing \mathcal{F} has Radon number ≤ 3 (see (4.4)). Consequently, a finite convexity space is a total order convexity if and only if it is a convex geometry with Radon number ≤ 3 and in which any two different points are contained in disjoint convexes whose union is the whole space. Again, the assumption "convex geometry" may be replaced by "coherent convexity". Compare with (8.6).

(9.3) ARBORESCENT CONVEXITIES. Let Arb denote the smallest variety (3.10) that contains tree convexities. Members of Arb ("arborescent convexity spaces") may be considered as a pertinent abstraction of what a tree-like structure should be. Downset convexities (8.4) and M -convexities of triangulated graphs (6.4) are examples of arborescent convexities which are not tree convexities.

PROBLEM 18 (Duchet) : Develop the theory of arborescent convexities (combinatorial invariants, partition problem, separation properties ...)

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