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Strict Differentiability via Differentiability

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Let \((X, |\cdot|), (Y, \|\cdot\|)\) be real normed linear spaces. A mapping \(F: X \to Y\) is said to be strictly differentiable at \(a \in X\) if there exists a continuous linear operator \(A: X \to Y\) such that for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
\|F(y) - F(x) - A(y - x)\| \leq \varepsilon |y - x|
\]

whenever \(|x - a| < \delta\) and \(|y - a| < \delta\). In this case the operator \(A\) is called a strict derivative of \(F\) at \(a\). Of course, \(A\) is a Frechet derivative of \(F\) at \(a\).

The natural and useful notion of a strict derivative is very old and well-known (cf. e.g. [5] for \(F: \mathbb{R} \to \mathbb{R}\), [1], [2], [4]).

It is well-known (see [3] or [6], p. 138) that for a continuous function \(F: \mathbb{R} \to \mathbb{R}\) the set of points at which \(F\) is differentiable and is not strictly differentiable is of the first category.

The aim of the present note is to prove that this assertion holds for quite arbitrary possibly discontinuous mappings \(F: X \to Y\).

We shall need the following essentially well-known lemma.

Lemma. Let \((X, |\cdot|), (Y, \|\cdot\|)\) be real normed linear spaces and \(F: X \to Y\) a mapping. Suppose that \(A: X \to Y\) is a linear mapping, \(c \in X, \varepsilon > 0, \delta > 0\) such that \(\|F(c + h) - F(c) - A(h)\| < \varepsilon |h|\) whenever \(|h| < \delta\). Then the inequalities \(|x - c| < \delta, |y - c| < \delta\) and \(|x - y| \geq |x - c|\) imply the inequality

\[
\|F(y) - F(x) - A(y - x)\| < 3\varepsilon |y - x|
\]

Proof. By the assumptions we have \(\|F(x) - F(c) - A(x - c)\| < \varepsilon |x - c|\) and \(\|F(y) - F(c) - A(y - c)\| < \varepsilon |y - c|\). Consequently

\[
\|F(y) - F(x) - A(y - x)\| < \varepsilon (|x - c| + |y - c|) \leq \varepsilon (|x - c| + |x - c| + |y - x|) \leq 3\varepsilon |y - x|
\]

The open ball with center \(x \in X\) and radius \(r > 0\) will be denoted by \(U(x, r)\). Further observe that the inequality (1) from the definition of strict differentiability

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is equivalent to
\[
(1') \quad \left\| \frac{F(y) - F(x)}{|y - x|} - A \left( \frac{y - x}{|y - x|} \right) \right\| \leq \varepsilon.
\]

Theorem. Let \((X, |\cdot|), (Y, \|\cdot\|)\) be real normed linear spaces and \(F: X \to Y\) be a mapping. Then the set \(V\) of points \(x \in X\) at which \(F\) is Frechet differentiable but is not strictly differentiable is of the first category.

Proof. Denote by \(V_{n,p}\) the set of all points \(a \in V\) for which
\[
(2) \quad \left\| F(a + h) - F(a) - (F'(a))(h) \right\| < |h|/p \quad \text{whenever} \quad |h| \leq 1/n \quad \text{and}
\]
\[
(3) \quad \text{for any} \ \delta > 0 \ \text{there exist points} \ x, y \in U(a, \delta) \ \text{such that}
\]
\[
\left\| F(y) - F(x) - (F'(a))(y - x) \right\| > (8/p) |y - x|.
\]

It is easy to see that \(V = \bigcup_{n,p=1}^{\infty} V_{n,p}\). Thus it is sufficient to prove that all sets \(V_{n,p}\) are nowhere dense. Suppose on the contrary that for some fixed \(n, p\) the set \(V_{n,p}\) is dense in a ball \(U(a, \delta), a \in V_{n,p}\). Put \(\delta = \min\left(\delta/4, 1/(8n)\right)\). By (3) we can find points \(x, y \in U(a, \delta)\) such that
\[
(4) \quad \left\| F(y) - F(x) - (F'(a))(y - x) \right\| > (8/p) |y - x|.
\]
Since \(|a - x| < \delta/4 \ \text{and} \ |y - x| < \delta/2\) we obtain \(U(x, |y - x|) \subseteq U(a, \delta)\); consequently we can choose a point \(\tilde{a} \in U(x, |y - x|) \cap V_{n,p}\). Since \(|y - x| < 1/(4n)\) and \(|\tilde{a} - x| < |y - x| < 1/(4n)|\), we have \(|y - \tilde{a}| < 1/(2n)|\. Clearly \(|x - y| \geq |x - \tilde{a}|\). On account of (2) we see that the assumptions of Lemma are satisfied for \(c = \tilde{a}, \varepsilon = 1/p, \delta = 1/n\) and \(A = F'(\tilde{a})\). Consequently Lemma implies
\[
(5) \quad \left\| F(y) - F(x) - F'(\tilde{a})(y - x) \right\| < (3/p) |y - x|.
\]
Put \(v = (y - x)/|y - x|\) and \(b = a + v/(2n)\). By (2) we obtain
\[
(6) \quad \left\| F(b) - F(a) - F'(\tilde{a})(b - a) \right\| < (1/p) |b - a|.
\]
Clearly \(|\tilde{a} - a| \leq |\tilde{a} - x| + |x - a| < 1/(4n) + 1/(8n) = 3/(8n)|\) and \(|\tilde{a} - b| \leq |\tilde{a} - a| + |a - b| < 3/(8n) + 1/(2n) < 1/n|. Further \(|a - b| = 1/(2n) > 3/(8n) > |\tilde{a} - a|\). Since \(\tilde{a} \in V_{n,p}\) we obtain by (2) and the above inequalities that the assumptions of Lemma are satisfied for \(c = \tilde{a}, \varepsilon = 1/p, \delta = 1/n, x = a, y = b\) and \(A = F'(\tilde{a})\). Consequently Lemma implies
\[
(7) \quad \left\| F(b) - F(a) - F'(\tilde{a})(b - a) \right\| < (3/p) |b - a|.
\]
The inequalities (4) and (5) clearly imply
\[
(8) \quad \left\| F'(a)(y - x) - F'(\tilde{a})(y - x) \right\| > (5/p) |y - x|.
\]
On account of (6) and (7) we obtain

\[ F'(a) (b - a) - F'(a) (b - a) \| < \frac{4}{p} |b - a|. \]

Now (8) implies \( \|F'(a)(v) - F'(a)(v)\| > \frac{5}{p} \) and (9) implies \( \|F'(a)(v) - F'(a)(v)\| < \frac{4}{p} \). This is a contradiction which completes the proof.

References