CONTINUITY OF THE IDENTITY EMBEDDING OF MUSIELAK-ORLICZ SEQUENCE SPACES

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Abstract. If the Musielak-Orlicz sequence spaces l^Φ, l^Ψ consist of real sequences or the functions Φ, Ψ are convex, then the inclusion $l^\Phi \subset l^\Psi$ implies the continuity of the identity embedding $i: l^\Phi \to l^\Psi$, $i(f)=f$, with respect to the usual norm topologies in these spaces [1], [5]. It is shown that this fact does not hold in general. In the main theorem a necessary and sufficient condition for the continuity of the embedding i is presented. Other notions of continuity with respect to the norm and modular convergences are also studied.

1. Introduction. Throughout this paper X will denote a real linear space.

1.1. DEFINITION. A function $\Phi = (\Phi_n)$, $\Phi_n : X \to [0, +\infty]$ is said to be a Φ-function if

a) $\Phi_n(0) = 0$, $\Phi_n(-x) = \Phi_n(x)$ for every $x \in X$, $n \in \mathbb{N}$,

b) $\lim\limits_{u \to 0} \Phi_n(ux) = 0$ for every $x \in \{y \in X: \Phi_n(y) < +\infty\}$, $n \in \mathbb{N}$,

c) $\Phi_n(ux+vy) \leq \Phi_n(x) + \Phi_n(y)$ for every $u, v \geq 0$, $u+v=1$, $x, y \in X$ and $n \in \mathbb{N}$.

If the functions Φ_n are convex on X for each $n \in \mathbb{N}$ then we shall shortly write: Φ is convex on X.

Let $\mathfrak{X}$ be the space of all sequences of elements of the space X. The functional $I_\Phi : \mathfrak{X} \to [0, +\infty]$ defined by

$$I_\Phi(f) = \sum_{n=1}^{+\infty} \Phi_n(f_n) \qquad \text{for} \quad f = (f_n) \in \mathfrak{X},$$

is a pseudomodular on $\mathfrak{X}$ [3].

1.2. DEFINITION. By the Musielak-Orlicz sequence space l^Φ we

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mean the space of all sequences $f \in \mathcal{X}$ such that $I_\Phi(af) < +\infty$ for some $a > 0$.

1.3. DEFINITION. A sequence $(f(m))$ of elements of $\mathcal{X}$ is said to, be modular [resp. norm] convergent to a sequence $f \in \mathcal{X}$ with respect to a Φ-function Φ (shortly: I_Φ-convergent [resp. N_Φ-convergent] to f) whenever $\lim_{m\to+\infty} I_\Phi(a(f(m)-f)) = 0$ for some $a>0$ [resp. for all a 0].

1.4. We say that N_Φ-convergence implies N_Ψ-convergence (shortly: N_Φ-conv. $\Rightarrow$ N_Ψ-conv.) whenever each sequence $(f(m))$ which is N_Φ-convergent to 0 is N_Ψ-convergent to 0 at the same time. In an analogous way we define the notions N_Φ-conv. $\Rightarrow$ I_Ψ-conv. , I_Φ-conv. $\Rightarrow$ I_Ψ-conv. , and I_Φ-conv. $\Rightarrow$ N_Ψ-conv. .

1.5. REMARK. The functional

$$|f|_\Phi = \inf\left\{u>0:\ I_\Phi\left(\frac{f}{u}\right)\leq u\right\}$$

is an F-pseudonorm on l^Φ. A sequence $f(m)$ is N_Φ-convergent to 0 if and only if $|f(m)|_\Phi \to 0$ as $m \to +\infty$.

Throughout this paper we shall use the following notations:

(1) $$P_n(a,c,K) = \{x\in X:\ \Phi_n(x)\leq a \text{ and } \Psi_n(cx) > K\,\Phi_n(x)\},$$

(2) $$\alpha_n(a,c,K) = \sup\{\Psi_n(cx):\ x\in P_n(a,c,K)\}$$

(with $\sup \emptyset = 0$) for every $a,c,K > 0$ and $n \in \mathbb{N}$.

2. <u>The inclusion $l^\Phi \subset l^\Psi$.</u>

The "inclusion" theorems play an important role in classical function spaces, in particular in Orlicz and Musielak-Orlicz sequence spaces (cf. [2]). For instance, it is well known ([1], [4]) that the inclusion $l^\Phi \subset l^\Psi$ holds if and only if

(Cn_o) there are numbers $a,c,K > 0$, $n_o \in \mathbb{N}$ and a sequence (α_n) of nonnegative numbers $\sum_{n=n_o}^{+\infty} \alpha_n < +\infty$, such that if $\Phi_n(x)\leq a$ then $\Psi_n(cx) \leq K\cdot\Phi_n(x) + \alpha_n$ for all $x\in X$ and $n\geq n_o$.

The above condition may be written concisely:

(Cn_o) there are numbers $a,c,K > 0$, $n_o\in\mathbb{N}$ such that

$$\sum_{n=n_o}^{+\infty} \alpha_n(a,c,K) < +\infty .$$

In this connection a very simple question arises: can the number n_o be replaced by 1 or not? In general the answer is no (cf. Examples 2.6, 2.7 below). However the conditions (Cn_o) and (C1) (i.e. (Cn_o) with $n_o=1$) are equivalent in most cases (Corollary 2.2) .

We start with the following:

2.1. PROPOSITION. If

$$(3)\qquad \forall_{n\in\mathbb{N}}\ \forall_{(x(m))}\left[\Phi_n(x(m))\xrightarrow[m\to+\infty]{}0 \Rightarrow \limsup_{m\to+\infty}\Psi_n\left(\frac{1}{m}x(m)\right)<+\infty\right]$$

then conditions (Cn_0) and $(C1)$ are equivalent.

Proof. Let conditions (3) and (Cn_0) (with numbers $a_{n_0}, c_{n_0}, K_{n_0}$) be satisfied. Suppose there is a number $1\le n<n_0$ such that

$$\forall_{m\in\mathbb{N}}\quad \alpha_n\left(\frac{1}{2^m},\frac{1}{m},2^m\right)=+\infty\ .$$

In virtue of (1) and (2) we can find a sequence $(x(m))$ of elements of X such that

$$\Phi_n(x(m))\le\frac{1}{2^m}\qquad\text{and}\qquad \Psi_n\left(\frac{1}{m}x(m)\right)\ge m\qquad\text{for all } m\in\mathbb{N}.$$

Therefore $\Phi_n(x(m))\to 0$ as $m\to+\infty$ and $\limsup_{m\to+\infty}\Psi_n\left(\frac{1}{m}x(m)\right)=+\infty$ – in contradiction to the assumption. Thus for each $1\le n<n_0$ we can find numbers $a_n,c_n,K_n>0$ such that $\alpha_n(a_n,c_n,K_n)<+\infty$.

Denote $a=\min_i a_i$, $c=\min_i c_i$, $K=\max_i K_i$, where $i\in\{1,2,\dots,n_0\}$. Since $P_n(a,c,K)\subset P_n(a_i,c_i,K_i)$ for all $n\in\mathbb{N}$ and $i=1,2,\dots,n_0$, $\alpha_n(a,c,K)\le\alpha_n(a_n,c_n,K_n)$ for each $n=1,2,\dots,n_0-1$ and $\alpha_n(a,c,K)\le\alpha_n(a_{n_0},c_{n_0},K_{n_0})$ for all $n\ge n_0$. Therefore the thesis is obvious.

2.2. COROLLARY. Suppose one of the following conditions is satisfied:

(4) $X=\mathbb{R}$ and the functions Φ_n are not identically equal to 0 for each $n\in\mathbb{N}$,

(5) $X=\mathbb{R}^k$ and $\liminf_{\|x\|\to+\infty}\Phi_n(x)>0$ for each $n\in\mathbb{N}$,

(6) $X=(X,\|\cdot\|)$ is a linear space with a p-homogeneous norm $\|\cdot\|$ $(p>0)$, $\liminf_{\|x\|\to+\infty}\Phi_n(x)>0$ and $\limsup_{\|x\|\to 0}\Psi_n(x)<+\infty$ for each $n\in\mathbb{N}$,

(7) at least one of the implications N_Φ-conv. $\Rightarrow$ N_Ψ-conv., N_Φ-conv. $\Rightarrow$ I_Ψ-conv., I_Φ-conv. $\Rightarrow$ N_Ψ-conv., I_Φ-conv. $\Rightarrow$ I_Ψ-conv. holds.

Then $l^\Phi\subset l^\Psi$ if and only if condition (C1) is fulfilled.

Proof. Let $n\in\mathbb{N}$ be fixed. To prove the corollary it suffices to show that condition (3) is satisfied.

The implication (4)⇒(5) is obvious. To prove (5)⇒(6) let us note that the following lemma holds:

2.3. LEMMA. If $X=\mathbb{R}^k$ then every Φ-function Ψ is continuous at 0, i.e. $\lim_{\|x\|\to 0}\Psi_n(x)=0$ for all $n=1,2,\dots$.

Proof. Let $\varepsilon > 0$ be fixed. Denote $e_i = (0,\ldots,0,\underbrace{1}_{i\text{-th place}},0,\ldots,0)$ for $i=1,2,\ldots,k$. In virtue of Definition 1.1.b. there is a number $u>0$ such that $\Psi_n(ue_i)<\frac{\varepsilon}{k}$ for $i=1,2,\ldots,k$.

Let $x = \sum_{i=1}^{k} x_i e_i$ be an element of $\mathbb{R}^k$ with $\|x\|^2 = \sum_{i=1}^{k}|x_i|^2 \leq \leq \frac{1}{4k^2}$. Hence, $|x_i|^2 \leq \frac{1}{4k^2}$ for $i=1,2,\ldots,k$, so $\sum_{i=1}^{k}|x_i| \leq \frac{1}{2k}k = \frac{1}{2}$. Thus

$$\Psi_n(ux) = \Psi_n\Big[u\sum_{i=1}^{k}|x_i|\cdot \operatorname{sgn} x_i\cdot e_i + \Big(1-\sum_{i=1}^{k}|x_i|\Big)\cdot 0\Big] \leq$$

$$\leq \sum_{i=1}^{k}\Psi_n(ue_i) < \varepsilon .$$

Therefore $\sup\{\Psi_n(x):\ \|x\|<\frac{u}{2k}\}\leq\varepsilon$, so $\lim_{\|x\|\to 0}\Psi_n(x) = 0$.

Proof of Corollary 2.2 (continued). (6)⇒(3). Let $(x(m))$ be a sequence such that $\lim_{m\to+\infty}\Phi_n(x(m)) = 0$. Then it is bounded, i.e. $\|x(m)\|<K$ for some $K>0$ and all $m\in\mathbb{N}$. By assumption, $\sup_{\|x\|<u}\Psi_n(x)<+\infty$ for some $u>0$. Let $m_0\in\mathbb{N}$ be a number such that $\left\|\frac{1}{m}\cdot x(m)\right\| \leq \frac{1}{m^p}\cdot K \leq u$ for all $m\geq m_0$. Then

$$\limsup_{m\to+\infty}\Psi_n\Big(\frac{1}{m}\cdot x(m)\Big) = \inf_{k\in\mathbb{N}}\sup_{m\geq k}\Psi_n\Big(\frac{1}{m}\,x(m)\Big) \leq$$

$$\leq \sup_{m\geq m_0}\Psi_n\Big(\frac{1}{m}\cdot x(m)\Big) \leq \sup_{\|x\|<u}\Psi_n(x) < +\infty ,$$

so (3) is satisfied.

(7)⇒(3). We have

$$\begin{array}{ccc}
 & N_\Phi\text{-conv.} \Rightarrow N_\Psi\text{-conv.} & \\
 \nearrow & & \searrow \\
I_\Phi\text{-conv.} \Rightarrow N_\Psi\text{-conv.} & & N_\Phi\text{-conv.} \Rightarrow I_\Psi\text{-conv.} \\
 \searrow & & \nearrow \\
 & I_\Phi\text{-conv.} \Rightarrow I_\Psi\text{-conv.} &
\end{array}$$

Therefore we may assume that N_Φ-conv. $\Rightarrow I_\Psi$-conv. . Suppose there is a sequence $(x(m))$ such that $I_\Phi(x(m))\to 0$ as $m\to+\infty$ and $\limsup_{m\to+\infty}\Psi_n\big(\frac{1}{m}\cdot x(m)\big) = +\infty$. For brevity, but without loss of generality, we shall assume that $\lim_{m\to+\infty}\Psi_n\big(\frac{1}{m}\cdot x(m)\big) = +\infty$. Let $f(m) = (f_i(m))_{i\in\mathbb{N}}$ be defined as follows

$$f_i(m) = \begin{cases} \frac{1}{m}\cdot x(m) & \text{for } i=n, \\ 0 & \text{otherwise.}\end{cases}$$

Let $a>0$ and let $m_o \in \mathbb{N}$ be such a number that $\sqrt{m} \geq a$ for all $m \geq m_o$. Then, for $m \geq m_o$,

$$I_{\Phi}(a \cdot f(m)) = \Phi_n(a \cdot f_n(m)) \leq \Phi_n\left(\sqrt{m}\frac{1}{\sqrt{m}}x(m)\right) \xrightarrow[m \to +\infty]{} 0 .$$

On the other hand, $a \geq \frac{1}{\sqrt{m}}$ for all $m \geq m_1$. Hence

$$I_{\Psi}(a \cdot f(m)) = \Psi_n\left(a \cdot \frac{1}{\sqrt{m}} x(m)\right) \geq \Psi_n\left(\frac{1}{m} \cdot x(m)\right) \xrightarrow[m \to +\infty]{} +\infty ,$$

i.e. $(f(m))$ is N_{Φ}-convergent to 0 and is not I_{Ψ}-convergent to 0 at the same time. This contradiction ends the proof of Corollary 2.2.

2.4. REMARK. Corollary 2.2 (with assumption (4)) implies results of Ph. Turpin [5, Theorem 3a] and J.Y.T. Woo [7, Proposition 2.1].

2.5. REMARK. The following result follows from the proof of Theorem 2.6 in [1]:

If X is a Banach space, Φ_n are lower-semicontinuous on X and $\liminf_{\|x\| \to +\infty} \Phi_n(x) > 0$ for each $n \in \mathbb{N}$, then $l^{\Phi} \subset l^{\Psi}$ if and only if condition (C1) holds.

Thus, the third assumption in (6) may be omitted in this case.

2.6. EXAMPLE. The implication $(Cn_o) \Rightarrow (C1)$ does not hold in general. Moreover, if the dimension of the normed space $(X, \|\cdot\|)$ is infinite then Lemma 2.3 is false.

Let $X = l^o$ be the space of all real sequences $x = (x_k)$ such that $x_k = 0$ for all sufficiently large $k \in \mathbb{N}$ with the norm $\|x\| = \max_{k \in \mathbb{N}} |x_k|$. Let us denote

$$\Phi_1(x) = \Phi_n(x) = \Psi_n(x) = \|x\| \qquad \text{for } n \geq 2,$$

$$\Psi_1(x) = \sum_{k=1}^{+\infty} k|x_k| .$$

Then $\liminf_{\|x\| \to +\infty} \Phi_n(x) = +\infty$ for all $n \in \mathbb{N}$ and $\sup_{\|x\| < u} \Psi_1(x) = +\infty$ for all $u>0$. Therefore neither condition (6) hold nor Ψ_1 is continuous at 0. It is easy to verify that $l^{\Phi} = l^{\Psi}$.

On the other hand, we have

$$P_1(a,c,K) = \left\{x \in l^o : \|x\| \leq a \text{ and } K \cdot \|x\| < c \sum_{k=1}^{+\infty} k|x_k|\right\} .$$

Taking $y(m) = (\underbrace{0, \ldots, 0, a}_{m\text{-th place}}, 0, \ldots)$ we obtain $\|y(m)\| = a$ and

$$\Psi_1(c \cdot y(m)) = \sum_{k=1}^{+\infty} ck|y_k(m)| = c \cdot m \cdot a > K \cdot a$$

for all $m \geq m_o \geq \frac{K}{c}$. Thus, $y(m) \in P_1(a,c,K)$ for $m \geq m_o$ and

$$\alpha_1(a,c,K) \geq \sup_{m \geq m_o} \Psi_1(c \cdot y(m)) = \sup_{m \geq m_o} c \cdot a \cdot m = +\infty .$$

The arbitrariness of numbers a,c,K implies that condition (C1) is not satisfied.

2.7. EXAMPLE. The assumption $\liminf_{\|x\|\to+\infty} \Phi_n(x) > 0$ cannot be omitted. Let $X = \mathbb{R}^2$, $\Phi_1(x,y) = |y|$, $\Phi_n(x,y) = \Psi_n(x,y) = \Psi_1(x,y) = \|(x,y)\| = \sqrt{x^2+y^2}$ for $n=2,3,\ldots$. Then $l^\Phi = l^\Psi$, $\liminf_{\|(x,y)\|\to+\infty} \Phi_1(x) = 0$ and $\limsup_{\|(x,y)\|\to 0} \Psi_1(x,y) = 0$. However, condition (C1) does not hold because taking the sequence $x_m = m$, $y_m = 0$ for $m=1,2,\ldots$ we obtain $(x_m,y_m) \in P_1(a,c,K)$ and

$$\alpha_1(a,c,K) \geqslant \sup_{m\in\mathbb{N}} c\sqrt{x_m^2+y_m^2} = \sup_{m\in\mathbb{N}} c\cdot m = +\infty$$

for all $a,c,K > 0$.

3. The identity embedding $i: l^\Phi \to l^\Psi$, $i(f)=f$.

The inclusion of Orlicz and Musielak-Orlicz sequence spaces may be considered both as inclusion of sets and as topological inclusion of F-normed spaces. Ph. Turpin [5] has pointed out that these notions coincide in the case $X = \mathbb{R}$. A similar result has been obtained by A. Kamińska [1], provided (among others) Φ, Ψ are convex Φ-functions and X is a Banach space. No any of these assumptions can be omitted, cf. Examples 2.6, 3.4. Therefore, it is worth to study the continuity of the identity embedding $i(f)=f$ in general.

In the following we shall consider four comprehensions of continuity with respect to the notions introduced in 1.5. We start with two lemmas which will be often used in the sequel.

3.1. LEMMA. Suppose there are sequences (a_m), (c_m), (K_m) of positive numbers and a number $0 < b < +\infty$ such that

$$\sum_{n=1}^{+\infty} \alpha_n(a_m,c_m,K_m) \geqslant b \qquad \text{for all } n\in\mathbb{N}.$$

Then there exist sequences $(g(m))$ of elements of $\mathfrak{X}$ and $(j_n(m))$ of numbers such that, for all $n,m \in \mathbb{N}$,

(8) $\quad 0 \leqslant j_n(m) < +\infty$,

(9) $\quad \Phi_n(g_n(m)) \leqslant a_m$,

(10) $\quad \Psi_n(c_m g_n(m)) \geqslant j_n(m) \geqslant K_m \cdot \Phi_n(g_n(m))$,

(11) $\quad \sum_{n=1}^{+\infty} j_n(m) \geqslant \frac{1}{2}\cdot b$.

Proof. Let $m\in\mathbb{N}$ be fixed. Denote $A_m = \{n\in\mathbb{N}: \alpha_n(a_m,c_m,K_m) > 0\}$ (it is nonempty), $A'_m = \{n\in A_m: \alpha_n(a_m,c_m,K_m) < +\infty\}$, $A''_m = A_m \setminus A'_m$.

Then for each $n \in A_m$ we can find an element $g_n(m) \in X$ such that

(12) $$g_n(m) \in P_n(a_m, c_m, K_m) \quad ,$$

and

$$\Psi_n(c_m g_n(m)) > \begin{cases} \alpha_n(a_m, c_m, K_m) - \frac{b}{2^{n+1}} & \text{for } n \in A'_m , \\ \max\{n, \frac{b}{2}\} & \text{for } n \in A''_m . \end{cases}$$

Let us define $g_n(m) = 0$ for $n \notin A_m$. Then

$$\sum_{n=1}^{+\infty} \Psi_n(c_m g_n(m)) \geq \frac{b}{2} \quad .$$

Further, denote $B_m = \{n \in \mathbb{N}: \Psi_n(c_m g_n(m)) = +\infty\}$ $(B_m \subset A''_m)$, and

(13) $$j_n(m) = \begin{cases} \Psi_n(c_m g_n(m)) & \text{for } n \notin B_m, \\ \max\{K_m \cdot \Phi_n(g_n(m)), \frac{b}{2}\} & \text{otherwise.} \end{cases}$$

Then (8) holds. Moreover, if $B_m \neq \emptyset$ then

$$\sum_{n=1}^{+\infty} j_n(m) = \sum_{n=1}^{+\infty} \Psi_n(c_m g_n(m)) \geq \frac{b}{2} \quad .$$

Since the above inequality is obvious for $B_m \neq \emptyset$, (11) is proved. Furthermore, (9) and (10) follow immediately from (12) and (13), so the proof is finished.

3.2. LEMMA. By the assumptions of Lemma 3.1 there exists a sequence $(f(m))$ of elements of $\mathfrak{X}$ such that $f_n(m) \in P_n(a_m, c_m, K_m)$ and

(14) $$I_\Phi(f(m)) \leq \frac{1}{2K_m} b + a_m \, ,$$

(15) $$I_\Psi(c_m f(m)) \geq \frac{1}{2} b$$

for all $n, m \in \mathbb{N}$.

Proof. Let $m \in \mathbb{N}$ be fixed. In virtue of (8) and (11) we can find a set $J_m = \{1, 2, \ldots, n_m\}$ such that

$$\sum_{n \in J_m} j_n(m) \geq \frac{b}{2} \quad \text{and} \quad \sum_{n \in J_m \setminus \{n_m\}} j_n(m) < \frac{b}{2}$$

(we assume $\sum_{n \in \emptyset} = 0$). Let us denote $f(m) = (f_n(m))$ by

$$f_n(m) = \begin{cases} g_n(m) & \text{for } n \in J_m , \\ 0 & \text{otherwise.} \end{cases}$$

By (9) and (10) we infer that

$$I_\Phi(f(m)) = \sum_{n=1}^{+\infty} \Phi_n(f_n(m)) = \sum_{n = J_m} \Phi_n(g_n(m)) \leq$$

$$\sum_{n \in J_m \setminus \{n_m\}} \frac{1}{K_m} \cdot j_n(m) + \Phi_{n_m}(g_{n_m}(m)) < \frac{b}{2K_m} + a_m \quad .$$

Furthermore, by (10) we obtain

$$I_{\Psi}(c_m f(m)) = \sum_{n=1}^{+\infty} \Psi_n(c_m f_n(m)) \geqslant \sum_{n\in J_m} j_n(m) \geqslant \frac{b}{2} .$$

3.3. THEOREM. The following conditions:

(i) $l^{\Phi} \subset l^{\Psi}$ and N_{Φ}-conv. $\Rightarrow$ N_{Ψ}-conv. ,

(ii) $l^{\Phi} \subset l^{\Psi}$ and N_{Φ}-conv. $\Rightarrow$ I_{Ψ}-conv. ,

(NN) $\displaystyle \forall_{\varepsilon>0} \ \exists_{a>0} \ \exists_{c>0} \ \exists_{K>0} \ \sum_{n=1}^{+\infty} \alpha_n(a,c,K) < \varepsilon$,

are pairwise equivalent.

Proof. The implication (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(NN). Suppose

$$\exists_{\varepsilon>0} \ \forall_{m\in\mathbb{N}} \ \sum_{n=1}^{+\infty} \alpha_n\left(\frac{1}{m},\frac{1}{m},m\right) \geqslant \varepsilon .$$

In virtue of Lemma 3.2, there is a sequence $(f(m))$ such that

$$I_{\Phi}(f(m)) \leqslant \frac{\varepsilon}{2m} + \frac{1}{m} \xrightarrow[m\to+\infty]{} 0$$

and

$$I_{\Psi}\left(\frac{1}{m}\cdot f(m)\right) \geqslant \frac{\varepsilon}{2} .$$

Define $g(m) = \frac{1}{\sqrt{m}} f(m)$. In an analogous fashion as in the proof of the implication (7)$\Rightarrow$(3) in Remark 2.2, we deduce that $(g(m))$ is N_{Φ}-convergent to 0 and is not I_{Ψ}-convergent to 0 at the same time, in contradiction to (ii).

(NN)$\Rightarrow$(i). It suffices to show that N_{Φ}-conv. $\Rightarrow$ N_{Ψ}-conv. . Let $(f(m))$ be an arbitrary sequence N_{Φ}-convergent to 0 in the space l^{Φ} and, moreover, let $u>0$, $\varepsilon>0$ be fixed. Then, by (NN), we can find numbers $a,c,K>0$ (depending on ε) such that

$$\sum_{n=1}^{+\infty} \alpha_n(a,c,K) < \frac{\varepsilon}{2} .$$

Denote $v=\frac{u}{c}$. Since $I_{\Phi}(vf(m)) \to 0$ as $m\to+\infty$, we infer that

$$\exists_{m(\varepsilon,u)} \ \forall_{m\geqslant m(\varepsilon,u)} \ I_{\Phi}(vf(m)) < \min\left\{\frac{\varepsilon}{2K}, a\right\} .$$

Hence $\Phi_n(vf(m)) \leqslant a$ for all $n\in\mathbb{N}$ and $m\geqslant m(\varepsilon,u)$. (1) and (2) imply

$$\Psi_n(cx) \leqslant K\cdot\Phi_n(x) + \alpha_n(a,c,K)$$

for all $n\in\mathbb{N}$ and $x\in\{y\in X: \Phi_n(y)\leqslant a\}$. Therefore, for $m\geqslant m(\varepsilon,u)$,

$$I_{\Psi}(uf(m)) = I_{\Psi}(v\cdot c\cdot f(m)) \leqslant K\cdot I_{\Phi}(vf(m)) + \sum_{n=1}^{+\infty} \alpha_n(a,c,K) < \varepsilon ,$$

so $(f(m))$ is N_{Ψ}-convergent to 0.

3.4. EXAMPLE. N_{Φ}-convergence may not imply N_{Ψ}-convergence even then $l^{\Phi} \subset l^{\Psi}$. Let $X = \mathbb{R}^2$, $\Phi_n(x,y) = \frac{|x|}{2^n}$, $\Psi_n(x,y) = \dfrac{|y|}{2^n(2+|y|)}$ for $n=1,2,\ldots$. Since

$$\sum_{n=1}^{+\infty} \alpha_n(a,c,K) \le \sum_{n=1}^{+\infty} \sup_{(x,y)\in\mathbb{R}^2} \Psi_n(cx,cy) \le \sum_{n=1}^{+\infty} \frac{1}{2^n} = 1 < +\infty$$

for all $a,c,K>0$, $l^\Phi \subset l^\Psi$ by Corollary 2.2.

On the other hand $(\frac{u}{2},\frac{1}{u}) \in P_n(u,u,\frac{1}{u})$, so

$$\alpha_n(u,u,\tfrac{1}{u}) = \sup_{(x,y)\in P_n(u,u,\frac{1}{u})} \Psi_n(u(x,y)) \ge \Psi_n(u(\tfrac{u}{2},\tfrac{1}{u})) = \Psi_n(\tfrac{u^2}{2},1) = \frac{1}{32^n}$$

for every $u>0$. Let a,c,K be arbitrary positive numbers. Then we can choose $u>0$ such that $0<u\le\min\{a,c,\frac{1}{K}\}$. Therefore

$$\sum_{n=1}^{+\infty} \alpha_n(a,c,K) \ge \sum_{n=1}^{+\infty} \alpha_n(u,u,\tfrac{1}{u}) \ge \frac{1}{3} .$$

In virtue of Theorem 3.3 N_Φ-convergence does not imply N_Ψ-convergence.

3.5. THEOREM. $l^\Phi \subset l^\Psi$ and I_Φ-conv. $\Rightarrow$ N_Ψ-conv. if and only if

$$\text{(IN)} \qquad \forall_{\varepsilon>0}\ \forall_{c\ 0}\ \exists_{a\ 0}\ \exists_{K\ 0}\ \sum_{n=1}^{+\infty} \alpha_n(a,c,K) < \varepsilon .$$

Proof. ($\Leftarrow$) Assume $I_\Phi(f(m))\to 0$ as $m\to+\infty$. Let c,ε be arbitrary positive numbers. Then

$$\sum_{n=1}^{+\infty} \alpha_n(a,c,K) < \frac{\varepsilon}{2}$$

for some $a,K>0$ (depending on ε and c). Moreover

$$I_\Phi(f(m)) \le \min\{a, \tfrac{\varepsilon}{2K}\}$$

for all $m \ge m(c,\varepsilon)$. Thus, by (1) and (2),

$$I_\Psi(cf(m)) \le K\cdot I_\Phi(f(m)) + \sum_{n=1}^{+\infty} \alpha_n(a,c,K) < \varepsilon$$

for all $m\ge m(c,\varepsilon)$. Therefore $(f(m))$ is N_Ψ-convergent to 0.

($\Rightarrow$) Suppose that

$$\exists_{\varepsilon>0}\ \exists_{c>0}\ \forall_{m\in\mathbb{N}}\ \sum_{n=1}^{+\infty} \alpha_n(\tfrac{1}{m},c,m) \ge \varepsilon .$$

In virtue of Lemma 3.2, there is a sequence $(f(m))$ such that

$$I_\Phi(f(m)) \le \frac{\varepsilon}{2m} + \frac{1}{m} \quad\text{and}\quad I_\Psi(cf(m)) \ge \frac{\varepsilon}{2}$$

for all $m\in\mathbb{N}$. Thus $(f(m))$ is I_Φ-convergent to 0 and is not N_Ψ-convergent to 0 - a contradiction.

3.6. REMARK. If $\Phi=\Psi$ then condition (IN) is equivalent to

$$(\delta_2^0) \qquad \forall_{\varepsilon>0}\ \exists_{a>0}\ \exists_{K>0}\ \sum_{n=1}^{+\infty} \alpha_n(a,2,K) < \varepsilon ,$$

(in other words: for every $\varepsilon>0$ there are $a,K>0$ and a sequence (α_n) of nonnegative numbers such that $\sum_{n=1}^{+\infty} \alpha_n < \varepsilon$ and

$$\Phi_n(x) \le a \quad\Rightarrow\quad \Psi_n(cx) \le 2\,\Phi_n(x) + \alpha_n$$

for all $x \in X$ and $n \in \mathbb{N}$.

3.7. THEOREM. $l^{\Phi} \subset l^{\Psi}$ and I_{Φ}-conv. $\subset I_{\Psi}$-conv. if and only if

(II) $$\exists_{c>0} \ \forall_{\varepsilon>0} \ \exists_{a>0} \ \exists_{K>0} \ \sum_{n=1}^{+\infty} \mathcal{L}_n(a,c,K) < \varepsilon \ .$$

Proof. ($\Leftarrow$) Assume $I_{\Phi}(f(m)) \to 0$ as $m \to +\infty$. Let $\varepsilon > 0$ be fixed. Then, by (II),

$$\sum_{n=1}^{+\infty} \mathcal{L}_n(a,c,K) < \frac{\varepsilon}{2}$$

for some $a, K > 0$ (depending on ε) and an absolute constant $c > 0$. Moreover, $I_{\Phi}(f(m)) \leqslant \min\{a, \frac{\varepsilon}{2K}\}$ for all $m \geqslant m(\varepsilon)$. Thus

$$I_{\Psi}(cf(m)) \leqslant K \cdot I_{\Phi}(f(m)) + \sum_{n=1}^{+\infty} \mathcal{L}_n(a,c,K) < \varepsilon$$

for $m \geqslant m(\varepsilon)$, so $(f(m))$ is I_{Ψ}-convergent to 0.

($\Rightarrow$) Suppose (II) does not hold. In particular,

$$\forall_{r \in \mathbb{N}} \ \exists_{1 > \varepsilon_r > 0} \ \forall_{m \in \mathbb{N}} \ \sum_{n=1}^{+\infty} \mathcal{L}_n\left(\frac{1}{m+r}, \frac{1}{r}, m+r\right) > \varepsilon_r \ .$$

In virtue of Lemma 3.2 there are sequences $g(m,r)$ such that

$$I_{\Phi}(g(m,r)) \leqslant \frac{\varepsilon_r}{2(m+r)} + \frac{1}{m+r} \quad \text{and} \quad I_{\Psi}\left(\frac{1}{r} \cdot g(m,r)\right) > \frac{1}{2}\varepsilon_r$$

for all $m, r \in \mathbb{N}$. Now, we shall construct one sequence $(f(k))$ from the sequences $(g(m,r))$. Denote $s_p = 1+2+\ldots+p$, $p \in \mathbb{N}$; $f(1) = g(1,1)$,

$$f(k) = g(p_k+2-l_k, \ l_k) \qquad \text{for } k=2,3,\ldots \ ,$$

where numbers $p_k, l_k \in \mathbb{N}$ are defined by

$$s_{p_k} < k \leqslant s_{p_{k+1}} \ , \quad l_k = k - s_{p_k} \quad \text{for } k=2,3,\ldots \ .$$

Then

$$I_{\Phi}(f(k)) = I_{\Phi}(g(p_k+2-l_k, l_k)) \leqslant \frac{1}{2(p_k+2)} + \frac{1}{p_k+2} \xrightarrow[k \to +\infty]{} 0 \ .$$

On the other hand, the sequence $f(k)$ is not I_{Ψ}-convergent to 0. Indeed, let $u>0$. Then $u > \frac{1}{r}$ for some $r \in \mathbb{N}$. Thus

$$I_{\Psi}(ug(m,r)) \geqslant I_{\Psi}\left(\frac{1}{r} \cdot g(m,r)\right) \geqslant \frac{1}{2}\varepsilon_r > 0 \ .$$

Since $(g(m,r))_{m \in \mathbb{N}}$ is a subsequence of $(f(k))$, $(f(k))$ is not I_{Ψ}-convergent to 0 - a contradiction.

3.8. REMARK. Analogous theorems concerning the continuity of the identity embedding $i(f) = f$ of Musielak-Orlicz spaces in the non-atomic measure case were presented in [6].

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