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On relations approximated by Continuous Functions

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Let $X$, $Y$ be metric spaces. By a relation in $X \times Y$ we mean a nonempty subset of the product. A relation $R$ is called closed if $R$ is a closed subset of $X \times Y$.

In the papers [1], [2], [8] are given some conditions under which there exist "well-behaved" functions that approximate closed relations.

This paper studies properties of closed relations that are approximate by continuous functions in the Hausdorff metric. Properties of a special class of such closed relations are also considered in [3]. We obtain a much more inclusive result.

Let $(Z, d)$ be a metric space. If $Z \ni E$ and $\varepsilon > 0$, let $B_\varepsilon[E]$ denote the union of all open $\varepsilon$-balls whose centers run over $E$ and $B_\varepsilon[x]$ denote the open $\varepsilon$-ball about a point $x$.

If $E$ and $F$ are nonempty subsets of $Z$ and for some $\varepsilon > 0$ both $B_\varepsilon[F] \ni E$ and $B_\varepsilon[E] \ni F$, then the Hausdorff distance $h_\varepsilon$ between them is given by $h_\varepsilon(E, F) = \inf \{\varepsilon : B_\varepsilon[E] \ni F \text{ and } B_\varepsilon[F] \ni E\}$. Otherwise we put $h_\varepsilon(E, F) = \infty$.

If we identify the sets with the same closure, then $h_\varepsilon$ is well defined on the equivalence classes so determined. Moreover, $h_\varepsilon$ defines an extended real valued metric on the class of nonempty closed subsets of $Z$, called the Hausdorff metric. Basic facts about this metric can be found in [7] Castaing and Valadier.

Now, let $(X, d_x)$ and $(Y, d_y)$ be metric spaces. We first need a metric on $X \times Y$ to induce the Hausdorff metric. For definiteness and computational simplicity, we take $\varrho$ defined by $\varrho((x_1, y_1), (x_2, y_2)) = \max \{d_x(x_1, x_2), d_y(y_1, y_2)\}$.

Denote $C(X, Y)$ the set of all continuous functions from $X$ to $Y$. Using the metric $\varrho$ we can restrict the Hausdorff metric $h_\varepsilon$ defined on the closed subsets of $X + Y$ to the graphs of functions in $C(X, Y)$. Denote this metric $d_2$.

Explicitly, if $f$ and $g$ are in $C(X, Y)$, let us represent their graphs by $G(f)$ and $G(g)$ respectively. Then $d_2(f, g)$ is defined by the formula $d_2(f, g) = \inf \{\varepsilon : B_\varepsilon[G(f)] \ni \supset G(g) \text{ and } B_\varepsilon[G(g)] \ni G(f)\}$.

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The Hausdorff metric on $C(X, Y)$ was studied by Beer [3], Naimpally [9], Waterhouse [10] and some other authors.

Let $F(X, Y)$ be the set of all functions from $X$ to $Y$. In the same way we can use $d_2$ to define the distance between any two functions from $F(X, Y)$: if $f$ and $g$ are two such functions denote the closures of their graphs by $\text{cl} \ G(f)$ and $\text{cl} \ G(g)$ respectively, and let $d_2(f, g)$ be the Hausdorff distance from $\text{cl} \ G(f)$ to $\text{cl} \ G(g)$. The function $d_2$ only defines a pseudometric on the space $F(X, Y)$.

The terminology and notation of J. Kelley will be used throughout. Moreover, we shall use the following notions and notations.

The closure of a subset $M$ of a topological space $X$ will be denoted by $\text{cl} \ M$.

Let $X, Y$ be topological spaces. Let $\mathcal{P}(Y)$ denote the collection of all subsets of $Y$. A multifunction $H$ from $X$ to $Y$ is a function $H: X \to \mathcal{P}(Y)$.

A multifunction $H$ is called closed if its graph $\{(x, y): x \in X \text{ and } y \in H(x)\}$ is a closed subset of $X \times Y$. We shall denote the graph of a multifunction $H$ by $G(H)$.

A multifunction $H$ from $X$ to $Y$ is called upper semicontinuous at $z$ in $X$ if whenever $V$ is an open subset of $Y$ that contains $H(z)$ then the set $\{x: H(x) \subseteq V\}$ contains a neighbourhood of $z$. It is called upper semicontinuous if it is upper semicontinuous at every $z \in X$.

Let $R$ be a relation in $X \times Y$. We shall use the following notation for vertical section at $x$ of $R$: $R(x) = \{y: (x, y) \in R\}$. Define the multifunction $H_R$ induced by $R$ by $H_R(x) = R(x)$. Then $G(H_R) = R$.

$\mathcal{N}$ will denote the set of positive integers.

Let $Y$ be a metric space. Let $\mathcal{X}$ be a functional defined on $\mathcal{P}(Y)$ as follows: $\mathcal{X}(\emptyset) = 0$ and if $A$ is a nonempty subset of $Y$, then $\mathcal{X}(A) = \inf \{\varepsilon: A \text{ has a finite } \varepsilon\text{-dense subset}\}$. In the literature $\mathcal{X}$ has been called the Hausdorff measure of noncompactness functional.

**Lemma 1.** (see [4]) The Hausdorff measure of non-compactness functional acts as follows:

(a) $\mathcal{X}(A) = \infty$ if and only if $A$ is unbounded
(b) $\mathcal{X}(A) = 0$ if and only if $A$ is totally bounded
(c) If $A \subseteq B$, then $\mathcal{X}(A) \leq 2\mathcal{X}(B)$
(d) If $A$ is totally bounded, then for each $\varepsilon > 0$, $\mathcal{X}(B_{\varepsilon}[A]) \leq \varepsilon$
(e) $\mathcal{X}(\text{cl} A) = \mathcal{X}(A)$.

**Theorem 1.** Let $X, Y$ be metric spaces. Let $X$ be a locally compact space and $Y$ be a complete metric space. Let $\{f_n\}$ be a sequence from $C(X, Y)$ such that the graphs of the terms of $\{f_n\}$ converge in the Hausdorff metric to a closed relation $R$ in $X \times Y$. Then the multifunction $H_R$ induced by $R$ is upper semicontinuous and $R(x)$ is a non-empty compact set for each $x \in X$.

**Proof.** Put $A = \{x \in X: R(x) \neq \emptyset\}$. The set $A$ is dense in $X$ (see [1]). Suppose that $A \neq X$. Let $x \in X \setminus A$. There is $\delta > 0$ such that $\text{cl} \ B_\delta[x]$ is compact. Put $B =$
We show that $\mathcal{K}(B) = 0$, where $\mathcal{K}$ is the Hausdorff measure of noncompactness functional. Let $\varepsilon > 0$. Put $\eta = \min \{ \varepsilon/2, \delta/2 \}$. There is $j \in \mathbb{N}$ such that $h_\varepsilon(R, G(f_n)) < \eta$ for every $n \geq j \ (1)$.

Let $n \geq j$. Then $B \subset B_{\delta/2}[x]$. Let $y \in B$. There is $a \in B_{\delta/2}[x]$ such that $((a, y), (b, f_n(b))) < \eta$. Then $y \in B_d[f_n(b)]$ and $b \in B_d[a] \subset B_{\delta/2} \subset B_d[x]$. Thus we have $B \subset B_{\eta}[f_n(B_{\delta/2}[x])]$.

Since $f_n(\text{cl} B[x])$ is compact, by (d) of Lemma 1 we have $\mathcal{K}(B_{\eta}[f_n(\text{cl} B_{\delta/2}[x])]) \leq \eta \leq \varepsilon/2$ and by (c) of Lemma 1 we have $\mathcal{K}(B) \leq \varepsilon$. Since $\mathcal{K}(B) \leq \varepsilon$ for any $\varepsilon > 0$, $\mathcal{K}(B) = 0$. By $(b)$ of Lemma 1, $B$ is a totally bounded set. The completeness of $Y$ implies that $\text{cl} B$ is compact.

There is a sequence $\{x_n\}$ of points of $A \cap B_{\delta/2}[x]$ such that $\{x_n\}$ converges to $x$. Let $\{y_n\}$ be a sequence of points of $Y$ such that $\text{cl} \{x_n, y_n\} \in R$. Since $\{y_n\}$ is a sequence of points of $B$ and $\text{cl} B$ is compact there is a cluster point $z$ of the sequence $\{y_n\}$. Then $(x, z)$ is a cluster point of the sequence $\{(x_n, y_n)\}$, i.e. $(x, z) \in \text{cl} R$. But $(x, z) \notin R$ contradicting to the fact that $R$ is closed.

For each $x \in X$ there are an open neighbourhood $V_x$ and a compact set $C_x$ such that $\bigcup \{ R(u): u \in V_x \} \subset C_x$. Let $x \in X$. There is $\delta_x > 0$ such that $\text{cl} B_{\delta_x}[x]$ is compact. Put $V_x = B_{\delta_x/2}[x]$ and $C_x = \text{cl} \cup \{ R(v): v \in V_x \}$. The proof of the compactness of $C_x$ is similar as above.

By result of Berge (see [6]) any closed multifunction with the compact range space is upper semicontinuous. Thus $H_R$ is upper semicontinuous on $V_x$ for each $x$. It is easy to see that then $H_R$ is upper semicontinuous. Since $R(x)$ is a compact subset of the compact set $C_x$ for each $x \in X$, $R(x)$ is a compact set for each $x \in X$.

Corollary 1. Let $X, Y$ be metric spaces. Let $X$ be a locally compact metric space and $Y$ be a complete metric space. Let $\{f_n\}$ be a sequence of functions from $C(X, Y)$ $d_2$-convergent to a function $f: X \to Y$ with a closed graph. Then $f$ is continuous.

The following example shows that the assumption of the locally compactness in Theorem 1 and Corollary 1 is essential.

Example 1. Let $Y$ be the set of real numbers with the usual metric. Let $n \in \mathbb{N}$. Let $\{x_j\}_{j=1}^\infty$ be a sequence of points of the open interval $(1/n, 1/n - 1)$ which is convergent to $1/n$. Put $X = \{0\} \cup \bigcup_{n=1}^\infty \{x_j: j = 1, 2, \ldots\}$ and consider $X$ with the usual metric. It is easy to verify that $X$ is not a locally compact space. Define the function $f$ by $f(x) = n$ for $x = x_j$ $j = 1, 2, \ldots$ and $f(0)$. Let $g_n$ $(n = 1, 2, \ldots)$ be a bijection from the set $\{x_j: j = 1, 2, \ldots\}$ to the set $\{j \in \mathbb{N}: j \geq n\}$ and define the functions $f_n (n = 1, 2, \ldots)$ as follows:

$$f_n(x) = \begin{cases} g_n(x) & \text{for } x = x_j \quad j = 1, 2, \ldots \\ f(x) & \text{for } x = x_j \quad m < n, \quad j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$
It is easy to verify that the sequence \( \{f_n\} \) is a sequence of continuous functions \( d_2 \)-convergent to the discontinuous function \( f \) with a closed graph.

**Proposition 1.** If a metric space \( Y \) is not complete, then there exist a compact metric space \( X \) and a sequence of continuous functions from \( X \) to \( Y \) \( d_2 \)-convergent to a discontinuous function with a closed graph.

Proof. There exists a Cauchy sequence \( \{y_n\} \) in \( Y \) which has no cluster point in \( Y \).
Let \( \overline{Y} \) be a completion of \( Y \). There exists \( y \in \overline{Y} \) such that \( \{y_n\} \) converges to \( y \) in \( \overline{Y} \).
Put \( X = \{y, y_1, y_2, \ldots, y_n, \ldots\} \) and consider \( X \) with the induced metric. Then \( X \) is compact. Define the functions \( f_n : X \to Y \) (\( n = 1, 2, \ldots \)) by \( f_n(y_i) = y_1 \) for \( i \leq n \) and \( f_n(x) = y_1 \) otherwise. It is easy to see that the functions \( f_n \) (\( n = 1, 2, \ldots \)) are continuous. Now define the function \( f : X \to Y \) as follows: \( f(y_i) = y_i \) and \( f(y) = y_1 \).
Since the sequence \( \{y_n\} \) has no cluster point in \( Y \) the function \( f \) has a closed graph.
But \( f \) is not continuous. (There exists an open set \( V \) in \( Y \) such that \( y_1 \in V \) and \( y_n \notin V \) for every \( n \geq 2 \). Then \( f^{-1}(V) = \{y, y_1\} \) is not open in \( X \).)

It remains to prove that \( \{f_n\} \) \( d_2 \)-converges to \( f \). Let \( \varepsilon > 0 \). There exists \( j \in \mathbb{N} \) such that for every \( n, m \geq j \) \( d_j(y_n, y_m) < \varepsilon \) and \( d_j(x, y_n) < \varepsilon/2 \). We show that \( G(f) = B_r[G(f)] \) and \( G(f_n) \subseteq B_r[G(f)] \) for every \( n \geq j \). Let \( x \in X \) and \( n \geq j \). If \( x = y_i \) for \( i \leq n \) or \( x = y \) then \( q((x, f(x)), (x, f_n(x))) = 0 \). Let \( x \in X \) and \( x = y_i \) for \( i > n \). For then \( d_x(y_i, y_n) < \varepsilon \) and thus \( q((y_i, f(y_i)), (y_n, f_n(y_n))) = q((y_i, y_i), (y_n, y_n)) < \varepsilon/2 \), i.e. \( (x, f(x)) \in B_r[G(f_n)] \).

Now choose \( (y_i, f_n(y_i)) \) for \( i > n \). Thus \( f_n(y_i) = y_1 \). Hence \( q((y_i, f_n(y_i)), (y, f(y))) = \max\{d_x(y_i, y), d_y(y_i, y_1)\} \leq \varepsilon/2 < \varepsilon \), i.e. \( (y_i, f_n(y_i)) \in B_r[G(f)] \) and thus \( G(f_n) \subseteq B_r[G(f)] \).

**Theorem 2.** Let \( X \) be a locally connected metric space and \( Y \) be a locally compact metric space. Let \( R \) be a closed relation in \( X \times Y \) such that \( R(x) \) is a nonempty compact set for each \( x \in X \). Let \( \{f_n\} \) be a sequence from \( C(X, Y) \) such that the graphs of the terms of the sequence \( \{f_n\} \) converge in the Hausdorff metric to \( R \). Then \( R(x) \) is a connected set for each \( x \in X \).

Proof. Fix \( x \in X \). If \( R(x) \) is a singleton, then \( R(x) \) is connected. Otherwise, suppose that \( R(x) \) contains at least two distinct points. Then \( x \) is not an isolated point of \( X \) (see [1]).

Suppose that \( R(x) \) is not connected. The compactness of \( R(x) \) implies that there are nonempty compact sets \( C, \ D \) such that \( C \cap D = \emptyset \) and \( R(x) = C \cup D \). Since \( Y \) is a locally compact metric space, there exists \( \varepsilon > 0 \) such that \( \text{cl} \ B_{\varepsilon}[C] \cap \text{cl} \ B_{\varepsilon}[D] = \emptyset \) and \( \text{cl} \ B_{\varepsilon}[C] \), \( \text{cl} \ B_{\varepsilon}[D] \) are compact sets. Fix \( u \in C, \ v \in D \). Let \( \{B_n\} \) be a sequence of connected neighbourhoods of \( x \) such that \( B_n \subset B_{1/n}[x] \) for each \( n \in \mathbb{N} \).

The convergence of the sequence \( \{f_n\} \) to \( R \) in the Hausdorff metric implies that there are an increasing sequence of positive integers \( \{k_n\} \) and sequences \( \{x_n\}, \{y_n\} \) of points of \( X \) such that \( q((x, u), (x_n, f_n(x_n))) < 1/n \), \( q((x, v), (y_n, f_n(y_n))) < 1/n \) and \( x_n, y_n \in B_n \) for each \( n \in \mathbb{N} \).
Put $L = \{y \in Y: \inf d_y(y, c) = \varepsilon/2\}$. The connectivity of sets $f_n(B_n)$ ($n = 1, 2, \ldots$) implies that there is $j \in \mathbb{N}$ such that $L \cap f_n(B_n) \neq \emptyset$ for each $n \geq j$. Let $\{v_n\}_{n=j}^{\infty}$ be a sequence of points of $Y$ such that $v_n \in L \cap f_n(B_n)$ for each $n \geq j$ and $\{a_n\}_{n=j}^{\infty}$ be a sequence of points of $X$ such that $f_n(a_n) = v_n$ and $a_n \in B_n$ for each $n \geq j$. Then $\{a_n\}_{n=j}^{\infty}$ converges to $x$.

Since $L$ is a closed subset of the compact set $cl\{B_n[C]\}$, $L$ is compact. Thus there exists a cluster point $z \in L$ of the sequence $\{v_n\}_{n=j}^{\infty}$, i.e., $(x, z)$ is a cluster point of the sequence $\{(a_n, v_n)\}_{n=j}^{\infty}$ (2).

We show that $(x, z) \in R$. Suppose that $(x, z) \notin R$. The closedness of $R$ implies that there is $\delta > 0$ for which $(B_0[x] \times B_0[z]) \cap R = \emptyset$. There is $l \in \mathbb{N}$ such that $h_\delta(R, G(f_n)) < \delta/2$ for every $n \geq l$ (3).

By (2) there is $m \in \mathbb{N}$ such that $k_m \geq l$ and $(a_m, f_k(a_m)) \in B_{\delta/2}[x] \times B_{\delta/2}[z]$. By (3) there is $(a, b) \in R$ such that $g((a, f_k(a_m)), (a, b)) < \delta/2$. But then $g((a, b), (x, z)) < \delta$ and that is a contradiction. Thus $(x, z) \in R$. Then $z \in C \cup D$. But $z \in L$. Thus $R(x)$ is connected.

**Theorem 3.** Let $X$ be a locally connected, locally compact metric space and $Y$ be a locally compact complete metric space. Let $\{f_n\}$ be a sequence from $C(X, Y)$ such that the graphs of the terms of the sequence $\{f_n\}$ converge in the Hausdorff metric to a closed relation $R$ in $X \times Y$. Then $H_R$ is an upper semicontinuous multifunction and $R(x)$ is a nonempty compact connected set for each $x \in X$.

**Proof.** By Theorem 1 $H_R$ is an upper semicontinuous closed multifunction and $R(x)$ is a nonempty compact set for each $x \in X$. By Theorem 2 $R(x)$ is a connected set for each $x \in X$.

Let $f \in F(X, Y)$. Define the limit set multifunction $H_f$ induced by $f$ (see [3]) as follows: $H_f(x) = \{y \in Y: (x, y) \in G(f)\}$ for each $x \in X$ and put $U(X, Y) = \{f \in F(X, Y): H_f$ is upper semicontinuous and $H_f(x)$ is a compact connected set for every $x \in X\}$.

From Theorem 3 we can obtain the following results

**Theorem 4.** Let $X$ be a locally compact, locally connected metric space and $Y$ be a locally compact complete metric space. Then the closure of $C(X, Y)$ in $(F(X, Y), d_2)$ is a subset of $U(X, Y)$.

**Proof.** Let $f \in F(X, Y)$ and $\{f_n\}$ be a sequence from $C(X, Y)$ $d_2$-convergent to $f$. The graphs of the sequence $\{f_n\}$ converge in the Hausdorff metric to the closed relation $G(H_f)$. By Theorem 3 $H_{G(H_f)}$ is upper semicontinuous and $G(H_f)(x)$ is a compact connected set for each $x \in X$. Since $H_{G(H_f)} = H_f$ and $G(H_f)(x) = H_f(x)$ for every $x \in X$ we have the assertion of Theorem.

If $Y$ is the set of real numbers, Theorem 4 is proved in [3].
References