Anna Tozzi; Oswald Wyler
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On Categories of Supertopological Spaces

A. TOZZI,*
L'Aquila

O. WYLER,**
Pennsylvania, U.S.A.

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Introduction

D. Doitchinov introduced the notion of supertopological spaces in 1964 [2],
in order to construct a unified theory of topological spaces, proximity spaces and
uniform spaces.

Thus it was natural to embed this concept in the more general context of topological
categories.

A. Tozzi and O. Wyler separately obtained some common results about this and
decided to publish them in a joint paper.

To obtain a topological category in the sense of [9], it was necessary to change
slightly the definition to have a unique supertopological structure on a singleton, and
on the empty set, in agreement with D. Doitchinov.

Thus a supertopology on a set \( X \) is a pair \((\mathcal{M}, \theta)\) where \( \mathcal{M} \) is a subset of the power
set \( \mathcal{P}X \) and \( \theta \) is a map \( \mathcal{M} \to \mathcal{P}X \) from the set \( \mathcal{M} \) to the set of the filters on \( X \), such that:
1) \( \mathcal{M} \) contains \( \mathcal{I}X = \{\emptyset\} \cup \{\{x\}, x \in X\} \) and \( \theta(\emptyset) = \mathcal{P}X \);
2) if \( A \in \mathcal{M} \) and \( U \in \theta(A) \), then \( A \subset U \);
3) if \( A \in \mathcal{M} \) and \( U \in \theta(A) \), then there exists \( V \in \theta(A) \) such that \( V \in \theta(B) \) for each \( B \in \mathcal{M} \)
with \( B \subset V \).

Usually another axiom is needed:
4) if \( A \in \mathcal{M} \) and \( A' \subset A \), then \( A' \in \mathcal{M} \).

D. Doitchinov embedded \( \text{TOP} \), the category of topological spaces, and \( \text{PROX} \),
the category of proximity spaces, into the category of supertopological spaces,
\( \text{STOP} \), by restriction \( \mathcal{M} = \mathcal{I}X \) and \( \mathcal{M} = \mathcal{P}X \) respectively.

*) Dipartimento di Matematica Pura e Applicata, Via Roma, 33, 67100 L'Aquila, Italy
**) Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213,
U.S.A.
We show that this is part of a diagram

\[
\begin{array}{ccc}
\text{PROX} & \rightarrow & \text{QPROX} \\
\downarrow & & \downarrow \\
\text{TOP} & \rightarrow & \text{ASTOP} \\
\end{array}
\]

of full embeddings, where the vertical arrows have left adjoints and the horizontal arrows right adjoints, which preserve underlying sets and functions.

We also show that TOP is a quotient reflective full subcategory of ASTOP and STOP, i.e., the embeddings TOP \rightarrow STOP and TOP \rightarrow ASTOP have left adjoint left inverses, with quotient maps as units.

0. Background

0.1. Topological categories. We recall that a topological category over a category C is a pair \((A, U)\) consisting of category A and a functor \(U: A \rightarrow C\), with the following properties:

0.1.1. \(U\) is faithful;
0.1.2. \(U\) reflects initial and terminal objects;
0.1.3. every \(U\)-source \(f_i: X \rightarrow UA_i\) (where \(X\) is an object of C, \(\{A_i\}\) is any collection of objects of \(A\) and \(\{f_i\}\) a collection of morphisms in C) has an initial lift, i.e., there exist an object \(A\) and a source \(g_i: A \rightarrow A_i\) in A such that \(UA = X\).

By the usual abuse de langage, we shall often say that \(A\) is a topological category over C, instead of \((A, U)\), if the functor \(U\) is clear from the context.

If \((A, U)\) is a topological category over a category C, then we can represent objects of \(A\) as pairs \((X, \alpha)\), consisting of an objects of C and a "structure" \(\alpha\) of \(X\), with \(U(X, \alpha) = X\), and morphisms \(f: (X, \alpha) \rightarrow (Y, \beta)\) of \(A\) as morphisms \(f: X \rightarrow Y\) of C which are "continuous" for the structures \(\alpha\) of \(X\) and \(\beta\) of \(Y\).

With this notation, 0.1.3 means that for an object \(X\) of C and a collection (which may be large) of objects \((X_i, \alpha_i)\) of \(A\) and morphisms \(f_i: X \rightarrow X_i\) of C, there is a unique \(A\)-structure \(\alpha\) of \(X\) such that \(g: (Y, \beta) \rightarrow (X, \alpha)\) in \(A\), for an object \((Y, \beta)\) of \(A\) and a morphism \(g: Y \rightarrow X\) of C, if and only if \(f_i \circ g: (Y, \beta) \rightarrow (X_i, \alpha_i)\) in \(A\) for every \(f_i\) and \(\alpha_i\) in the collection.

0.2. The structure order of the U-fibres. Applied to \(id_X: X \rightarrow X\) and an object \((X, \alpha)\) of \(A\), the property 0.1.3 implies that \(\alpha = \alpha'\) if \(id_X: (X, \alpha) \rightarrow (X, \alpha')\) is an isomorphism. Thus 0.1.2 reduces to the requirement that an initial or terminal object \(T\) of C has exactly one \(A\)-structure. It follows from 0.1.3 that \(T\) with this structure is an initial or terminal object respectively of \(A\).
It is well known that \((A^{\text{op}}, U^{\text{op}})\) is a topological category over \(C^{\text{op}}\) if \((A, U)\) is a topological category over \(C\). We recall that an initial source for \(A^{\text{op}}\) is called a final sink for \(A\).

If \(A\) is a topological category over a category \(C\), then we order \(A\)-structures of an object \(X\) of \(C\) by putting \(\alpha \leq \beta\) if \(\text{id}_X: (X, \alpha) \to (X, \beta)\) in \(A\), and we say that \(\alpha\) is finer than \(\beta\), and \(\beta\) coarser than \(\alpha\), if \(\alpha \leq \beta\). With this order, structures of \(X\) form a complete lattice, with infima obtained as initial sources, and suprema as final sinks.

0.3. Topological functors. If \((A, U)\) and \((A_1, U_1)\) are topological categories over a category \(C\), then a topological functor \(T: (A, U) \to (A_1, U_1)\) is a functor \(T: A \to A_1\) such that \(U_1 \circ T = U_1\), and \(T\) preserves initial sources. Dually, \(T\) is cotopological if \(T \circ U = U\) and \(T\) preserves final sinks.

By the Taut Lifting Theorem [12], a functor \(T: A \to A_1\) of topological categories with \(U_1 \circ T = U\) is topological if and only if \(T\) has a left adjoint \(T_1: A_1 \to A\) such that \(U \circ T_1 = T\). This left adjoint is uniquely determined, and cotopological.

In particular, we define a topological subcategory of a topological category \(A\) as a full subcategory with topological embedding functor. A cotopological subcategory is defined dually.

A topological functor \(T: A \to A_1\) over \(C\), and its cotopological left adjoint \(T_1: A_1 \to A\), are given by assignments \(T_1(X, \alpha) = (X, T_1 \alpha)\) and \(T(Y, \beta) = (Y, T \beta)\), with \(f: (X, \alpha) \to (Y, T \beta)\) in \(A_1\), for a morphism \(f: X \to Y\) of \(C\), if and only if \(f: (X, T_1 \alpha) \to (Y, \beta)\) in \(A\). If \(T\) is a full embedding, then \(T_1 \alpha\) is called the \(A\)-structure induced by \(\alpha\), and if \(T_1\) is a full embedding, then \(T \beta\) is called the \(A_1\)-modification of \(\beta\), for objects \((X, \alpha)\) of \(A_1\) and \((Y, \beta)\) of \(A\).

We note that \((C, \text{Id}_C)\) always is a topological category over \(C\). If \((A, U)\) is a topological category over \(C\), then \(U: (A, U) \to (C, \text{Id}_C)\) is a topological functor. The topological right adjoint of \(U\) assigns to an object \(X\) of \(C\) the coarsest or trivial structure of \(X\), and the cotopological left adjoint of \(U\) assigns to \(X\) the finest or discrete structure of \(X\).

0.4. Notations. For a set \(X\), we denote by \(\mathcal{P}X\) the power set of \(X\), by \(\mathcal{I}X\) the set consisting of \(\emptyset\) and all \(\{x\}\) with \(x \in X\), and by \(\mathcal{F}X\) the set of all filters on \(X\), including the null filter \(\mathcal{O}_X = \emptyset\). For a mapping \(f: X \to Y\), we denote by \(f^\rightarrow: \mathcal{P}X \to \mathcal{P}Y\) the induced direct image mapping, by \(f^\leftarrow: \mathcal{P}Y \to \mathcal{P}X\) the induced inverse image mapping, and we define \(\mathcal{F}f: \mathcal{F}X \to \mathcal{F}Y\) by putting \(B \in (\mathcal{F}f)(\phi)\), for \(B \subseteq Y\) and \(\phi \in \mathcal{F}X\), if and only if \(f^\leftarrow(B) \in \phi\). It follows that \((\mathcal{F}f)(\phi)\) is the filter on \(Y\) with the sets \(f^\leftarrow(A)\), \(A \in \phi\), as a basis.

If \(A \subseteq X\), then we denote by \([A]_X\) the principal filter on \(X\) with basis \(\{A\}\). We order \(\mathcal{P}X\) by inclusion and \(\mathcal{F}X\) dually to inclusion, putting \(\phi \leq \psi\) if \(\phi\) is finer than \(\psi\). With this order, introduced by Kowalsky [11], the mapping \(\mathcal{P}X \to \mathcal{F}X\) \((A)\mapsto \ldots\).
irreducible and preserves finite meets and all suprema, and $\phi \leq [A]_X$, for $A \subseteq X$ and $\phi \subseteq \mathcal{P}X$, if and only if $A \in \phi$.

0.5. B-sets. We define a B-set (with B for "bounded") as a pair $(X, \mathcal{M})$ with $X$ a set and $\mathcal{M} \subseteq \mathcal{P}X$, satisfying two conditions

0.5.1. $\mathcal{I}X \subseteq \mathcal{M}$.

0.5.2. If $A \in \mathcal{M}$ and $A' \subseteq A$, then $A' \in \mathcal{M}$.

A map $f: (X, \mathcal{M}) \rightarrow (Y, \mathcal{M'})$ of B-sets is a mapping $f: X \rightarrow Y$ which satisfies the following condition.

0.5.3. If $A \in \mathcal{M}$, then $f^{-1}(A) \in \mathcal{M'}$.

0.5.4. Theorem. B-sets and their maps form a topological category over SET.

Proof. For B-sets $(X_i, \mathcal{M}_i)$ and mappings $f_i: X_i \rightarrow X_i$, it is easily seen that the initial B-set-structure $\mathcal{M}$ of $X$ is obtained by putting $A \in \mathcal{M}$, for $X \subseteq X$, if and only if $f_i^{-1}(A) \in \mathcal{M}_i$ for every $i$.

The empty set has only one B-set structure $\mathcal{O}$, and $(\mathcal{O}, \mathcal{O})$ clearly is an initial object for B-sets. A singleton $X$ has also only one B-set structure $\mathcal{P}X$, and $(X, \mathcal{P}X)$ is then a terminal objects for B-sets. This completes the proof.

The final B-sets structure of a set $X$, for B-set $(X_i, \mathcal{M}_i)$ and mappings $f_i: X_i \rightarrow X$, consists of $\mathcal{I}X$ and all sets $f_i^{-1}(A')$, for some $f_i: X_i \rightarrow X$ and some $A' \in \mathcal{M}_i$.

B-set structures of the set $X$ are ordered by inclusion, with set intersections as infima, and set unions as suprema of non-empty families. $\mathcal{I}X$ is discrete B-set structure, and $\mathcal{P}X$ the trivial structure of $X$.

1. Neighborhood structures and supertopologies

1.1. Definitions. We define a neighborhood structure of a B-set $(X, \mathcal{M})$ as a mapping $\theta: \mathcal{M} \rightarrow \mathcal{P}X$ which satisfies:

1.1.1. $\theta(\mathcal{O}) = [\mathcal{O}]_X$;

1.1.2. If $A \in \mathcal{M}$ and $U \in \theta(A)$, then $A \subseteq U$;

1.1.3. If $A \in \mathcal{M}$, $U \in \theta(A)$, and $A' \subseteq A$, then $U \in \theta(A')$.

A topology of $(X, \mathcal{M})$ is a neighborhood structure $\theta$ of $(X, \mathcal{M})$, which satisfies the following condition:

1.1.4. If $A \in \mathcal{M}$ and $U \in \theta(A)$, then there is a set $V$ in $\theta(A)$ such that always $U \in \theta(B)$ if $B \in \mathcal{M}$ and $B \subseteq V$.

We note that 1.1.2 and 1.1.3 state that always $[A]_X \leq \theta(A)$, and that $\theta(A') \leq \theta(A)$ if $A' \subseteq A$ and $A \in \mathcal{M}$. It is easily seen that 1.1.3 is a consequence of 1.1.2 and 1.1.4. If $\theta$ is a neighborhood structure of $(X, \mathcal{M})$, then we call $(\mathcal{M}, \theta)$ a neighborhood
structure of $X$, and $(X, \mathcal{M}, \theta)$ a neighborhood space. It $\theta$ is a topology of $(X, \mathcal{M})$, then we call $(\mathcal{M}, \theta)$ a supertopology of $X$, and $(X, \mathcal{M}, \theta)$ a supertopological space.

We define a continuous map $f: (X, \mathcal{M}, \theta) \to (Y, \mathcal{M}', \theta')$ of neighborhood spaces as a map $f: (X, \mathcal{M}) \to (Y, \mathcal{M}')$ of $B$-sets which satisfies:

1.1.5. If $A \in \mathcal{M}$ and $V \in \theta'(f^{-1}(A))$, then $f^{-1}(V) \in \theta(A)$.

In other words, we require that $\theta'(f^{-1}(A)) \supseteq (\mathcal{F}f)(\theta(A))$ for $A \in \mathcal{M}$.

We denote by $\text{SNBD}$ the category of neighborhood space and their continuous maps, and by $\text{STOP}$ the full subcategory of $\text{SNBD}$ with supertopological spaces as objects.

1.2. Theorem. $\text{SNBD}$ is a topological category, over $B$-sets and over sets, and $\text{STOP}$ is a topological subcategory of $\text{SNBD}$.

Proof. For a $B$-set $(X, \mathcal{M})$ and for neighborhood spaces $(X_i, \mathcal{M}_i, \theta_i)$ and maps $f_i: (X, \mathcal{M}) \to (X_i, \mathcal{M}_i)$, we obtain the initial neighborhood structure $\theta$ of $(X, \mathcal{M})$ as follows. For $A \in \mathcal{M}$, the finite intersections of the form $\bigcap f_i^{-1}(U_i)$, with $U_i \in \theta_i(f_i^{-1}(A))$ for each $i$, form a filter basis on $X$. We let $\theta(A)$ be the filter with this basis; it is easily verified that $\theta$ thus defined is the desired initial structure.

Over $\text{SET}$, with spaces $(X_i, \mathcal{M}_i, \theta_i)$ and mappings $f_i: X \to X_i$ given, we first construct the initial $B$-set structure $\mathcal{M}$ for the given data, and then the initial neighborhood structure $\theta$ of $X$ for the maps $f_i: (X, \mathcal{M}) \to (X_i, \mathcal{M}_i)$. It is easily verified that $(\mathcal{M}, \theta)$ thus constructed is the initial neighborhood structure over $\text{SET}$.

The empty set $\emptyset$ has only one neighborhood structure $(\emptyset, \emptyset)$, with $\emptyset(\emptyset) = \emptyset$. This clearly is a supertopology, and $(\emptyset, \emptyset, \emptyset)$ is an initial object in $\text{SNBD}$ and $\text{STOP}$. A singleton $T$ also has only one neighborhood structure $(\mathcal{F}T, \emptyset)$, with $\emptyset(A) = [A]_T$ for $A \subset T$. This is also a supertopology, and $(T, \mathcal{F}T, \emptyset)$ is a terminal object for $\text{SNBD}$ and for $\text{STOP}$.

It remains to show that the initial structure $\theta$, as constructed above, is a topology of $(X, \mathcal{M})$ if each $\theta_i$ is a topology of $(X_i, \mathcal{M}_i)$. For $A \in \mathcal{M}$ and basis set $U = \bigcap f_i^{-1}(U_i)$ of $\theta(A)$, we have for each $i$ a set $V_i$ in $\theta_i(f_i^{-1}(A))$ such that $U \in \theta_i(f_i^{-1}(A))$ for $B \in \mathcal{M}_i$ with $B \subset V_i$, if each $\theta_i$ is a topology. If $V = \bigcap f_i^{-1}(V_i)$, then $V \in \theta(A)$, and if $B \in \mathcal{M}$, $B \subset V$, then $f_i^{-1}(B) \subset V_i$, hence $U \in \theta_i(f_i^{-1}(B))$, for each $i$, and thus $U \in \theta(B)$. This shows that $\theta$ is a topology of $(X, \mathcal{M})$ and completes the proof.

1.3. Induced topologies. By 1.2 and 0.3 the full embedding $\text{STOP} \to \text{SNBD}$ has a left adjoint which preserves underlying $B$-sets and assigns to every neighborhood structure $\theta$ of a $B$-set $(X, \mathcal{M})$ and induced topology of $(X, \mathcal{M})$. We denote this topology by $\theta$ and construct it by transfinite recursion as follows.

We begin with $\theta_0 = \emptyset$. If $\theta_\lambda$ is constructed, then we construct $\theta_{\lambda+1}(A)$ for $A \in \mathcal{M}$, by putting $U \in \theta_{\lambda+1}(A)$, for $U \subset X$, if there is a set $V$ in $\theta_i(A)$ such that $U \in \theta_i(B)$ for every $B \in \mathcal{M}$ with $B \subset V$. If $\lambda$ is a limit ordinal, and $\theta_\mu$ is constructed for each $\mu < \lambda$, then we put $U \in \theta_{\lambda}(A)$, for $A \in \mathcal{M}$ and $U \subset X$, if and only if $U \in \theta_{\mu}(A)$ for each $\mu < \lambda$. This completes the proof.
\( \mu < \lambda \). It is easily verified by transfinite induction that this defines a neighborhood structure \( \theta_\mu \) of \((X, \mathcal{M})\) for every ordinal \( \lambda \), with \( \theta_\lambda \subseteq \theta_\mu \) if \( \lambda < \mu \).

For an ordinal \( \lambda \), let \( T_\lambda \) be the set of all pairs \((A, \mathcal{U})\) in \( \mathcal{M} \times \mathcal{P}X \) such that \( U \cap \theta(A) \neq \emptyset \). Since \( \theta_\lambda \subseteq \theta_\mu \) for \( \lambda < \mu \), the sets \( T_\lambda \) are mutually disjoint. As ordinals form a proper class, it follows that \( T_\lambda = \emptyset \) for some ordinal \( \lambda \). Then \( \theta_\lambda \) is a topology of \((X, \mathcal{M})\), and it is easily seen that \( \theta_\mu = \theta_\lambda \) for all \( \mu > \lambda \). We claim that \( \theta_\lambda \), for an ordinal \( \lambda \) with \( T_\lambda = \emptyset \), is the desired topology \( \theta \).

Since \( \theta \subseteq \theta_\lambda \), we have \( f : (X, \mathcal{M}, \theta) \rightarrow (Y, \mathcal{M}', \theta') \) for \( f : X \rightarrow Y \) if \( f : (X, \mathcal{M}, \theta_\lambda) \rightarrow (Y, \mathcal{M}', \theta') \). Thus it suffices to show that \( f : (X, \mathcal{M}, \theta_\lambda) \rightarrow (Y, \mathcal{M}', \theta') \) for every \( \lambda \) if \( f : (X, \mathcal{M}, \theta) \rightarrow (Y, \mathcal{M}', \theta') \) and \( \theta' \) is a topology of \((Y, \theta')\), i.e., that then \( f^{-1}(\mathcal{V}) \in \theta_\lambda(A) \) for every \( \lambda \) if \( \mathcal{A}, \mathcal{M} \) and \( \mathcal{V} \theta'((f^{-1}(A))) \).

By assumption \( f^{-1}(\mathcal{V}) \in \theta_0(A) \), and there is \( W \in \theta'((f^{-1}(A))) \) such that \( W \theta'(B') \) for all \( B' \mathcal{M} \) with \( B' \subseteq W \). If \( f : (X, \mathcal{M}, \theta_\lambda) \rightarrow (Y, \mathcal{M}', \theta') \), then \( f(W) \in \theta_\lambda(A) \). If \( B \mathcal{M} \) with \( B \subseteq f^{-1}(W) \), then \( f^{-1}(B') \subseteq W \) and thus \( W \theta'((f^{-1}(B))) \). But then \( f^{-1}(\mathcal{V}) \in \theta_\lambda(B) \) and hence \( f^{-1}(\mathcal{V}) \in \theta_{\lambda+1}(A) \). Thus \( f : (X, \mathcal{M}, \theta_{\lambda+1}) \rightarrow (Y, \mathcal{M}', \theta') \). For a limit ordinal \( \lambda \), with \( f^{-1}(\mathcal{V}) \in \theta_\mu(A) \) for each \( \mu < \lambda \), we have \( f^{-1}(\mathcal{V}) = \theta_\lambda(A) \); thus \( f \) is continuous for \( \theta_\lambda \) if \( f \) is continuous for each \( \theta_\mu \) with \( \mu < \lambda \). This completes the proof.

1.4. Final structures. For a \( B \)-set \((X, \mathcal{M})\) and a collection of neighborhood spaces \((X_i, \mathcal{M}_i, \theta_i)\) and maps \( f_i : (X_i, \mathcal{M}_i) \rightarrow (X, \mathcal{M}) \), we obtain a neighborhood structure \( \theta \) of \((X, \mathcal{M})\) by putting \( U \in \theta(A) \), for \( \mathcal{A}, \mathcal{M} \) and \( U \subseteq X \), if and only if \( A \subseteq U \), and \( f_i^{-1}(U) \in \theta(A') \), for every \( i \) and every \( A' \) in \( \mathcal{M} \) such that \( f^{-1}(A') \subseteq A \). It is easily verified that this is indeed a neighborhood structure of \((X, \mathcal{M})\), and in fact the final neighborhood structure for the given data.

Over sets, with neighborhood spaces \((X_i, \mathcal{M}_i, \theta_i)\) and mappings \( f_i : X_i \rightarrow X \) given, we first construct the final \( B \)-set structure for the \( B \)-sets \((X_i, \mathcal{M}_i)\) and the mappings \( f_i : X_i \rightarrow X \), and then the final neighborhood structure for the spaces \((X_i, \mathcal{M}_i, \theta_i)\) and the maps \( f_i : (X_i, \mathcal{M}_i) \rightarrow (X, \mathcal{M}) \). This clearly produces the final structure \((X, \theta)\) over sets for the given data.

Even if each \( \theta_i \) is a topology of \((X_i, \mathcal{M}_i)\), the final neighborhood structure \( \theta \) is in general not a topology. However, the induced topology functor \( \text{SNBD} \rightarrow \text{STOP} \), left adjoint to the embedding functor \( \text{STOP} \rightarrow \text{SNBD} \), preserves final structures. Thus the induced topology \( \theta \) of \((X, \mathcal{M})\) is the desired final structure in \( \text{STOP} \).

1.5. The embedding \( \text{STOP} \rightarrow \text{SNBD} \). We have \( \theta \subseteq \theta_1 \), for neighborhood structures or topologies of a \( B \)-set \((X, \mathcal{M})\), if and only if \( \theta(A) \subseteq \theta_1(A) \) for every \( \mathcal{A}, \mathcal{M} \). Over sets, we have \((\mathcal{M}, \theta) \subseteq (\mathcal{M}_1, \theta_1) \) if and only if \( \mathcal{M} \subseteq \mathcal{M}_1 \), and \( \theta(A) \subseteq \theta_1(A) \) for every \( \mathcal{A}, \mathcal{M} \). Neighborhood structures and topologies of a \( B \)-set \((X, \mathcal{M})\), and neighborhood structures and supertopologies of a set \( X \), form complete lattices. The embedding \( \text{STOP} \rightarrow \text{SNBD} \) preserves infima and categorical limits, but not suprema and colimits.

Every \( B \)-set \((X, \mathcal{M})\) has a finest or discrete neighborhood structure \( \theta_0 \), with \( \theta_0(A) = [A]_X \) for every \( \mathcal{A}, \mathcal{M} \), and a coarsest or trivial neighborhood structure \( \theta_1 \),
with \( \theta_0(\emptyset) = [\emptyset]_x \), and \( \theta_1(A) = \{X\} \) for \( A \neq \emptyset \) in \( \mathcal{M} \). It is easily seen that \( \theta_d \) and \( \theta_t \) are topologies of \((X, \mathcal{M})\).

Over sets, every set \( X \) has a discrete neighborhood structure \((\mathcal{X}, \theta_d)\) and a trivial neighborhood structure \((\mathcal{P}X, \theta_t)\). Both of these structures are supertopologies of \( X \).

2. Topologies versus supertopologies

2.1. Pretopologies. A neighborhood structure \( \theta \) of a discrete \( B \)-set \((X, \mathcal{J}X)\) assigns to every \( x \in X \) a filter \( \theta(\{x\}) \) of neighborhoods of \( x \). We denote by \( \text{PRT} \) the category of pretopological spaces thus obtained and their continuous maps, and by \( I: \text{PRT} \to \text{SNBD} \) the full embedding.

The filters \( \theta'(\{x\}) \) are neighborhood filters of points \( x \in X \) for a topology of \( X \) if and only if \( \theta \) is a topology of \((X, \mathcal{J}X)\); thus we obtain a full embedding \( I: \text{TOP} \to \text{STOP} \).

In the other direction, we have functors \( J: \text{SNBD} \to \text{STOP} \) and \( J: \text{STOP} \to \text{TOP} \) which preserve underlying sets and mappings, with \( \theta'(\{x\}) \) the filter of neighborhoods of \( x \) in \( J(X, \mathcal{M}, \theta) \), for \( x \in X \).

2.2. Theorem. The full embeddings \( I: \text{TOP} \to \text{STOP} \) and \( I: \text{PRT} \to \text{SNBD} \) are cotopological, with the functors \( J: \text{STOP} \to \text{TOP} \) and \( J: \text{SNBD} \to \text{PRT} \) as right adjoints.

Proof. For a pretopological space or topological space \((X, \tau)\), a neighborhood space or supertopological space \((Y, \mathcal{M}, \theta)\), and a mapping \( f: X \to Y \), we claim that \( f: I(X, \tau) \to (Y, \mathcal{M}, \theta) \) if and only if \( f: (X, \tau) \to J(Y, \mathcal{M}, \theta) \). This is easily verified.

2.2'. Corollary. \( I \) preserves final structures and categorical colimits; \( J \) preserves initial structures and categorical limits.

2.3. Theorem. The full embeddings \( I: \text{TOP} \to \text{STOP} \) and \( I: \text{PRT} \to \text{SNBD} \) have left adjoints \( Q: \text{STOP} \to \text{TOP} \) and \( Q: \text{SNBD} \to \text{PRT} \), with units given by quotient maps.

Proof. If \( f: (X, \mathcal{M}, \theta) \to I(Y, \tau) \), for a supertopological space \((X, \mathcal{M}, \theta)\) and a topological space \((Y, \tau)\), then \( f'^{-1}(A) \) must be a singleton for every set \( A \neq \emptyset \) in \( \mathcal{M} \). Thus if \( q_{\mathcal{M}}: X \to \bar{X} \) is a quotient mapping for the finest partition of \( X \) such that every \( A \neq \emptyset \) is a subset of one partition set, then \( f = f' \circ q_{\mathcal{M}} \) for a mapping \( f': \bar{X} \to Y \). If we provide \( \bar{X} \) with the quotient supertopology \((\bar{\mathcal{M}}, \bar{\theta})\) for \( q_{\mathcal{M}} \), then \( f': (\bar{X}, \bar{\mathcal{M}}, \bar{\theta}) \to (Y, \tau) \) becomes continuous. But then \( \bar{\mathcal{M}} = \mathcal{F}\bar{X} \), and thus \((\bar{X}, \bar{\mathcal{M}}, \bar{\theta}) = IQ(X, \mathcal{M}, \theta) \) for a topological space \( Q(X, \mathcal{M}, \theta) \) with underlying set \( \bar{X} \), and with \( f': Q(X, \mathcal{M}, \theta) \to (Y, \tau) \) continuous. This provides the desired left adjoint \( Q \) of \( I \).

The proof for pretopological spaces is exactly analogous.

2.3'. Corollary. The embeddings \( I: \text{TOP} \to \text{STOP} \) and \( I: \text{PRT} \to \text{SNBD} \) preserve categorical limits, and collectively injective initial sources.
3. Additive supertopologies

3.1. Definition. A neighborhood structure \( \theta \) of a B-set \( (X, \mathcal{M}) \) is called additive if it satisfies the following condition:

3.1.1. For sets \( A \) and \( B \) such that \( A \cup B \in \mathcal{M} \), and for sets \( U \in \theta(A) \) and \( V \in \theta(B) \), we always have \( U \cup V \in \theta(A \cup B) \).

Combining this with 1.1.3, we get \( \theta(A \cup B) = \theta(A) \cup \theta(B) \) if \( A \cup B \in \mathcal{M} \).

We note that a neighborhood structure of a discrete B-set \( (X, \mathcal{F}X) \) always is additive.

A neighborhood space \( (X, \mathcal{M}, \theta) \) with additive structure is also called additive; we denote by ASTOP and ASNBD the categories of additive supertopological spaces and of additive neighborhood spaces, with their continuous maps.

3.2. Additive modifications. We construct the additive modification \( \theta_a \) of a neighborhood structure \( \theta \) of a B-set \( (X, \mathcal{M}) \) by putting \( U \in \theta_a(A) \), for \( A \in \mathcal{M} \) and \( U \subseteq X \), if \( A \) and \( U \) are finite set unions. \( A = \bigcup A_i \) and \( U = \bigcup U_i \), indexed by the same finite set, such that \( U \in \theta(A_i) \) for each \( i \).

3.2.1. Lemma. \( \theta_a \) is a neighborhood structure of \( (X, \mathcal{M}) \).

Proof. Clearly \( \theta_a(\emptyset) = \emptyset \) and \( A \subseteq U \) if \( U \in \theta(A) \). If \( U \in \theta_a(A) \) with \( U = \bigcup U_i \) and \( A = \bigcup A_i \), and if \( U \subseteq U' \), then \( U' = \bigcup U'_i \) for sets \( U'_i \) with \( U_i \subseteq U'_i \) for each \( i \); thus \( U' \in \theta_a(A) \). If \( U \in \theta_a(A) \), with \( U = \bigcup U_i \), \( A = \bigcup A_i \), and \( U \in \theta_a(A_i) \) for each \( i \), and if \( V \in \theta_a(A) \), with \( V = \bigcup V_j \), \( A = \bigcup B_j \), and \( V \in \theta_a(B_j) \) for each \( j \), then \( U \cap V = \bigcup (U_i \cap V_j) \), \( A = \bigcup (A_i \cap B_j) \), with \( U_i \cap V \in \theta_a(A_i \cap B_j) \) for each pair \((i, j)\) by 1.1.3. Thus \( U \cap V \in \theta_a(A) \). As also \( X \in \theta_a(A) \), this shows that \( \theta_a(A) \) is a filter. If \( U \in \theta_a(A) \), with \( U = \bigcup U_i \), \( A = \bigcup A_i \), and \( \theta_a(A_i) \) for each \( i \), and if \( A' \subseteq A \), then \( A' = \bigcup (A_i \cap A') \), with \( U \in \theta_a(A_i \cap A') \) for each \( i \) by 1.1.3. Thus \( U \in \theta_a(A') \), and \( \theta_a \) satisfies 1.1.3.

3.2.2. Lemma. If \( \theta \) is a topology of \( (X, \mathcal{M}) \), then \( \theta_a \) is a topology of \( (X, \mathcal{M}) \).

Proof. Let \( U \in \theta_a(A) \), with \( U = \bigcup U_i \), \( A = \bigcup A_i \), and \( U \in \theta_a(A) \) for each \( i \). If \( \theta \) is a topology, then there are sets \( V_i \) with \( V_i \in \theta(A_i) \), and with \( U \in \theta(B) \) for \( B \in \mathcal{M} \), \( B \subseteq V_i \). If \( V = \bigcup V_i \), then \( V \in \theta_a(A) \). If \( B \in \mathcal{M} \), with \( B \subseteq V_i \), then \( B \in \bigcup (B \cap V_i) \), with \( \theta_a(B \cap V_i) \) for each \( i \). But then \( U \in \theta_a(B) \), so that \( \theta_a \) is a topology as claimed.

3.2.3. Lemma. If \( f : (Y, \mathcal{M}', \theta') \to (X, \mathcal{M}, \theta) \) for an additive neighborhood space \( (Y, \mathcal{M}', \theta') \), then \( f : (Y, \mathcal{M}', \theta') \to (X, \mathcal{M}, \theta_a) \).

Proof. Let \( B \in \mathcal{M}' \) and \( U \in \theta_a(f(B)) \). If \( f(B) = \bigcup A_i \) and \( U = \bigcup U_i \), with \( U \in \theta(A_i) \), then \( B = \bigcup (B \cap f^{-1}(A_i)) \), with \( f(B \cap f^{-1}(A_i)) = A_i \) for each \( i \). Thus \( f^{-1}(U_i) \in \theta'(B \cap f^{-1}(A_i)) \) for each \( i \) if \( f \) is continuous for \( \theta' \) and \( \theta \). Since \( f^{-1}(U) = \bigcup f^{-1}(U_i) \), it follows that \( f^{-1}(U) \in \theta'(B) \), and \( f \) remains continuous for \( \theta' \) and \( \theta_a \), if \( \theta' \) is additive.
3.3. Theorem. The full embeddings ASTOP → STOP and ASNBD → SNBD are cotopological, preserving colimits and final sinks.

Proof. If \((Y, \mathcal{M}', \theta')\) is additive, then \(f: (Y, \mathcal{M}', \theta') \rightarrow (X, \mathcal{M}, \theta)\) in STOP or SNBD, for \(f: X \rightarrow Y\), if and only if \((Y, \mathcal{M}', \theta') \rightarrow (X, \mathcal{M}, \theta_a)\), by 3.2.3 and 3.2.2. and the fact that \(\theta_a \leq \theta\).

3.4. Proposition. If \(\theta\) is an additive neighborhood structure of a \(B\)-set \((X, \mathcal{M})\), then the induced topology \(\bar{\theta}\) of \((X, \mathcal{M})\) is additive.

Proof. It suffices to prove that each \(\theta_x\) in the construction of 1.3 is additive. This is true for \(\theta_0\) if \(\theta\) is additive. If \(\theta_x\) is additive, and if \(U \in \theta_{x+1}(A)\) and \(U' \in \theta_{x+1}(B)\), then \(A \cup B \in \mathcal{M}\) with \(U \in \theta_x(A)\) and \(U' \in \theta_x(B)\), with \(U \in \theta_x(C)\) for \(C \in V\), and \(U' \in \theta_x(C')\) for \(C' \in V'\). Then \(V \cup V' \in \theta_x(A \cup B)\), if \(C \in V \cup V'\), since \(V \cup V' \in \theta_x(C \cup C')\) if \(C \in V \cup V'\). Hence \(\theta_x(A \cup B)\) if and only if \(A \cup B \in \mathcal{M}\) if and only if \(V \cup V' \in \theta_x(C \cup C')\) if and only if \(C \cup C' \in V \cup V'\). Thus \(\theta_x(A \cup B)\) is additive.

3.5. Additive \(B\)-set structures. A \(B\)-set structure \(\mathcal{M}\) on a set \(X\) generates a dual filter on \(X\) which we denote by \(\mathcal{M}^*\), consisting of all finite set unions \(\bigcup A_i\) of sets \(A_i\) in \(\mathcal{M}\), and containing all finite subsets of \(X\). If \(f: (X, \mathcal{M}) \rightarrow (X, \mathcal{M}_1)\) is a map of \(B\)-sets, then \(f: (X, \mathcal{M}^*) \rightarrow (X, \mathcal{M}_1^*)\) is also a map of \(B\)-sets.

\(B\)-sets \((X, \mathcal{M})\) with \(\mathcal{M}\) a dual filter on \(X\) have been called bornological sets [10].

For a \(B\)-set \((X, \mathcal{M})\), a bornological set \((Y, \mathcal{M}_1)\) and a mapping \(f: X \rightarrow Y\), we clearly have \(f: (X, \mathcal{M}) \rightarrow (Y, \mathcal{M}_1)\) if and only if \(f: (X, \mathcal{M}^*) \rightarrow (X, \mathcal{M}_1^*)\). Thus bornological sets determine a topological subcategory of the category of \(B\)-sets.

3.6. Theorem. An additive neighborhood structure \(\theta\) of a \(B\)-set \((X, \mathcal{M})\) has a unique extension to an additive neighborhood structure \(\theta^*\) of \((X, \mathcal{M}^*)\). If \(\theta\) is a topology, then \(\theta^*\) is a topology. If \(f: (X, \mathcal{M}, \theta) \rightarrow (Y, \mathcal{M}_1, \theta_1)\) is a continuous map of additive neighborhood spaces, then \(f: (X, \mathcal{M}^*, \theta^*) \rightarrow (X, \mathcal{M}_1^*, \theta_1^*)\) is continuous.

Proof. If \(\theta^*\) is an extension of \(\theta\) to \((X, \mathcal{M}^*)\), and if \(A \in \mathcal{M}^*\) and \(U \in \theta^*(A)\), then \(U \in \theta(\theta^*(A'))\) for every \(A' \in \mathcal{M}\) with \(A' \subset A\). Conversely, if \(A = \bigcup A_i\), a finite union of sets \(A_i\) in \(\mathcal{M}\), and if \(U \in \theta(A_i)\) for each \(i\), then \(U \in \theta^*(A)\) if \(\theta^*\) is an additive extension of \(\theta\). Thus we must define \(\theta^*\) by putting \(U \in \theta^*(A)\), for \(A \in \mathcal{M}^*\), if and only if \(U \in \theta(\theta^*(A'))\) for each \(A' \in \mathcal{M}\) with \(A' \subset A\). It is easily seen that this defines an extension \(\theta^*\) of \(\theta\) to a neighborhood structure \(\theta^*\) of \(\mathcal{M}^*\). If \(U \in \theta^*(A)\) and \(V \in \theta^*(B)\), for sets \(A\) and \(B\) in \(\mathcal{M}^*\), and if \(C \in \mathcal{M}\), \(C \subset A \cup B\), then \(U \in \theta(C \cap A)\), and \(V \in \theta(C \cap B)\), then \(U \cup V \in \theta(C)\) if \(\theta\) is additive, as \(C = (C \cap A) \cup (C \cap B)\). But then \(U \cup V \in \theta^*(A \cup B)\), and \(\theta^*\) is additive if \(\theta\) is additive. Assume now that \(\theta\) is an additive topology, and let \(U \in \theta^*(A)\), for \(A = \bigcup A_i\), a finite union of sets \(A_i\) in \(\mathcal{M}\). Then \(U \in \theta(A_i)\) for each \(i\),
and for each \( i \) there is \( V_i \in \theta(A_i) \) with \( U_i \in \theta(B) \) for each \( B \in \mathcal{M} \) with \( B \subseteq V_i \). If \( V = \bigcup V_i \), then \( V \in \theta^*(A) \) by additivity of \( \theta^* \). If \( A_\varepsilon \in \mathcal{M} \) with \( B \subseteq V \), and \( B' \in \mathcal{M} \) with \( B' \subseteq V \), then \( U_\varepsilon \in \theta(B' \cap V_i) \) for each \( i \), and \( U \in \theta(B') \) follows since \( B' = \bigcup (B' \cap V_i) \). But then \( U \in \theta^*(B) \), and \( \theta^* \) is a topology. Let now \( f: (X, \mathcal{M}, \theta) \to (Y, \mathcal{M}', \theta') \). and let \( A_\varepsilon \mathcal{M} \) and \( U \in \theta^*(f^{-1}(A)) \). If \( A' \mathcal{M} \), with \( A' \subseteq A \), then \( U \in \theta((f^{-1}(A'))) \), and hence \( f^{-1}(U) \in \theta(A') \). Thus \( f^{-1}(U) \in \theta^*(A) \), and \( f: (X, \mathcal{M}^*, \theta^*) \to (Y, \mathcal{M}'^*, \theta'^*) \) remains continuous.

4. Supertopologies and proximities

4.1. Generalized proximity relations. Every topology \( \theta \) of a \( B \)-set \( (X, \mathcal{M}) \) induces a generalized proximity relation \( p \) from \( \mathcal{M} \) to \( PX \), with \( A \mathcal{P} B \), for \( A \subseteq X \), if and only if every set \( U \) in \( \theta(A) \) intersects \( B \). In other words, \( A \mathcal{P} B \iff X - B \notin \theta(A) \). It follows that the relation \( p \) characterizes the topology.

In terms of the generalized proximity \( p \), and its negation \( \overline{p} \), the axioms of 1.1 become:

4.1.1. If \( A \subseteq \mathcal{M} \) and \( B \subseteq X \), then \( \overline{A} \mathcal{P} \emptyset \) and \( \emptyset \mathcal{P} B \); 
4.1.2. If \( A \mathcal{P} (B \cup C) \), then \( A \mathcal{P} B \) or \( A \mathcal{P} C \), for \( b \in M \) and for subsets \( B \) and \( C \) of \( X \); 
4.1.3. If \( A \subseteq \mathcal{M} \), \( B \subseteq X \), and \( A \cap B = \emptyset \), then \( A \mathcal{P} B \); 
4.1.4. If \( A \mathcal{P} B \) and \( A \subseteq A' \), with \( A' \subseteq \mathcal{M} \), then \( A' \mathcal{P} B \); 
4.1.5. If \( A \mathcal{P} B \), with \( A \subseteq \mathcal{M} \) and \( B \subseteq X \), then there is a set \( V \subseteq X \) such that \( A \mathcal{P} X - V \), and \( C \mathcal{P} B \) for every \( C \mathcal{M} \) with \( C \subseteq V \).

The first two of these axioms state that \( \theta(A) \) is a filter on \( X \), for \( A \subseteq \mathcal{M} \), with \( \theta(\emptyset) = \emptyset \), and the other three axioms translate 1.1.2, 1.1.3 and 1.1.4.

In terms of induced generalized proximity relations \( p \) and \( p' \), the continuity condition 1.1.5 for \( f: (X, \mathcal{M}, \theta) \to (Y, \mathcal{M}', \theta') \) becomes:

4.1.6. \( A \mathcal{P} B \) then \( f^{-1}(A) \mathcal{P} f^{-1}(B) \).

4.2. Symmetric supertopologies. A topology \( \theta \) of a \( B \)-set \( (X, \mathcal{M}) \) is called symmetric if its induced generalized proximity satisfies the following condition.

4.2.1. If \( A, B \) are in \( \mathcal{M} \) and \( A \mathcal{P} B \), then \( B \mathcal{P} A \).

For topological spaces, i.e., for \( \mathcal{M} = \mathcal{F}X \), this is symmetry in the usual sense. The symmetric \( T_0 \) spaces are the \( T_1 \) spaces. It is well known that symmetric topological spaces define a topological subcategory of \( \text{TOP} \). 4.6 shows that this result—cannot be generalized to supertopological spaces.

4.3. Preproximities, quasiproximities and proximities. If \( \mathcal{M} = PX \), then 4.1.5 becomes: If \( A \mathcal{P} B \), for subsets \( A \) and \( B \) of \( X \), then there is a subset \( V \) of \( X \) such that \( A \mathcal{P} X - V \) and \( V \mathcal{P} B \). Unlike 4.1.5, this does not imply 4.1.4.

We say that a topology \( \theta \) of a \( B \)-set \( (X, \mathcal{M}) \) is a prepropximity of \( X \). An additive prepropximity is called a quasiproximity, and a symmetric prepropximity is called a proximity.
It is clear from 4.1 that proximities and quasiproximities thus defined are proximi-
ties as defined by Efremović [6] and quasiproximities as defined by Fletcher and
Lindgren [7], respectively. Furthermore quasiproximities are exactly topogenous
structures as defined by Császár [1] and proximity structures as defined by Dowker
[5].

If \((PX, \theta)\) is a preproximity, quasiproximity or proximity, then we call \((X, \theta)\)
or equivalently \((X, p)\), a preproximity space, a quasiproximity space or a proximity
space.

As 4.1.6 shows, continuity for supertopological spaces \((X, PX, \theta)\) is proximal
continuity for preproximity spaces \((X, \theta)\). We denote by PROX, OPROX and
PPROX the categories of proximity spaces, of quasiproximity spaces and of pre-
proximity spaces, with continuous maps.

For supertopologies \((PX, \theta)\), symmetry clearly implies additivity. Thus PROX
is a full subcategory of PPROX.

4.4. Symmetric neighborhood structures. Symmetry can be defined for neigh-
borhood structures as well as for topologies. If \(\theta\) is a symmetric neighborhood
structure of a \(B\)-set \((X, PX)\), then \(\theta\) is called a semiproximity of \(X\), and \((X, \theta)\) is
called a semiproximity space. We need the following result.

4.4.1. Lemma. If \(\theta\) is a semiproximity of a set \(X\), then the induced topology \(\theta\) of
\((X, PX)\) is a proximity of \(X\).

**Proof:** It suffices to show that every structure \(\theta_\lambda\), in the construction of 1.3, is
symmetric, i.e., that always \(X \setminus A \varepsilon \theta_{\lambda}(B)\) if \(X \setminus B \varepsilon \theta_{\lambda}(A)\). This is true by assumption
for \(\theta_0\). If \(X \setminus B \varepsilon \theta_{\lambda+1}(A)\), then \(V \varepsilon \theta_{\lambda}(A)\) and \(X \setminus B \varepsilon \theta_{\lambda}(V)\) for some set \(V \subset X\). If
\(\theta_\lambda\) is symmetric, then \(X \setminus A \varepsilon \theta_{\lambda}(X \setminus V)\), and \(X \setminus V \varepsilon \theta_{\lambda}(B)\). Thus \(X \setminus A \varepsilon \theta_{\lambda+1}(B)\),
and \(\theta_{\lambda+1}\) is symmetric. For a limit ordinal \(\lambda\), with \(\theta_\mu\) symmetric for every \(\mu < \lambda\),
and \(X \setminus B \varepsilon \theta_{\lambda}(A)\), we have \(X \setminus B \varepsilon \theta_\mu(A)\), hence \(X \setminus A \varepsilon \theta_\mu(B)\), for every \(\mu < \lambda\). But
then \(X \setminus A \varepsilon \theta_\lambda(B)\), and \(\theta_\lambda\) is symmetric.

4.5. Induced preproximities. For a supertopology \((\mathcal{M}, \theta)\) of a set \(X\), we construct the
induced preproximity \(\theta_p\) of \(X\) as follows.

We extend \(\theta\) to a neighborhood structure \(\theta_1\) of \((X, PX)\), by putting \(U \varepsilon \theta_1(A)\),
for subsets \(A, U\) of \(X\), if and only if \(U \varepsilon \theta(A')\) for every \(A' \subset A\). It is
easily seen that \(\theta_1\) is indeed a neighborhood structure, with \(\theta_1(A) = \theta(A)\) for \(A \in \mathcal{M}\).
We cannot expect \(\theta_1\) to be a topology of \((X, PX)\); thus we put \(\theta_p = \theta_1\). We now
show that this works.

4.5.1. Lemma. If \(f: (X, \mathcal{M}, \theta) \to (Y, Y, \theta')\) for a preproximity space \((Y, \theta')\), then
\(f: (X, PX, \theta_p) \to (Y, PY, \theta')\).
4.5.2. Lemma. If \((\mathcal{M}, \theta)\) is additive, then \(\theta_p\) is a quasiproximity.

Proof. By 3.4, it is sufficient to show that \(\theta_1\) is additive. Thus assume \(U \in \theta_1(A)\) and \(V \in \theta_1(B)\). If \(C \subseteq A \cup B\), then \(U \in \theta(A \cap C)\) and \(V \in \theta(B \cap C)\) thus \(U \cup V \in \theta(C)\) since \(\theta\) is additive and \(C = (A \cap C) \cup (B \cap C)\). But then \(U \cup V \in \theta(A \cup B)\), and \(\theta_1\) is additive.

4.6. Theorem. PPROX is a topological subcategory of STOP, and QPROX is a topological subcategory of ASTOP. PPROX and PROX are cotopological subcategories of PPROX, and PROX is a cotopological subcategory of QPROX.

Proof. In 4.5, we constructed \(\theta_p\), for a supertopology \((\mathcal{M}, \theta)\) of a set \(X\), so that \(f: (X, PX, \theta) \to (Y, PY, \theta')\) if, and since \((\mathcal{M}, \theta) \leq (PX, \theta_p)\) only if \(f: (X, \mathcal{M}, \theta) \to (Y, PY, \theta')\). This shows that the embedding PPROX \(\to\) STOP is a topological functor. Since \(\theta_p\) is a quasiproximity if \((\mathcal{M}, \theta)\) is additive, the same construction show that the embedding QPROX \(\to\) ASTOP is a topological functor. The additive modification \(\theta_s\) of a preproximity is a quasiproximity. Thus \(f: (Y, PY, \theta') \to (X, PX, \theta)\), for a preproximity space \((X, PX, \theta)\) and a quasiproximity space \((Y, PY, \theta')\), if and only if \(f: (Y, PY, \theta') \to (X, PX, \theta)\). This show that the embedding QPROX \(\to\) PROX is a cotopological functor. We prove the last part of the theorem by showing that the final preproximity od a set \(X\), for proximity spaces \((X_i, \theta_i)\) and mappings \(f_i: X_i \to X\), is a proximity. By 1.4 this final preproximity is the induced supertopology \((PX, \theta)\) of the final neighborhood structure \((PX, \theta)\) for the given data. Thus it suffices, by 4.4, to show that \((PX, \theta)\) is symmetric. Indeed, if \(X \setminus B \in \theta(A)\) for this structure, for subsets \(A\) and \(B\) of \(X\), i.e., \(X_i \setminus f_i^{-1}(B) \in \theta_i(A')\) for all \(i\) and all \(A' \subseteq X_i\) with \(f_i^{-1}(A') \subseteq A\), then in particular \(X_i \setminus f_i^{-1}(B) \in \theta_i(f_i^{-1}(A))\) for all \(i\). But then \(X_i \setminus f_i^{-1}(A) \in \theta_i(B)\) for all \(i\) since the \(\theta_i\) are symmetric. By 1.1.3, it follows that \(X_i \setminus f_i^{-1}(A) \in \theta(A')\) for all \(i\) and all \(B' \subseteq X_i\) with \(f_i^{-1}(B') \subseteq B\). But then \(X \setminus A \in \theta(B)\), and \(\theta\) is symmetric.

References


