Acta Universitatis Carolinae. Mathematica et Physica

Francis Borceux; Maria Cristina Pedicchio

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*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 28 (1987), No. 2, 9--12

Persistent URL: `http://dml.cz/dmlcz/701918`

# De Morgan's and Strong De Morgan's Laws in a Topos of Sheaves

FRANCIS BORCEUX,*)

Louvaine-la-Neuve, Belgium

M. CRISTINA PEDICCHIO,**)

Trieste, Italy

*Received 25 March, 1987*

We investigate some consequences of adding De Morgan's law $\neg x \vee \neg y = \neg(x \wedge y)$ or strong De Morgan's law $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ to the axioms of a topos of sheaves on a locale $L$. In the case of a topological space $X$, these conditions correspond to some properties of disconnectedness on $X$.

## 0. Introduction

In [3] and [4], the authors have investigated properties of Boolean algebras in a topos *Sh L* of sheaves on a locale *L*. Problems of this kind have also been studied by B. Banaschewski and K. R. Bhutani in [1] and [2].

The main difficulty that arises, working in *Sh L*, is that, classical algebraic theorems don't remain in general true, their validity strictly depends on some suitable properties of the locale *L*. While studying this kind of properties, we have seen that De Morgan's and strong De Morgan's laws very often occur in the hypotheses we need, and that they really force some interesting results, as the Stone theorem for Boolean algebras (see [4]). This note wants to be a short survey on the consequences of adding De Morgan's or strong De Morgan's law to the axioms of a topos of sheaves. The principal aim is to give a list of conditions that we have proved to be equivalent to the above mentioned logical principles; these results and their complete constructive proofs are essentially part of [4] and [5]. Many interesting results on the subject are contained in P. Johnstone [7] and [8]. De Morgan's and strong De Morgan's laws also appear in S. B. Niefield - K. I. Rosenthal [10] and [11], in connection with the spectrum of a commutative ring.

*) Institut de Mathématique Pure et Appliquée, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgique.

**) Dipartimento di Scienze Matematiche, Università degli Studi, 34100 Trieste, Italia.

## 1. On De Morgan's and strong De Morgan's laws

Consider a locale $L$ and the topos $Sh\ L$ of sheaves on $L$. It is well known, that the two De Morgan's laws are the following

(A) $$\neg x \wedge \neg y = \neg(x \vee y) \quad \forall x, y \in L$$

(B) $$\neg x \vee \neg y = \neg(x \wedge y) \quad \forall x, y \in L$$

and that, while the first one holds in any Heyting algebra (hence in any topos), the second one is not in general true. For strong De Morgan's law, we mean the following

(C) $$(x \Rightarrow y) \vee (y \Rightarrow x) = 1 \quad \forall x, y \in L$$

where 1 is the top element of the locale.

Of course (C) $\Rightarrow$ (B).

In the case of a topos of sheaves on a topological space $X$ (then $L \simeq \{$open subsets of $X\}$), P. Johnstone proved the following

**Proposition 1.1** (see [7]).

i) *$L$ satisfies* (B) *iff $X$ is extremally disconnected.*
ii) *$L$ satisfies* (C) *iff any closed subset of $X$ is extremally disconnected.*

Let us recall now some definitions.

**Definition 1.2** (see [4]). An element $x \in L$ is *internal prime* iff $x$ is prime as a global element $x \in \Omega$ in $Sh\ L$ (in the internal logic of the topos).

It is easy to see that $x$ is internal prime iff the following holds in $L$:

$$\forall u, v \in L, \quad u \wedge v \leqq x \quad \text{implies} \quad (u \Rightarrow x) \vee (v \Rightarrow x) = 1\,.$$

Denote by $D$ a distributive lattice in $Sh\ L$, and by $B$ a Boolean algebra in $Sh\ L$. The sheaves $Idl(D)$ and $Filt(D)$ defined by $Idl(D)\ (u) = \{$ideals for $D_{|u}\}$ and $Filt(D)\ (u) = \{$filters for $D_{|u}\}$, $\forall u \in L$, are respectively the internal object of ideals for $D$, and the internal object of filters for $D$ (for more details see [1]).

**Definition 1.3.** A $D$-filter $F$ is *internal prime* iff $F$ is prime as a global element of $Filt(D) \in Sh\ L$ (i.e. iff – internally – $\forall D$-filters $F_1, F_2$, $F_1 \cap F_2 \subseteq F$ implies $F_1 \subseteq F$ or $F_2 \subseteq F$).

**Definition 1.4** (see [4]). A $D$-filter $F$ is *prime* iff (internally) $\forall a, b \in D$, $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Of course, internal prime implies prime.

**Definition 1.5** (see [5]). A $D$-filter is *relatively maximal* iff (internally) $\forall D$-filter $G$, $G \supseteq F$ and $(0 \in G \Rightarrow 0 \in F)$, implies $G = F$.

Note that in Definition 1.3, 1.4 and 1.5, the filters are not supposed to be proper. Similarly all the previous definitions can be given in the case of ideals instead of filters.

**Definition 1.6.** A Boolean algebra $B$ is *prime* (we say *uniform* in [4]) iff the ideal 0 is internal prime (i.e. $\forall B$-ideals $I$, $J$, $I \cap J = 0$ implies $I = 0$ or $J = 0$).

Now, we can give the main results.

**Theorem 1.7.** *The following are equivalent.*

1) *$L$ satisfies De Morgan's law.*
2) *$0 \in L$ is internal prime ($0$ is the bottom element of $L$).*
3) *$2 \in Sh\, L$ is a prime Boolean algebra ($2$ denotes the coproduct of two copies of $1 \in Sh\, L$).*
4) *$2 \in Sh\, L$ is an injective Boolean algebra.*
5) *$\forall$ distributive lattice $D \in Sh\, L$, any proper maximal filter in internal prime.*
6) *$\forall$ Boolean algebra $B \in Sh\, L$, any proper maximal filter is internal prime.*
7) *$\forall D \in Sh\, L$, any proper maximal filter is prime.*
8) *$\forall B \in Sh\, L$, any proper maximal filter is prime.*
9) *$\forall B \in Sh\, L$, any proper prime filter is internal prime.*

**Proof.** For what concernes the proof, points 2) and 3) are contained in [4], 4) in [7], 7) and 8) in [8], and 5), 6) and 9) in [5]. In [7] and [8] a few more conditions that we don't mention here, are given.

**Theorem 1.8.** *The following are equivalent.*

1) *$L$ satisfies strong De Morgan's law.*
2) *$\Omega$ is (internally) totally ordered in $Sh\, L$.*
3) *$\uparrow x \subseteq L$ satisfies De Morgan's law, $\forall x \in L$.*
4) *$x \in L$ is internal prime, $\forall x \in L$.*
5) *$2_x \in Sh\, L$ is a prime Boolean algebra, $\forall x \in L$ ($2_x$ denotes the sheaf associated to the following presheaf $V$,*

$$V(u) = \begin{cases} \{0, 1\} & \text{if } u \nleq x \\ \{0 = 1\} & \text{if } u \leq x \end{cases} \quad (\textit{see } [4]).$$

6) *$\forall D \in Sh\, L$, any relatively maximal filter is internal prime.*
7) *$\forall B \in Sh\, L$, any relatively maximal filter is internal prime.*
8) *$\forall D \in Sh\, L$, any relatively maximal filter is prime.*
9) *$\forall B \in Sh\, L$, any relatively maximal filter is prime.*
10) *$\forall B \in Sh\, L$, any prime filter is internal prime.*

**Proof.** 2) is trivial. 3), 4) and 5) are in [4]; 6), 7), 8), 9) and 10) are in [5].

Note that in [5], as in [7] and [8], proofs are given in the case of an arbitrary topos.

**Remark 1.9.** Theorem 1.7 involves only proper $D$-filters, i.e. filters $F$ such that (internally) $0 \notin F$; so $\forall u \in L$, $u \neq 0$, $F(u) \neq D(u)$. Of course in this case we must consider only on trivial distributive lattices $D$.

In Theorem 1.8 any filter is considered.

**Remark 1.10.** An interesting relation between some conditions of Theorem 1.7 and of Theorem 1.8, is obtained by considering the following morphism of locales

$$L \xrightarrow{-\vee x} \uparrow x,$$

and its corresponding geometric morphism.

$$Sh\, L \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} Sh \uparrow x .$$

Since $- \vee x$ is an inclusion of locales, $Sh \uparrow x$ is a subtopos of $Sh\, L$ (it is also more); a sheaf $F \in Sh \uparrow x$ becomes an element of $Sh\, L$ by putting $F(u) = F(u \vee x)$, $u \in L$, and a sheaf $G \in Sh\, L$ is in $Sh \uparrow x$ iff $G(u) = \{*\}$, $\forall u \leqq x$.

So, since $Sh\, L$ satisfies strong De Morgan's law iff $Sh \uparrow x$ satisfies De Morgan's law for any $x$, we get a connection between the two theorems. For example, the sheaf $2_x \in Sh\, L$ can be seen as $\phi_*(2)$ where now $2 \in Sh \uparrow x$; in this way properties of $2_x$ (Theorem 1.8, 5)) can be asily related to properties of the initial Boolean algebra 2 (Theorem 1.7, 3)).

**Remark 1.11.** Note that 9) of Theorem 1.7 and 10) of Theorem 1.8 hold only in the case of filters for Boolean algebras. A similar result is true for any distributive lattice iff the topos is Boolean (see [5]). Of course $L$ Boolean implies that $L$ satisfies strong De Morgan's law.

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