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Symmetric Selective Derivatives

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The notion of a selective derivative is generalized in a natural way and it is shown to have a pathological property.

Selective derivatives were introduced by R. J. O'Malley in [2]. The natural way of generalizing this notion is a symmetric selective derivative. In this paper we show that symmetric selective derivatives depend substantially on the selection (Corollary). Moreover we can find a function f such that every function which is a derivative (in ordinary sense) is a symmetric selective derivative of f with respect to a suitable selection (Theorem).

In this paper we deal with finite real valued functions defined on the real open interval $\mathrm{I} = (0, 1)$. Denote by Δ the set of all derivatives on I and by C(I) the set of all continuous functions on I.

By a selection s we mean an interval function $\mathrm{s}(x, y)$ such that

$$x < \mathrm{s}(x, y) < y \quad \text{for every} \quad 0 < x < y < 1 .$$

It seems natural to define symmetric selective derivatives as finite limits of the form

$$\operatorname{sym} \mathrm{s}f'(x) \overset{\mathrm{def}}{=} \lim_{\delta \to 0^+} \frac{f(\mathrm{s}(x, x + \delta)) - f(\mathrm{s}(x - \delta, x))}{\mathrm{s}(x, x + \delta) - \mathrm{s}(x - \delta, x)},$$

where s is a selection and f is defined on I.

An important property of selective derivatives is that they are "almost" independent of the selection. More exactly, if $\mathrm{s}_1 f'$ and $\mathrm{s}_2 f'$ exist in I, then $\mathrm{s}_1 f' = \mathrm{s}_2 f'$ holds apart from a countable set (see [1]. Theorem 3).

As our Corollary shows, the symmetric selective derivative does not possess this property.

Theorem. *There is a function f such that for each $h \in \Delta$ there is a selection* s *for which* $\operatorname{sym} \mathrm{s}f' = h$ *holds.*

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Proof. The function f can be defined as follows. For arbitrary $x \in I$ we put $G_x = \{y \in I;\ y - x$ is a rational number$\}$. Of course, if $G_x \cap G_y \neq \emptyset$, then $G_x = G_y$ and $\bigcup_{x \in I} G_x = I$.

The class $\{G_x;\ x \in I\}$ has the cardinality continuum. Since C(I) has the cardinality continuum too, there is a bijection $\varphi: \{G_x;\ x \in I\} \to C(I)$.

Put

$$f(x) = \varphi(G_x)(x) \quad \text{for} \quad x \in I .$$

Now, let h be an arbitrary function in Δ. Then there is $g \in C(I)$ such that $h = g'$. A suitable selection s will be

$$s(x, y) \in (x, y) \cap \phi^{-1}(g) \quad \text{for each} \quad 0 < x < y < 1 .$$

(Observe that the set $(x, y) \cap \varphi^{-1}(g) \neq \emptyset$ because $\varphi^{-1}(g) = G_z$ for some $z \in I$ and G_z is a dense set in I.)

Let $x \in I$. Then for arbitrary $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ such that $u_n \nearrow x$ and $v_n \searrow x$ for for $n \to \infty$ the following is true

$$\frac{g(v_n) - g(u_n)}{v_n - u_n} \to g'(x) = h(x) \quad \text{for} \quad n \to \infty .$$

And hence

$$\lim_{\delta \to 0^+} \frac{f(s(x, x + \delta)) - f(s(x - \delta, x))}{s(x, x + \delta) - s(x - \delta, x)} = h(x) . \qquad \text{QED}$$

Corollary. *There are a function f and selections s_1 and s_2 such that*

$$\{x \in I;\ \text{sym}\, s_1 f'(x) \neq \text{sym}\, s_2 f'(x)\} = I .$$

Proof. Let $h_1(x) = 2x$ and $h_2(x) = 2x + 1$ for each $x \in I$. The assertion follows from Theorem immediately. QED

Problem. *Does* sym sf' *have more decent properties if $f \in C(I)$ or $f \in \mathscr{D}B_1$?*

References

[1] Laczkovich M., On the Baire class of selective derivatives, Acta Math. Acad. Sci. Hungar. 29 (1977), 99—105.

[2] O'Malley R. J., Selective derivatives, ibid. 29 (1977), 77—97.