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Some Combinatorial Problems, Connected with Product-isomorphisms of Binary Relations

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Received 1 March, 1988

Let \( E \) be a base set; two subsets \( A \) and \( B \) of the set \( E^2 \) are said to be product-isomorphic (denoted by \( \Pi(A, B) \)) if there exists a bijection \( f: E \to E \) such that \( f(A) = B \) (where \( f((x, y)) = (f(x), f(y)) \)).

In the book of S. Ulam [1] the following two problems are raised:

**Problem 1.** Let \( E \) be an infinite set and \( A \subset E^2 \); find the cardinality of the set of all subsets in \( E^2 \), which are product-isomorphic with the set \( A \).

**Problem 2.** Assume that \( E \) is a continual set, and \( n \) is a natural number. Does there exist, for every \( n \), a set having exactly \( n \) product-automorphisms?

Introduce the notations:

\[
\begin{align*}
  p(A) &= \text{Card} \left\{ X \mid X \subset E^2 \& \Pi(X, A) \right\}; \\
  p_a(A) &= \text{Card} \left\{ f \mid f: E \to E \& f(A) = A \right\}.
\end{align*}
\]

In connection with Problem 1 we should mention the paper by Kharazishvili [2] where, in particular, for the validity of GCH we have the following relation

\[
(\forall A) \left( A \subset E^2 \Rightarrow p(A) \in \{1, \text{Card } E, 2^{\text{Card } E}\} \right).
\]

The paper deals also with geometric characteristics of types of \( A \) sets for which the function \( p(\cdot) \) assumes, respectively, values 1, \( \text{Card } E \) and \( 2^{\text{Card } E} \).

Since the general solution of Problem 1 depends on the generalized continuum hypothesis, the consideration of some particular cases of this problem is of a certain interest.

Suppose we have mapping \( f: E \to E \); it is well known that the tree whose vertices are the elements of the set \( E \), corresponds to this mapping. It is also evident that the mapping graph \( f \) is a uniform set in \( E^2 \). Hence, the questions of product-isomorphisms of uniform sets are closely connected with similar questions on tree isomorphisms. The following theorem (Kipiani, Tsakadze) holds.

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Theorem 1. Let $E$ be an infinite base set, $\text{Card } E = a$, let $A_E$ be diagonal of $E^2$ and let $U$ be a uniform with respect to the $i$-th direction subset in $E^2$ ($i = 1, 2$). Then

1) if $\text{Card } (A_E \cap U) = a$ & $\text{Card } (A_E \setminus U) = b$ then $p(U) = a^b$;
2) if $(\exists l) (l \in E^2$ & $(l = \{x\} \times E \lor l = E \times \{x\})$ & $\text{Card } (l \cap U) = a$ & $\text{Card } (l \setminus U) = b$, then $p(U) = a^{b+1}$;
3) if $\text{Card } \text{pr}_1 U = b < a$, then $p(U) = a^b$;
4) in all the remaining cases $p(U) = 2^b$.

Let us further identify the mapping $f: E \to E$ with the graph and denote by $\mathcal{S}(E)$ the group of all permutations of the set $E$. Then as easily seen, if $f \in \mathcal{S}(E)$, the cardinal number $p(f)$ coincides with the cardinality of the set of all elements conjugated to $f$ element in the group $\mathcal{S}(E)$ and $\text{Card } (\mathcal{S}(E)/\Pi(f, ))$ is the cardinality of maximal (with respect to inclusion) family of pairwise nonconjugated elements of the group $\mathcal{S}(E)$.

In [3] it is proved that if $\text{Card } E = a \geq \omega_0$, $b = \text{Card } (f \cap A_E)$ and $c = \text{Card } (A_E \setminus f)$ then $p(f) = a^{\min\{b, c\}} \cdot 2^c$.

The equality $\text{Card } (\mathcal{S}(E)/\Pi(f, )) = (\text{Card } a + \omega_0)^{\omega_0}$ (where $\text{Card } E = \omega_0$) is also proved there.

The paper [2] contains the following result: for any group $G$, whose cardinality is less or equal to $\text{Card } E$, one can find a digraph $A \subset E^2$ such that the group of all automorphisms of this digraph is isomorphic with the group $G$ (see the proof in [4], p. 54–60). This statement implies the following

Corollary. For any infinite base set $E$ and for any cardinal number $\tau \in ]0$, $\text{Card } E]$ there exists a digraph $A \subset E^2$ with exactly $\tau$ product-automorphisms.

This result for natural $\tau$ may be easily proved directly. Any such proof however uses the axiom of choice. The question naturally arises: may Problem 2 be solved effectively, i.e. without the axiom of choice?

It turns out that the proof of the following result may be effectively carried out.

Theorem 2. (ZF) Assume that $E$ is a base set, and $n$ is a positive natural number. Then if $\text{Card } E \in \{\omega_n, 2^{\omega_n}, 2^{2\omega_n}, \ldots\}$, there exists a digraph $A \subset E^2$, with exactly $n$ product-automorphisms.

The following result, in spite of simplicity of the proof, is useful for applications.

Theorem 3. Let $E$ be a base set and $A \subset E^2$. Then

$$p(A) \cdot p_\alpha(A) = \begin{cases} 2^{\text{Card } E} & \text{if } \text{Card } E \geq \omega_0 \\ ((\text{Card } E)! & \text{if } \text{Card } E < \omega_0 \end{cases}$$

Corollary 1. If $R$ is the well ordering relation on the infinite set $E$, then $p(R) = 2^{\text{Card } E}$. 


Corollary 2. The cardinality of the set of all subsets $A$, which are the solutions of Problem 2, is equal to the $2^{\text{Card}E}$.

It should be noted finally that any algebraic system

$$\mathcal{A} = (E; f_1, f_2, \ldots, f_k; r_1, r_2, \ldots, r_n)$$

may be represented in $E^m$ for some $m \geq 1$. Thus, two algebraic systems on the base set $E$ will be isomorphic if and only if the corresponding subsets will be product-isomorphic in $E^m$ (see [1], p. 18 – 19). Hence, the $m$-dimensional analogue of Theorem 3 which is also a true statement, asserts that the product of two cardinal numbers, first of which is the number of all isomorphic with $\mathcal{A}$ systems on $E$, and the second is the number of all automorphisms of the system $\mathcal{A}$, is equal to the $2^{\text{Card}E}$ (or $(\text{Card}E)!$ if $\text{Card} E < \omega_0$).

References