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Period Structure for Pointwise Periodic Isometries of Continua

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Let (X, d) be a compact metric space. An isometry T of X is said to be pointwise periodic if for every $x \in X$ there exists $n \in \mathbb{N}$ such that $T^n x = x$. We denote by $P(x)$ the minimal (or essential) period of x. We let

$$S(X, T) = \{P(x): x \in X\} .$$

As in [1], we say that a subset S of $\mathbb{N}$ is *finitely based* if it contains finitely many numbers $n_1, \ldots, n_k$ such that each element of S is a multiple of one of the n_j's. This means S has finitely many minimal elements when partially ordered by the divisibility relation $m \mid n$.

It was shown in [1] $S(X, T)$ must be finitely based and, conversely, any finitely based subset S of $\mathbb{N}$ is equal to an $S(X, T)$ for some pointwise periodic isometry T on a compact space X. (In fact the more general compact equicontinuous systems are dealt with in [1]. In the metrizable case equicontinuous system becomes isometric provided a suitable equivalent metric is chosen.)

In the present note we consider pointwise periodic isometries of continua, i.e. connected compact metric spaces. If X is not only connected but also locally Euclidean then $S(X, T)$ is known to be finite [2]. As we shall see, for pointwise periodic isometries of general continua the set of periods need not be finite. On the other hand, not all finitely based sets can occur.

We say that a subset S of $\mathbb{N}$ is *connected* if for any $m, n \in S$ there exists a path $n_0, \ldots, n_k$ in S such that $n_0 = m$, $n_k = n$, and either $n_{j-1} \mid n_j$ or $n_j \mid n_{j-1}$ holds for each $j = 1, \ldots, k$. This clearly means that S forms a connected subgraph of the (graph of) partially ordered set $(\mathbb{N}, \mid)$.

Theorem. If $\emptyset \neq S \subset \mathbb{N}$ is finitely based and connected then there exists a continuum X and a pointwise periodic isometry T such that $S = S(X, T)$. Conversely, if T is a pointwise periodic isometry of a continuum X then $S(X, T)$ is finitely based and connected.

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Proof. To prove the first part of the theorem we first observe that for any pair m, n of natural numbers such that $m \mid n$ there exists a continuum X with a pointwise periodic isometry T such that $S(X, T) = \{m, n\}$. To this end consider a k-fold covering $h: X_1 \to X_2$ of the unit circle by itself where $k = n/m$. Let T_1, T_2 be rational rotations of the circles with periods n, m, respectively and such that $hT_1 = T_2h$. Now we can connect X_1 with X_2 in order to obtain a continuum X and a pointwise periodic isometry T extending T_1 and T_2. This can be done as follows. First define $Y = X_1 \times [0, 1)$ and extend T_1 in a natural way by letting $\tilde{T}_1(x, t) = (T_1x, t)$. Now compactify Y by adjoining $X_2 = X_2 \times \{1\}$ so that

$$\lim (x, t) = h(x)$$

as $t \to 1$. The system (X, T), where $T = \tilde{T}_1 \cup T_2$, is clearly equicontinuous, hence isometric for an appropriate choice of the metric. Evidently $S(X, T) = \{m, n)$.

The above construction easily extends to any finite chain $n_0, \ldots, n_k$ such that $n_{j-1} \mid n_j$ $(j = 1, \ldots, k)$. More generally, if $\emptyset \neq S_0 \subset \mathbb{N}$ is finite and connected then there exists a continuum X_0 with a pointwise periodic isometry T such that $S(X_0, T) = S_0$. Finally, if $S \neq \emptyset$ is finitely based and connected, then there exists S_0 as above such that each element of S is a multiple of a number from S_0. It now suffices to enlarge (X_0, T) by adding new multiple cycles in the same way as it was done in the beginning of the proof. If the number of added cycles is infinite we take care of compactness by gradually decreasing to zero the distance between X_0 and the newly added cycles (this can be accomplished in the Hilbert cube). The pointwise periodic equicontinuous system (X, T) obtained in this manner becomes a continuum with $S(X, T) = S$.

Now we prove the second part of the theorem (note that this part is valid for a general non-metrizable case). For each $x \in X$ denote by $\langle x \rangle$ the set of all $y \in X$ such that the numbers $P(x)$, $P(y)$ can be joined by a path in the partially ordered set $S(X, T)$. It is easy to see that the sets $\langle x \rangle$, $x \in X$ form a partition of X. Besides, if $y \in \langle x \rangle$ then $[y] \subset \langle x \rangle$ where

$$[y] = \{z \in X\colon\ P(y) \mid P(z)\}$$

as in [1]. Since $[y]$ is an open set ([1], Lemma 1), all the sets $\langle x \rangle$ are open. By connectedness, $X = \langle x_0 \rangle$ for a single x_0, which implies that $S(X, T)$ is connected in $(\mathbb{N}, |)$. Theorem 1 in [1] says that $S(X, T)$ is finitely based, which concludes the proof of the theorem.

We would like to remark that an intrinsic characterization of the sets of periods for arbitrary isometries of compact metric spaces (with the presence of aperiodic points) seems to remain an open problem. In fact the period structure for arbitrary isometry can be more complicated than incorrectly stated in Theorem 3 of [1]. To see this we fix an infinite set of odd primes and arrange it in two disjoint arrays (p_{ij}), (q_{ij}). Now we construct a compact isometric system (X, T) with $S(X, T) = \{2^k p_{i1} \ldots p_{ij} q_{ij}\colon\ i, j, k \in \mathbb{N}\}$. This can be done by modifying the construction in

[1], Proposition 2. For each $i \in \mathbb{N}$ we obtain a system (X_i, T_i) with $S(X_i, T_i) = \{a_{i1}b_{i1}, a_{i1}a_{i2}b_{i2}, \ldots, \infty\}$ where $(a_{i1}, a_{i2}, \ldots) = (2, \ldots, 2, p_{i1}, p_{i2}, \ldots)$, the initial 2 being repeated i times, and $b_{ij} = q_{ij}$. Now (X, T) is the disjoint union of the systems (X_i, T_i) plus the compact group of dyadic integers with translation by its topological generator $(1, 0, 0, \ldots)$. On the other hand, $S(X, T) \cap \mathbb{N}$ cannot be represented as a finite union of a finitely based set and sets of type ∞.

References

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