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On Categories of Supertopological Spaces

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Introduction

D. Doitchinov introduced the notion of supertopological spaces in 1964 [2], in order to construct a unified theory of topological spaces, proximity spaces and uniform spaces.

Thus it was natural to embed this concept in the more general context of topological categories.

A. Tozzi and O. Wyler separately obtained some common results about this and decided to publish them in a joint paper.

To obtain a topological category in the sense of [9], it was necessary to change slightly the definition to have a unique supertopological structure on a singleton, and on the empty set, in agteement with D. Doitchinov.

Thus a supertopology on a set X is a pair $(\mathscr{M}, \theta)$ where $\mathscr{M}$ is a subset of the power set $\mathscr{P}X$ and θ is a map $\mathscr{M} \to \mathscr{P}X$ from the set $\mathscr{M}$ to the set of the filters on X, such that:

1) $\mathscr{M}$ contains $\mathscr{S}X = \{\emptyset\} \cup \{\{x\}, x\varepsilon X\}$ and $\theta(\emptyset) = \mathscr{P}X$;

2) if $A\varepsilon\mathscr{M}$ and $U\varepsilon\, \theta(A)$, then $A \subset U$;

3) if $A\varepsilon\mathscr{M}$ and $U\varepsilon\, \theta(A)$, then there exists $V\varepsilon\, \theta(A)$ such that $V\varepsilon\, \theta(B)$ for each $B\varepsilon\mathscr{M}$ with $B \subset V$.

Usually another axiom is needed:

4) if $A\varepsilon\mathscr{M}$ and $A' \subset A$, then $A'\varepsilon\mathscr{M}$.

D. Doitchinov embedded TOP, the category of topological spaces, and PROX, the category of proximity spaces, into the category of supertopological spaces, STOP, by restriction $\mathscr{M} = \mathscr{S}X$ and $\mathscr{M} = \mathscr{P}X$ respectively.

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We show that this is part of a diagram

$$\begin{array}{ccccc} \mathrm{PROX} & \to & \mathrm{QPROX} & \to & \mathrm{PPROX} \\ & & \downarrow & & \downarrow \\ \mathrm{TOP} & \to & \mathrm{ASTOP} & \to & \mathrm{STOP} \end{array}$$

of full embeddings, where the vertical arrows have left adjoints and the horizontal arrows right adjoints, which preserve underlying sets and functions.

We also show that TOP is a quotient reflective full subcategory of ASTOP and STOP, i.e., the embeddings TOP → STOP and TOP → ASTOP have left adjoint left inverses, with quotient maps as units.

0. Background

0.1. Topological categories. We recall that a *topological category* over a category C is a pair (A, U) consisting of category A and a functor U: A → C, with the following properties:

0.1.1. U is faithful;

0.1.2. U reflects initial and terminal objects;

0.1.3. every U-source $f_i: X \to UA_i$ (where X is an object of C, $\{A_i\}$ is any collection of objects of A and $\{f_i\}$ a collection of morphisms in C) has an initial lift, i.e., there exist an object A and a source $g_i: A \to A_i$ in A such that $UA = X$. $Ug_i = f_i$, for every i, and for each object B in A and morphism h- $UB \to X$ in C, $h = Uk$, where $k: B \to A$, iff $f_i \circ h = Uk_i$ with $k_i: B \to A_i$ in A.

By the usual abus de langage, we shall often say that A is a topological category over C, instead of (A, U), if the functor U is clear from the context.

If (A, U) is a topological category over a category C, then we can represent objects of A as pairs (X, α), consisting of an objects of C and a "structure" α of X, with $U(X, \alpha) = X$, and morphisms $f: (X, \alpha) \to (Y, \beta)$ of A as morphisms $f: X \to Y$ of C which are "continuous" for the structures α of X and β of Y.

With this notation, 0.1.3 means that for an object X of C and a collection (which may be large) of objects (X_i, α_i) of A and morphisms $f_i: X \to X_i$ of C, there is a unique A-structure α of X such that $g: (Y, \beta) \to (X, \alpha)$ in A, for an object (Y, β) od A and a morphism $g: Y \to X$ of C, if and only if $f_i \circ g: (Y, \beta) \to (X_i, \alpha_i)$ in A for every f_i and α_i in the collection.

0.2. The structure order of the U-fibres. Applied to $id_X: X \to X$ and an object (X, α) of A, the property 0.1.3 implies that $\alpha = \alpha'$ if $id_X: (X, \alpha) \to (X, \alpha')$ is an isomorphism. Thus 0.1.2 reduces to the requirement that an initial or terminal object T of C has exactly one A-structure. It follows from 0.1.3 that T with this structure is an initial or terminal object respectively of A.

It is well known that (A^{op}, U^{op}) is a topological category over C^{op} if (A, U) is a topological category over C. We recall that an initial source for A^{op} is called a final sink for A.

If A is a topological category over a category C, then we order A-structures of an object X of C by putting $\alpha \leqq \beta$ if $id_X: (X, \alpha) \to (X, \beta)$ in A, and we say that α is finer than β, and β coarser than α, if $\alpha \leqq \beta$. With this order, structures of X form a complete lattice, with infima obtained as initial sources, and suprema as final sinks.

0.3. Topological functors. If (A, U) and (A_1, U_1) are topological categories over a category C, then a *topological functor* $T: (A, U) \to (A_1, U_1)$ is a functor $T: A \to A_1$ such that $U_1 \circ T = U$, and T preserves initial sources. Dually, T is cotopological if T^{op} is topological, i.e., if $U_1 \circ T = U$ and T preserves final sinks.

By the Taut Lifting Theorem [12], a functor $T: A \to A_1$ of topological categories with $U_1 \circ T = U$ is topological if and only if T has a left adjoint $T_1\ A_1 \to A$ such that $U \circ T_1 = T$. This left adjoint is uniquely determined, and cotopological.

In particular, we define a *topological subcategory* of a topological category A as a full subcategory with topological embedding functor. A *cotopological subcategory* is defined dually.

A topological functor T:A $\to A_1$ over C, and its cotopological left adjoint T_1: $A_1 \to A$, are given by assignments $T_1(X, \alpha) = (X, T_1\alpha)$ and $T(Y, \beta) = (Y, T\beta)$, with f: $(X, \alpha) \to (Y, T\beta)$ in A_1, for a morphism f: $X \to Y$ of C, if and only if $f: (X, T_1\alpha) \to (Y, \beta)$ in A. If T is a full embedding, then $T_1\alpha$ is called the A-structure *induced* by α, and if T_1 is a full embedding, then $T\beta$ is called the *A_1-modification* of β, for objects (X, α) of A_1 and (Y, β) of A.

We note that (C, Id_H) always is a topological category over C. If (A, U) is a topological category over C, then U: $(A, U) \to (C, Id_H)$ is a topological functor. The topological right adjoint of U assigns to an object X of C the coarsest or *trivial* structure of X, and the cotopological left adjoint of U assigns to X the finest or *discrete* structure of X.

0.4. Notations. For a set X, we denote by $\mathscr{P}X$ the power set of X, by $\mathscr{S}X$ the set consisting of $\emptyset$ and all $\{x\}$ with $x\varepsilon X$, and by $\mathscr{F}X$ the set of all filters on X, including the *null filter* $[\emptyset]_X = \mathscr{P}X$. For a mapping $f: X \to Y$, we denote by $f^{\to}: \mathscr{P}X \to \mathscr{P}Y$ the induced direct image mapping, by $f^{\leftarrow}$: $\mathscr{P}Y \to \mathscr{P}X$ the induced inverse image mapping, and we define $\mathscr{F}f: \mathscr{F}X \to \mathscr{F}Y$ by putting $B\varepsilon(\mathscr{F}f)(\phi)$, for $B \subset Y$ and $\phi\varepsilon\mathscr{F}X$, if and only if $f^{\leftarrow}(B)\ \varepsilon\phi$. It follows that $(\mathscr{F}f)(\phi)$ is the filter on Y with the sets $f^{\to}(A)$, $A\varepsilon\phi$, as a basis.

If $A \subset X$, then we denote by $[A]_X$ the *principal filter* on X with basis $\{A\}$. We order $\mathscr{P}X$ by inclusion and $\mathscr{F}X$ dually to inclusion, putting $\phi \leqq \psi$ if ϕ is finer than ψ. With this order, introduced by Kowalsky [11], the mapping $\mathscr{P}X \to \mathscr{F}X$ $(A \mapsto$

$\mapsto [A]_X$) preserves finite meets and all suprema, and $\phi \leqq [A]_X$, for $A \subset X$ and $\phi \varepsilon \mathscr{F}X$, if and only if $A \varepsilon \phi$.

0.5. B-sets. We define a *B*-set (with *B* for "bounded") as a pair $(X, \mathscr{M})$ with X a set and $\mathscr{M} \subset \mathscr{P}X$, satisfying two conditions

0.5.1. $\mathscr{I}X \subset \mathscr{M}$.

0.5.2. If $A \varepsilon \mathscr{M}$ and $A' \subset A$, then $A' \varepsilon \mathscr{M}$.

A *map* f: $(X, \mathscr{M}) \to (Y, \mathscr{M}')$ of *B*-sets is a mapping $f: X \to Y$ which satisfies the following condition.

0.5.3. If $A \varepsilon \mathscr{M}$, then $f^{\to}(A) \, \varepsilon \mathscr{M}'$.

0.5.4. Theorem. B-sets and their maps form a topological category over SET.

Proof. For *B*-sets $(X_i, \mathscr{M}_i)$ and mappings f_i: $X \to X_i$, it is easily seen that the initial *B*-set-structure $\mathscr{M}$ of X is obtained by putting $A \varepsilon \mathscr{M}$, for $X \subset X$, if and only if $f_i^{\to}(A)$ . . $\varepsilon \mathscr{M}_i$ for every i.

The empty set has only one *B*-set structure $\mathscr{P}\emptyset$, and $(\emptyset, \mathscr{P}\emptyset)$ clearly is an initial object for *B*-sets. A singleton X has also only one *B*-set structure $\mathscr{P}X$, and $(X, \mathscr{P}X)$ is then a terminal objects for *B*-sets. This completes the proof.

The final *B*-sets structure of a set X, for *B*-set $(X_i, \mathscr{M}_i)$ and mappings f_i: $X_i \to X$, consists of $\mathscr{I}X$ and all sets $f_i^{\to}(A')$, for some f_i: $X_i \to X$ and some $A' \varepsilon \mathscr{M}_i$.

B-set structures of the set X are ordered by inclusion, with set intersections as infima, and set unions as suprema of non-empty families. $\mathscr{I}X$ is discrete *B*-set structure, and $\mathscr{P}X$ the trivial structure of X.

1. Neighborhood structures and supertopologies

1.1. Definitions. We define a *neighborhood structure* of a *B*-set $(X, \mathscr{M})$ as a mapping θ: $\mathscr{M} \to \mathscr{F}X$ which satisfies:

1.1.1. $\theta(\emptyset) = [\emptyset]_X$;

1.1.2. If $A \varepsilon \mathscr{M}$ and $U \varepsilon\, \theta(A)$, then $A \subset U$;

1.1.3. If $A \varepsilon \mathscr{M}$, $U \varepsilon\, \theta(A)$, and $A' \subset A$, then $U \varepsilon\, \theta(A')$.

A *topology* of $(X, \mathscr{M})$ is a neighborhood structure θ of $(X, \mathscr{M})$, which satisfies the following condition:

1.1.4. If $A \varepsilon \mathscr{M}$ and $U \varepsilon\, \theta(A)$, then there is a set V in $\theta(A)$ such that always $U \varepsilon\, \theta(B)$ if $B \varepsilon \mathscr{M}$ and $B \subset V$.

We note that 1.1.2 and 1.1.3 state that always $[A]_X \leqq \theta(A)$, and that $\theta(A') \leqq \theta(A)$ if $A' \subset A$ and $A \varepsilon \mathscr{M}$. It is easily seen that 1.1.3 is a consequence of 1.1.2 and 1.1.4. If θ is a neighborhood structure of $(X, \mathscr{M})$, then we call $(\mathscr{M}, \theta)$ a *neighborhood* *structure* of X, and $(X, \mathcal{M}, \theta)$ a *neighborhood space.* It θ is a topology of $(X, \mathcal{M})$, then we call $(\mathcal{M}, \theta)$ a *supertopology* of X, and $(X, \mathcal{M}, \theta)$ a *supertopological space.*

We define a *continuous map* $f\colon (X, \mathcal{M}, \theta) \to (Y, \mathcal{M}', \theta')$ of neighborhood spaces as a map $f\colon (X, \mathcal{M}) \to (Y, \mathcal{M}')$ of B-sets which satisfies:

1.1.5. If $A \varepsilon \mathcal{M}$ and $V \varepsilon\, \theta'(f^{\to}(A))$, then $f^{\leftarrow}(V) \,\varepsilon\, \theta(A)$.

In other words, we require that $\theta'(f^{\to}(A)) \geqq (\mathscr{F}f)(\theta(A))$ for $A \varepsilon \mathcal{M}$.

We denote by SNBD the category of neighborhood space and their continuous maps, and by STOP the full subcategory of SNBD with supertopological spaces as objects.

1.2. Theorem. SNBD is a topological category, over B-sets and over sets, and STOP is a topological subcategory of SNBD.

Proof. For a B-set $(X, \mathcal{M})$ and for neighborhood spaces $(X_i, \mathcal{M}_i, \theta_i)$ and maps $f_i\colon (X, \mathcal{M}) \to (X_i, \mathcal{M}_i)$, we obtain the initial neighborhood structure θ of $(X, \mathcal{M})$ as follows. For $A \varepsilon \mathcal{M}$, the finite intersections of the form $\bigcap f_i^{\leftarrow}(U_i)$, with $U_i \varepsilon\, \theta_i(f_i^{\to}(A)$ for each i, form a filter basis on X. We let $\theta(A)$ be the filter with this basis; it is easily verified that θ thus defined is the desired initial structure.

Over SET, with spaces $(X_i, \mathcal{M}_i, \theta_i)$ and mappings $f_i\colon X \to X_i$ given, we first construct the initial B-set structure $\mathcal{M}$ for the given data, and then the initial neighborhood structure θ of X for the maps $f_i\colon (X, \mathcal{M}) \to (X_i, \mathcal{M}_i)$. It is easily verified that $(\mathcal{M}, \theta)$ thus constructed is the initial neighborhood structure over SET.

The empty set $\emptyset$ has only one neighborhood structure $(\mathscr{P}\emptyset, \theta)$, with $\theta(\emptyset) = \mathscr{P}\emptyset$. This clearly is a supertopology, and $(\emptyset, \mathscr{P}\emptyset, \theta)$ is an initial object in SNBD and STOP. A singleton T also has only one neighborhood structure $(\mathscr{F}T, \theta)$, with $\theta(A) = [A]_T$ for $A \subset T$. This is also a supertopology, and $(T, \mathscr{P}T, \theta)$ is a terminal object for SNBD and for STOP.

It remains to show that the initial structure θ, as constructed above, is a topology of $(X, \mathcal{M})$ if each θ_i is a topology of $(X_i, \mathcal{M}_i)$. For $A \varepsilon \mathcal{M}$ and basis set $U = \bigcap f_i^{\leftarrow}(U_i)$ of $\theta(A)$, we have for each i a set V_i in $\theta_i(f_i^{\to}(A))$ such that $U_i \varepsilon\, \theta_i(B')$ for $B' \varepsilon \mathcal{M}_i$ with $B' \subset V_i$, if each θ_i is a topology. If $V = \bigcap f_i^{\leftarrow}(V_i)$, then $V \varepsilon\, \theta(A)$, and if $B \,\varepsilon\, \mathcal{M}$, $B \subset V$, then $f_i^{\to}(B) \subset V_i$, hence $U_i \varepsilon\, \theta_i(f_i^{\to}(B))$, for each i, and thus $U \varepsilon\, \theta(B)$. This shows that θ is a topology of $(X, \mathcal{M})$ and completes the proof.

1.3. Induced topologies. By 1.2 and 0.3 the full embedding STOP $\to$ SNBD has a left adjoint which preserves underlying B-sets and assigns to every neighborhood structure θ of a B-set $(X, \mathcal{M})$ and induced topology of $(X, \mathcal{M})$. We denote this topology by $\tilde{\theta}$ and construct it by transfinite recursion as follows.

We begin with $\theta_0 = \theta$. If θ_λ is constructed, then we construct $\theta_{\lambda+1}(A)$ for $A \varepsilon \mathcal{M}$, by putting $U \varepsilon \theta_{\lambda+1}(A)$, for $U \subset X$, if there is a set V in $\theta_\lambda(A)$ such that $U \varepsilon\, \theta_\lambda(B)$ for every $B \varepsilon \mathcal{M}$ with $B \subset V$. If λ is a limit ordinal, and θ_μ is constructed for each $\mu < \lambda$, then we put $U \varepsilon\, \theta_\lambda(A)$, for $A \varepsilon \mathcal{M}$ and $U \subset X$, if and only if $U \varepsilon\, \theta_\lambda(A)$ for each

$\mu < \lambda$. It is easily verified by transfinite induction that this defines a neighborhood structure θ_λ of $(X, \mathscr{M})$ for every ordinal λ, with $\theta_\lambda \leqq \theta_\mu$ if $\lambda < \mu$.

For an ordinal λ, let T_λ be the set of all pairs (A, U) in $\mathscr{M} \times \mathscr{P}X$ such that $U \varepsilon\, \theta_\lambda(A) \setminus \theta_{\lambda+1}(A)$. Since $\theta_\lambda \leqq \theta_\mu$ for $\lambda < \mu$, the sets T_λ are mutually disjoint. As ordinals form a proper class, it follows that $T_\lambda = \emptyset$ for some ordinal λ. Then θ_λ is a topology of $(X, \mathscr{M})$, and it is easily seen that $\theta_\mu = \theta_\lambda$ for all $\mu > \lambda$. We claim that θ_λ, for an ordinal λ with $T_\lambda = \emptyset$, is the desired topology $\tilde{\theta}$.

Since $\theta \leqq \theta_\lambda$, we have $f\colon (X, \mathscr{M}, \theta) \to (Y, \mathscr{M}', \theta')$ for $f\colon X \to Y$ if $f\colon (X, \mathscr{M}, \theta_\lambda) \to$ $\to (Y, \mathscr{M}', \theta')$. Thus it suffices to show that $f\colon (X, \mathscr{M}, \theta_\lambda) - (Y, \mathscr{M}', \theta')$ for every λ if $f\colon (X, \mathscr{M}, \theta) \to (Y, \mathscr{M}', \theta')$ and θ' is a topology of (Y, θ'), i.e., that then $f^{\leftarrow}(V) \,\varepsilon\, \theta_\lambda(A)$ for every λ if $A \varepsilon \mathscr{M}$ and $V \varepsilon\, \theta'(f^{\rightarrow}(A))$.

By assumption $f^{\leftarrow}(V) \,\varepsilon\, \theta_0(A)$, and there is W in $\theta'(f^{\rightarrow}(A))$ such that $V \varepsilon\, \theta'(B')$ for all $B' \varepsilon \mathscr{M}'$ with $B' \subset W$. If $f\colon (X, \mathscr{M}, \theta_\lambda) \to (Y, \mathscr{M}', \theta')$, then $f(W) \,\varepsilon\, \theta_\lambda(A)$. If $B \varepsilon \mathscr{M}$, with $B \subset f^{\leftarrow}(W)$, then $f^{\leftarrow}(B) \subset W$, and thus $V \varepsilon\, \theta'(f^{\rightarrow}(B))$. But then $f^{\leftarrow}(V) \,\varepsilon\, \theta_\lambda(B)$ and hence $f^{\leftarrow}(V) \,\varepsilon\, \theta_{\lambda+1}(A)$. Thus $f\colon (X, \mathscr{M}, \theta_{\lambda+1}) \to (Y, \mathscr{M}', \theta')$. For a limit ordinal λ, with $f^{\leftarrow}(V) \,\varepsilon\, \theta_\mu(A)$ for each $\mu < \lambda$, we have $f^{\leftarrow}(V) \,\varepsilon\, \theta_\lambda(A)$; thus f is continuous for θ_λ if f is continuous for each θ_μ with $\mu < \lambda$. This completes the proof.

1.4. Final structures. For a B-set $(X, \mathscr{M})$ and a collection of neighborhood spaces $(X_i, \theta_i, \mathscr{M}_i)$ and maps $f_i\colon (X_i, \mathscr{M}_i) \to (X, \mathscr{M})$, we obtain a neighborhood structure θ of $(X, \mathscr{M})$ by putting $U \varepsilon\, \theta(A)$, for $A \varepsilon \mathscr{M}$ and $U \subset X$, if and only if $A \subset U$, and $f_i^{\leftarrow}(U) \,\varepsilon\, \theta_i(A')$, for every i and every A' in $\mathscr{M}_i$ such that $f^{\rightarrow}(A') \subset A$. It is easily verified that this is indeed a neighborhood structure of $(X, \mathscr{M})$, and in fact the final neigborhood structure for the given data

Over sets, with neighborhood spaces $(X_i, \mathscr{M}_i, \theta_i)$ and mappings $f_i\colon X_i \to X$ given, we first construct the final B-set structure for the B-sets $(X_i, \mathscr{M}_i)$ and the mappings $f_i\colon X_i \to X$, and then the final neighborhood structure for the spaces $(X_i, \mathscr{M}_i, \theta_i)$ and the maps $f_i\colon (X_i, \mathscr{M}_i) \to (X, \mathscr{M})$. This clearly produces the final structure (X, θ) over sets for the given data.

Even if each θ_i is a topology of $(X_i, \mathscr{M}_i)$, the final neighborhood structure θ is in general not a topology. However, the induced topology functor SNBD $\to$ STOP, left adjoint to the embedding functor STOP $\to$ SNBD, preserves final structures. Thus the induced topology $\tilde{\theta}$ of $(X, \mathscr{M})$ is the desired final structure in STOP.

1.5. The embedding STOP$\to$SNBD. We have $\theta \leqq \theta_1$, for neighborhood structures or topologies of a B-set $(X, \mathscr{M})$, if and only if $\theta(A) \leqq \theta_1(A)$ for every $A \varepsilon \mathscr{M}$. Over sets, we have $(\mathscr{M}, \theta) \leqq (\mathscr{M}_1, \theta_1)$ if and only if $\mathscr{M} \subset \mathscr{M}_1$, and $\theta(A) \leqq \theta_1(A)$ for every $A \varepsilon \mathscr{M}$. Neighborhood structures and topologies of a B-set $(X, \mathscr{M})$, and neighborhood structures and supertopologies of a set X, form complete lattices. The embedding STOP $\to$ SNDB preserves infima and categorical limits, but not suprema and colimits.

Every B-set $(X, \mathscr{M})$ has a finest or discrete neighborhood structure θ_d, with $\theta_d(A) =$ $= [A]_X$ for every $A \varepsilon \mathscr{M}$, and a coarsest or trivial neighborhood structure θ_t, with $\theta_t(\emptyset) = [\emptyset]_X$, and $\theta_t(A) = \{X\}$ for $A \neq \emptyset$ in $\mathscr{M}$. It is easily seen that θ_d and θ_t are topologies of $(X, \mathscr{M})$.

Over sets, every set X has a discrete neighborhood structure $(\mathscr{I}X, \theta_d)$ and a trivial neighborhood structure $(\mathscr{P}X, \theta_t)$. Both of these structures are supertopologies of X.

2. Topologies versus supertopologies

2.1. Pretopologies. A neighborhood structure θ of a discrete B-set $(X, \mathscr{I}X)$ assigns to every $x \varepsilon X$ a filter $\theta(\{x\})$ of neighborhoods of x. We denote by PRT the category of pretopological spaces thus obtained and their continuous maps, and by I: PRT $\to$ SNBD the full embedding.

The filters $\theta'(\{x\})$ are neighborhood filters of points $x \varepsilon X$ for a topology of X if and only if θ is a topology of $(X, \mathscr{I}X)$; thus we obtain a full embedding I: TOP $\to$ $\to$ STOP.

In the other direction, we have functors J: SNBD $\to$ STOP and J: STOP $\to$ TOP which preserve underlying sets and mappings, with $\theta(\{x\})$ the filter of neighborhoods of x in $J(X, \mathscr{M}, \theta)$, for $x \varepsilon X$.

2.2 Theorem. The full embeddings I: TOP $\to$ STOP and I: PRT $\to$ SNBD are cotopological, with the functors J: STOP $\to$ TOP and J: SNBD $\to$ PRT as right adjoints.

Proof. For a pretopological space or topological space (X, τ), a neighborghood space or supertopological space $(Y, \mathscr{M}, \theta)$, and a mapping $f: X \to Y$, we claim that $f: I(X, \tau) \to (Y, \mathscr{M}, \theta)$ if and only if $f: (X, \tau) \to J(Y, \mathscr{M}, \theta)$. This is easily verified.

2.2'. Corollary. I preserves final structures and categorical colimits; J preserves initial structures and categorical limits.

2.3. Theorem. The full embeddings I: TOP $\to$ STOP and I: PRT $\to$ SNBD have left adjoints Q: STOP $\to$ TOP and Q: SNBD $\to$ PRT, with units given by quotient maps.

Proof. If $f: (X, \mathscr{M}, \theta) \to I(Y, \tau)$, for a supertopological space $(X, \mathscr{M}, \theta)$ and a topological space (Y, τ), then $f^{\to}(A)$ must be a singleton for every set $A \neq \emptyset$ in $\mathscr{M}$. Thus if $q_{\mathscr{M}}: X \to \overline{X}$ is a quotient mapping for the finest partition of X such that every $A \neq \emptyset$ is a subset of one partition set, then $f = f' \circ q_{\mathscr{M}}$ for a mapping $f': \overline{X} \to Y$. If we provide $\overline{X}$ with the quotient supertopology $(\overline{\mathscr{M}}, \overline{\theta})$ for $q_{\mathscr{M}}$, then $f': (\overline{X}, \overline{\mathscr{M}}, \overline{\theta}) \to$ $\to I(Y, \tau)$ becomes continuous. But then $\overline{\mathscr{M}} = \mathscr{I}\overline{X}$, and thus $(\overline{X}, \overline{\mathscr{M}}, \overline{\theta}) = IQ(X, \mathscr{M}, \theta)$ for a topological space $Q(X, \mathscr{M}, \theta)$ with underlying set $\overline{X}$, and with $f': Q(X, \mathscr{M}, \theta) \to (Y, \tau)$ continuous. This provides the desired left adjoint Q of I.

The proof for pretopological spaces is exactly analogous.

2.3'. Corollary. The embeddings I: TOP $\to$ STOP and I: PRT $\to$ SNBD preserve categorical limits, and collectively injective initial sources.

3. Additive supertopologies

3.1. Definition. A neighborhood structure θ of a B-set $(X, \mathcal{M})$ is called additive if it satisfies the following condition:

3.1.1. For sets A and B such that $A \cup B \varepsilon \mathcal{M}$, and for sets $U \varepsilon\, \theta(A)$ and $V \varepsilon\, \theta(B)$, we always have $U \cup V \varepsilon\, \theta(A \cup B)$.

Combining this with 1.1.3, we get $\theta(A \cup B) = \theta(A) \cup \theta(B)$ if $A \cup B \varepsilon \mathcal{M}$.

We note that a neighborhood structure of a discrete B-set $(X, \mathcal{S}X)$ always is additive.

A neighborhood space $(X, \mathcal{M}, \theta)$ with additive structure is also called *additive*; we denote by ASTOP and ASNBD the categories of additive supertopological spaces and of additive neighborhood spaces, with their continuous maps.

3.2. Additive modifications. We construct the *additive modification* θ_a of a neighborhood structure θ of a B-set $(X, \mathcal{M})$ by putting $U \varepsilon\, \theta_a(A)$, for $A \varepsilon \mathcal{M}$ and $U \subset X$, if A and U are finite set unions. $A = \bigcup A_i$ and $U = \bigcup U_i$, indexed by the same finite set, such that $U_i \varepsilon\, \theta(A_i)$ for each i.

3.2.1. Lemma. θ_a is a neighborhood structure of $(X, \mathcal{M})$.

Proof. Clearly $\theta_a(\emptyset) = [\emptyset]_X$. and $A \subset U$ if $U \varepsilon\, \theta(A)$. If $U \varepsilon\, \theta_a(A)$ with $U = \bigcup U_i$ and $A = \bigcup A_i$, and if $U \subset U'$, then $U' = \bigcup U_i'$ for sets U_i' with $U_i \subset U_i'$ for each i; thus $U' \varepsilon\, \theta_a(A)$. If $U \varepsilon\, \theta_a(A)$, with $U = \bigcup U_i$, $A = \bigcup A_i$, and $U_i \varepsilon\, \theta_a(A_i)$ for each i, and if $V \varepsilon\, \theta_a(A)$, with $V = \bigcup V_j$, $A = \bigcup B_j$, and $V_j \varepsilon\, \theta_a(B_j)$ for each j, then $U \cap V = \bigcup (U_i \cap V_j)$, $A = \bigcup (A_i \cap B_j)$, with $U_i \cap V_j \varepsilon\, \theta(A_i \cap B_j)$ for each pair (i, j) by 1.1.3. Thus $U \cap V \varepsilon\, \theta_a(A)$. As also $X \varepsilon\, \theta_a(A)$, this shows that $\theta_a(A)$ is a filter. If $U \varepsilon\, \theta_a(A)$, with $U = \bigcup U_i$, $A = \bigcup A_i$, and $U_i \varepsilon\, \theta_a(A_i)$ for each i, and if $A' \subset A$, then $A' = \bigcup (A_i \cap A')$, with $U_i \varepsilon\, \theta(A_i \cap A')$ for each i by 1.1.3. Thus $U \varepsilon\, \theta_a(A')$, and θ_a satisfies 1.1.3.

3.2.2. Lemma. If θ is a topology of $(X, \mathcal{M})$, then θ_a is a topology of $(X, \mathcal{M})$.

Proof. Let $U \varepsilon\, \theta_a(A)$, with $U = \bigcup U_i$, $A = \bigcup A_i$, and $U_i \varepsilon\, \theta(A_i)$ for each i. If θ is a topology, then there are sets V_i with $V_i \varepsilon\, \theta(A_i)$, and with $U_i \varepsilon\, \theta(B)$ for $B \varepsilon \mathcal{M}$, $B \subset V_i$. If $V = \bigcup V_i$, then $V \varepsilon\, \theta_a(A)$. If $B \varepsilon \mathcal{M}$, with $B \subset V$, then $B = \bigcup (B \cap V_i)$, with $U_i \varepsilon\, \theta(B \cap V_i)$ for each i. But then $U \varepsilon\, \theta_a(B)$, so that θ_a is a topology as claimed.

3.2.3. Lemma. If $f\colon (Y, \mathcal{M}', \theta') \to (X, \mathcal{M}, \theta)$ for an additive neighborhood space $(Y, \mathcal{M}', \theta')$, then $f\colon (Y, \mathcal{M}', \theta') \to (X, \mathcal{M}, \theta_a)$.

Proof. Let $B \varepsilon \mathcal{M}'$ and $U \varepsilon\, \theta_a(f^{\rightarrow}(B))$. If $f^{\rightarrow}(B) = \bigcup A_i$ and $U = \bigcup U_i$, with $U_i \varepsilon\, \theta(A_i)$, then $B = \bigcup (B \cap f^{\leftarrow}(A_i))$, with $f^{\rightarrow}(B \cap f^{\leftarrow}(A_i)) = A_i$ for each i. Thus $f^{\leftarrow}(U_i) \varepsilon\, \theta'(B \cap f^{\leftarrow}(A_i))$ for each i if f is continuous for θ' and θ. Since $f^{\leftarrow}(U) = \bigcup f^{\leftarrow}(U_i)$, it follows that $f^{\leftarrow}(U) \varepsilon\, \theta'(B)$, and f remains continuous for θ' and θ_a, if θ' is additive.

3.3. Theorem. The full embeddings ASTOP $\to$ STOP and ASNBD $\to$ SNBD are cotopological, preserving colimits and final sinks.

Proof. If $(Y, \mathcal{M}', \theta')$ is additive, then $f: (Y, \mathcal{M}', \theta') \to (X, \mathcal{M}, \theta)$ in STOP or SNBD, for $f: X \to Y$, if and only $f: (Y, \mathcal{M}', \theta') \to (X, \mathcal{M}, \theta_a)$, by 3.2.3 and 3.2.2. and the fact that $\theta_a \leqq \theta$.

3.4. Proposition. If θ is an additive neighborhood structure of a B-set $(X, \mathcal{M})$, then the induced topology $\tilde{\theta}$ of $(X, \mathcal{M})$ is additive.

Proof. It suffices to prove that each θ_λ in the construction of 1.3 is additive. This is true for θ_0 if θ is additive. If θ_λ is additive, and if $U \varepsilon\, \theta_{\lambda+1}(A)$ and $U' \varepsilon\, \theta_{\lambda+1}(B)$, with $A \cup B \varepsilon \mathcal{M}$, let $V \varepsilon\, \theta_\lambda(A)$ and $V' \varepsilon\, \theta_\lambda(B)$, with $U \varepsilon\, \theta_\lambda(C)$ for $C \varepsilon \mathcal{M}$ if $C \subset V$, and $U' \varepsilon\, \theta_\lambda(C)$ for $C \varepsilon \mathcal{M}$ if $C \subset V'$. Then $V \cup V' \varepsilon\, \theta_\lambda(A \cup B)$. If $C \varepsilon \mathcal{M}$ with $C \subset V \cup V'$, then $U \varepsilon\, \theta_\lambda(C \cap V)$ and $U' \varepsilon\, \theta_\lambda(C \cap V')$. Since $(C \cap V) \cup (C \cap V') = C$, we have $V \cup V' \varepsilon\, \theta_\lambda(C)$; thus $\theta_{\lambda+1}$ is additive. If λ is a limit ordinal, with θ_μ additive for each $\mu < \lambda$, and if $U \varepsilon\, \theta_\lambda(A)$ and $V \varepsilon\, \theta_\lambda(B)$, with $A \cup B \varepsilon \mathcal{M}$, then $U \varepsilon\, \theta_\mu(A)$ and $V \varepsilon\, \theta_\mu(B)$ for each $\mu < \lambda$. But then $U \cup V \varepsilon\, \theta_\mu(A \cup B)$ for each $\mu < \lambda$. Thus $U \cup V \varepsilon\, \theta_\lambda(A \cup B)$, and θ_λ is additive.

3.5. Additive B-set structures. A B-set structure $\mathcal{M}$ on a set X generates a dual filter on X which we denote by $\mathcal{M}^*$, consisting of all finite set unions $\bigcup A_i$ of sets A_i in $\mathcal{M}$, and containing all finite subsets of X. If $f: (X, \mathcal{M}) \to (X, \mathcal{M}_1)$ is a map of B-sets, then $f: (X, \mathcal{M}^*) \to (X, \mathcal{M}_1^*)$ is also a map of B-sets.

B-sets $(X, \mathcal{M})$ with $\mathcal{M}$ a dual filter on X have been called *bornological sets* [10] For a B-set $(X, \mathcal{M})$, a bornological set $(Y, \mathcal{M}_1)$ and a mapping $f: X \to Y$, we clearly have $f: (X, \mathcal{M}) \to (Y, \mathcal{M}_1)$ if and only if $f: (X, \mathcal{M}^*) \to (X, \mathcal{M}_1)$. Thus bornological sets determine a topological subcategory of the category of B-sets.

3.6. Theorem. An additive neighborhood structure θ of a B-set $(X, \mathcal{M})$ has a unique extension to an additive neighborhood structure θ^* of $(X, \mathcal{M}^*)$. If θ is a topology, then θ^* is a topology. If $f: (X, \mathcal{M}, \theta) \to (Y, \mathcal{M}_1, \theta_1)$ is a continuous map of additive neighborhood spaces, then $f: (X, \mathcal{M}^*, \theta^*) \to (X, \mathcal{M}_1^*, \theta_1^*)$ is continuous.

Proof. If θ^* is an extension of θ to $(X, \mathcal{M}^*)$, and if $A \varepsilon \mathcal{M}^*$ and $U \varepsilon\, \theta^*(A)$, then $U \varepsilon\, \theta(A')$ for every $A' \varepsilon \mathcal{M}$ with $A' \subset A$. Conversely, if $A = \bigcup A_i$, a finite union of sets A_i in $\mathcal{M}$, and if $U \varepsilon\, \theta(A_i)$ for each i, then $U \varepsilon\, \theta^*(A)$ if θ^* is an additive extension of θ. Thus we must define θ^* by putting $U \varepsilon\, \theta^*(A)$, for $A \varepsilon \mathcal{M}^*$, if and only if $U \varepsilon\, \theta(A')$ for each $A' \varepsilon \mathcal{M}$ with $A' \subset A$. It is easily seen that this defines an extension θ^* of θ to a neighborhood structure θ^* of $\mathcal{M}^*$. If $U \varepsilon\, \theta^*(A)$ and $V \varepsilon\, \theta^*(B)$, for sets A and B in $\mathcal{M}^*$, and if $C \varepsilon \mathcal{M}$, $C \subset A \cup B$, then $U \varepsilon\, \theta(C \cap A)$, and $V \varepsilon\, \theta(C \cap B)$, then $U \cup$ $\cup\, V \varepsilon\, \theta(C)$ if θ is additive, as $C = (C \cap A) \cup (C \cap B)$. But then $U \cup V \varepsilon\, \theta^*(A \cup B)$, and θ^* is additive if θ is additive. Assume now that θ is an additive topology, and let $U \varepsilon\, \theta^*(A)$, for $A = \bigcup A_i$ a finite union of sets A_i in $\mathcal{M}$. Then $U \varepsilon\, \theta(A_i)$ for each i, and for each i there is $V_i \varepsilon\, \theta(A_i)$ with $U_i \varepsilon\, \theta(B)$ for each $B \varepsilon \mathscr{M}$ with $B \subset V_i$. If $V = \bigcup V_i$, then $V \varepsilon\, \theta^*(A)$ by additivity of θ^*. If $B \varepsilon \mathscr{M}^*$ with $B \subset V$, and $B' \varepsilon \mathscr{M}$ with $B' \subset V$, then $U_i \varepsilon\, \theta(B' \cap V_i)$ for each i, and $U \varepsilon\, \theta(B')$ follows since $B' = \bigcup(B' \cap V_i)$. But then $U \varepsilon\, \theta^*(B)$, and θ^* is a topology. Let now $f: (X, \mathscr{M}, \theta) \to (Y, \mathscr{M}_1, \theta_1)$. and let $A \varepsilon \mathscr{M}^*$ and $U \varepsilon\, \theta_1^*(f^{\to}(A))$. If $A' \varepsilon \mathscr{M}$, with $A' \subset A$, then $U \varepsilon\, \theta_1(f^{\to}(A'))$, and hence $f^{\leftarrow}(U) \varepsilon\, \theta(A')$. Thus $f^{\leftarrow}(U)\, \varepsilon\, \theta^*(A)$, and $f: (X, \mathscr{M}^*, \theta^*) \to (Y, \mathscr{M}_1^*, \theta_1^*)$ remains continuous.

4. Supertopologies and proximities

4.1. Generalized proximity relations. Every topology θ of a B-set $(X, \mathscr{M})$ induces a generalized proximity relation $\boldsymbol{p}$ from $\mathscr{M}$ to $\boldsymbol{P}X$, with $A\boldsymbol{p}B$, for $A \varepsilon \mathscr{M}$ and $B \subset X$, if and only if every set U in $\theta(A)$ intersects B. In other words, $A\boldsymbol{p}B \Leftrightarrow X \setminus B \notin \theta(A)$. It follows that the relation $\boldsymbol{p}$ characterizes the topology.

In terms of the generalized proximity $\boldsymbol{p}$, and its negation $\bar{\boldsymbol{p}}$, the axioms of 1.1 become:

4.1.1. If $A \varepsilon \mathscr{M}$ and $B \subset X$, then $A\bar{\boldsymbol{p}}\emptyset$ and $\emptyset\bar{\boldsymbol{p}}B$;

4.1.2. $A\boldsymbol{p}(B \cup C) \Leftrightarrow A\boldsymbol{p}B$ or $A\boldsymbol{p}C$, for $A \varepsilon \mathscr{M}$ and for subsets B and C of X;

4.1.3. If $A \varepsilon \mathscr{M}, B \subset X$, and $A \cap B \neq \emptyset$, then $A\boldsymbol{p}B$;

4.1.4. If $A\boldsymbol{p}B$ and $A \subset A'$, with $A' \varepsilon \mathscr{M}$, then $A'\boldsymbol{p}B$;

4.1.5. If $A\boldsymbol{p}B$, with $A \varepsilon \mathscr{M}$ and $B \subset X$, then there is a set $V \subset X$ such that $A\bar{\boldsymbol{p}}X \setminus V$, and $C\bar{\boldsymbol{p}}B$ for every $C \varepsilon \mathscr{M}$ with $C \subset V$.

The first two of these axioms state that $\theta(A)$ is a filter on X. for $A \varepsilon \mathscr{M}$, with $\theta(\emptyset) = [\emptyset]_X$, and the other three axioms translate 1.1.2, 1.1.3 and 1.1.4.

In terms of induced generalized proximity relations $\boldsymbol{p}$ and $\boldsymbol{p}'$, the continuity condition 1.1.5 for $f: (X, \mathscr{M}, \theta) \to (Y, \mathscr{M}'\theta')$ becomes:

4.1.6. $A\boldsymbol{p}B$ then $f^{\to}(A)\, \boldsymbol{p}'\, f^{\to}(B)$.

4.2. Symmetric supertopologies. A topology θ of a B-set $(X, \mathscr{M})$ is called symmetric if its induced generalized proximity satisfies the following condition.

4.2.1. If A, B are in $\mathscr{M}$ and $A\boldsymbol{p}B$, then $B\boldsymbol{p}A$.

For topological spaces, i.e., for $\mathscr{M} = \mathscr{I}X$, this is symmetry in the usual sense. The symmetric T_0 spaces are the T_1 spaces. It is well known that symmetric topot logical spaces define a topological subcategory of TOP. 4.6 shows that this resul- cannot be generalized to supertopological spaces.

4.3. Preproximaties, quasiproximities and proximities. If $\mathscr{M} = \boldsymbol{P}X$, then 4.1.5 becomes: If $A\bar{\boldsymbol{p}}B$, for subsets A and B of X, then there is a subset V of X such that $A\bar{\boldsymbol{p}}X \setminus V$ and $V\bar{\boldsymbol{p}}B$. Unlike 4.1.5, this does not imply 4.1.4.

We say that a topology θ of a B-set $(X, \boldsymbol{P}X)$ is a *preproximity* of X. An additive preproximity is called a *quasiproximity*, and a symmetric preproximity is called *proximity*.

It is clear from 4.1 that proximities and quasiproximities thus defined are proximities as defined by Efremovič [6] and quasiproximities as defined by Fletcher and Lindgren [7], respectively. Furthermore quasiproximities are exactly topogenous structures as defined by Császár [1] and proximity structures as defined by Dowker [5].

If $(\mathbf{P}X, \theta)$ is a preproximity, quasiproximity or proximity, then we call (X, θ) or equivalently $(X, \mathfrak{p})$, a *preproximity space*, a *quasiproximity space* or *a proximity space.*

As 4.1.6 shows, continuity for supertopological spaces $(X, \mathbf{P}X, \theta)$ is proximal continuity for preproximity spaces (X, θ). We denote by PROX, QPROX and PPROX the categories of proximity spaces, of quasiproximity spaces and of preproximity spaces, with continuous maps.

For supertopologies $(\mathbf{P}X, \theta)$, symmetry clearly implies additivity. Thus PROX is a full subcategory of QPROX.

4.4. Symmetric neighborhood structures. Symmetry can be defined for neighborhood structures as well as for topologies. If θ is a symmetric neighborhood structure of a B-set $(X, \mathbf{P}X)$, then θ is called a *semiproximity of* X, and (X, θ) is called a *semiproximity space.* We need the following result.

4.4.1. Lemma. If θ is a semiproximity of a set X, then the induced topology $\tilde{\theta}$ of $(X, \mathbf{P}X)$ is a proximity of X.

Proof: It suffices to show that every structure θ_λ, in the construction of 1.3, is symmetric, i.e., that always $X \setminus A \varepsilon\, \theta_\lambda(B)$ if $X \setminus B \varepsilon\, \theta_\lambda(A)$. This is true by assumption for θ_0. If $X \setminus B \varepsilon\, \theta_{\lambda+1}(A)$, then $V \varepsilon\, \theta_\lambda(A)$ and $X \setminus B \varepsilon\, \theta_\lambda(V)$ for some set $V \subset X$. If θ_λ is symmetric, then $X \setminus A \varepsilon\, \theta_\lambda(X \setminus V)$, and $X \setminus V \varepsilon\, \theta_\lambda(B)$. Thus $X \setminus A \varepsilon\, \theta_{\lambda+1}(B)$, and $\theta_{\lambda+1}$ is symmetric. For a limit ordinal λ, with θ_μ symmetric for every $\mu < \lambda$, and $X \setminus B \varepsilon\, \theta_\lambda(A)$, we have $X \setminus B \varepsilon\, \theta_\mu(A)$, hence $X \setminus A \varepsilon\, \theta_\mu(B)$, for every $\mu < \lambda$. But then $X \setminus A \varepsilon\, \theta_\lambda(B)$, and θ_λ is symmetric.

4.5. Induced preproximities. For a supertopology $(\mathscr{M}, \theta)$ of a set X, we construct the *induced preproximity* $\theta_{\mathfrak{p}}$ of X as follows.

We extend θ to a neighborhood structure θ_1 of $(X, \mathbf{P}X)$, by putting $U \varepsilon\, \theta_1(A)$, for subsets A, U of X, if and only if $U \varepsilon\, \theta(A')$ for every A' in $\mathscr{M}$ with $A' \subset A$. It is easily seen that θ_1 is indeed a neighborhood structure, with $\theta_1(A) = \theta(A)$ for $A \varepsilon \mathscr{M}$. We cannot expect θ_1 to be a topology of $(X, \mathbf{P}X)$; thus we put $\theta_{\mathfrak{p}} = \tilde{\theta}_1$. We now show that this works.

4.5.1. Lemma. If f: $(X, \mathscr{M}, \theta) \to \mathbf{P}(Y, Y, \theta')$ for a preproximity space (Y, θ'), then f: $(X, \mathbf{P}X, \theta_{\mathfrak{p}}) \to (Y, \mathbf{P}Y, \theta')$.

Proof: By 1.2 and 1.3, it suffices to prove that $f: (X, \mathbf{P}X, \theta_1) \to (Y, \mathbf{P}Y, \theta')$. For $A \subset X$ and $U \varepsilon \theta'(f^{\to}(A))$, we have $U \varepsilon \theta'(f^{\to}(A'))$, and hence $f^{\leftarrow}(U) \varepsilon \theta(A')$ if $f: (X, \mathcal{M}, \theta) \to (Y, \mathbf{P}Y, \theta')$, for every $A' \varepsilon \mathcal{M}$ with $A' \subset A$. But then $f^{\leftarrow}(U) \varepsilon \theta(A)$, and we are done.

4.5.2. Lemma. If $(\mathcal{M}, \theta)$ is additive, then θ_p is a quasiproximity.

Proof. By 3.4, it is sufficient to show that θ_1 is additive. Thus assume $U \varepsilon \theta_1(A)$ and $V \varepsilon \theta_1(B)$. If $C \varepsilon \mathcal{M}$ with $C \subset A \cup B$, then $U \varepsilon \theta(A \cap C)$ and $V \varepsilon \theta(B \cap C)$ thus $U \cup V \varepsilon \theta(C)$ since θ is additive and $C = (A \cap C) \cup (B \cap C)$. But then $U \cup V \varepsilon$ $\varepsilon . \theta_1(A \cup B)$, and θ_1 is additive.

4.6. Theorem. PPROX is a topological subcategory of STOP, and QPROX is a topological subcategory of ASTOP. QPROX and PROX are cotopological subcategories of PPROX, and PROX is a cotopological subcategory of QPROX.

Proof: In 4.5, we constructed θ_p, for a supertopology $(\mathcal{M}, \theta)$ of a set X, so that $f: (X, \mathbf{P}X, \theta_p) \to (Y, \mathbf{P}Y, \theta')$ if, and since $(\mathcal{M}, \theta) \leqq (\mathbf{P}X, \theta_p)$ only if, $f: (X, \mathcal{M}, \theta) \to (Y, \mathbf{P}Y, \theta')$. This shows that the embedding PPROX $\to$ STOP is a topological functor. Since θ_p is a quasiproximity if $(\mathcal{M}, \theta)$ is additive, the same construction show that the embedding QPROX $\to$ ASTOP is a topological functor. The additive modification θ_a of a preproximity is a quasiproximity. Thus $f: (Y, \mathbf{P}Y, \theta') \to (X, \mathbf{P}X, \theta)$, for a preproximity space $(X, \mathbf{P}X, \theta)$ and a quasiproximity space $(Y, \mathbf{P}Y, \theta')$, if and only if $f: (Y. \mathbf{P}Y, \theta') \to (X, \mathbf{P}X, \theta_a)$. This show that the embedding QPROX $\to$ PPROX is a cotopological functor. We prove the last part of the theorem by showing that the final preproximity od a set X, for proximity spaces (X_i, θ_i) and mappings f_i: $X_i \to X$, is a proximity. By 1.4 this final preproximity is the induced supertopology $(\mathbf{P}X, \tilde{\theta})$ of the final neighborhood structure $(\mathbf{P}X, \theta)$ for the given data. Thus it suffices, by 4.4, to show that $(\mathbf{P}X, \theta)$ is symmetric. Indeed, if $X \setminus B \varepsilon \theta(A)$ for this structure, for subsets A and B of X, i.e., $X_i \setminus f_i^{\leftarrow}(B) \varepsilon \theta_i(A')$ for all i and all $A' \subset X_i$ with $f_i^{\to}(A') \subset A$, then in particular $X_i \setminus f_i^{\leftarrow}(B) \varepsilon \theta_i(f_i^{\leftarrow}(A))$ for all i. But then $X_i \setminus f_i^{\leftarrow}(A) \varepsilon$ $\theta_i(f_i^{\leftarrow}(B))$ for all i since the θ_i are symmetric. By 1.1.3, it follows that $X_i \setminus f_i^{\leftarrow}(A) \varepsilon$ $\theta_i(B')$ for all i and all $B' \subset X_i$ with $f_i^{\to}(B') \subset B$. But then $X \setminus A \varepsilon \theta(B)$, and θ is symmetric.

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