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On Some Notions Related to Compactness for Locales

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There are four possible ways of saying what it means for a topological space $X$ to be locally compact:

1. Every point of $X$ has a compact closed neighbourhood (or, a neighbourhood whose closure is compact).
2. Every point of $X$ has a compact neighbourhood.
3. Every point of $X$ has a base of compact neighbourhoods (i.e., given $x \in U$ open in $X$, there exists a compact $K$ with $x \in K \subseteq U$).
4. Every point of $X$ has a base of compact closed neighbourhoods.

For Hausdorff space $X$, there are all equivalent, of course; and many textbooks on topology, whose authors aren’t particularly interested in compactness in non-Hausdorff spaces, tend to give (1) or (2) as the definition of local compactness. The condition (3) is the correct and usual notion of local compactness for not-necessarily-Hausdorff spaces, because it conforms to the general scheme for defining local version of topological properties and, as it is well known (see e.g. [4]), locally compact locales in this sense are exactly the distributive continuous lattices. In this paper we will study the locale-theoretic analogue of the condition (1) called weak local compactness.

A locale $L$ is compact iff $L$ is weakly locally compact and almost compact. Weakly locally compact locales are closed under closed sublocales and finite products. An arbitrary product $\prod L_\gamma$ of locales is weakly locally compact iff each $L_\gamma$ is weakly locally compact and $L_\gamma$ is compact for all but finitely many $\gamma$. A sum $\Sigma L_\gamma$ is weakly locally compact iff each $L_\gamma$ is weakly locally compact.

In the second part we investigate almost compact locales. A product $\prod L_\gamma$ is almost compact iff any $L_\gamma$ is almost compact. A Hausdorff locale $L$ is compact iff $\uparrow a$ is almost compact for all $a \in L$. If $L$ is a regular locally almost compact locale then $L$ is weakly locally compact.

The notion of the one-point extension may be adapted to locales (for spaces see [1]) and we consider some connections between locales and their one-point extensions

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concerning separation axioms. We investigate also the one-point compactification of locales, which coincides with the Alexandroff extension on topological spaces. Using the one-point compactification, we can prove that every weakly locally compact regular locale is spatial. Some of these results are generalized from known results for spaces (for example, see [1] and [12]).

All unexplained facts concerning locales can be found in P. T. Johnstone [5]. Recall that a frame is a complete lattice \( L \) in which the infinite distributive law
\[
a \land \bigvee S = \bigvee \{a \land s : s \in S\}
\]
holds for all \( a \in L \), \( S \subseteq L \). A frame homomorphism \( K \to L \) is a map preserving finite meets and arbitrary joins. Let \( \text{Frm} \) be the category of frames. Many facts (see [5]) indicate the importance of the opposite category \( \text{Loc} = \text{Frm}^{\text{op}} \). Objects of \( \text{Loc} \) are called locales. Of course, sublocales correspond to quotient frames and products of locales correspond to sums of frames. If \( T \) is a topological space then the lattice \( O(T) \) of all open sets of \( T \) is a locale. These locales and locales isomorphic with them are called spatial or topologies. A continuous map \( f : S \to T \) of topological spaces determines a frame homomorphism \( O(f) : O(T) \to O(S) \) sending \( V \in O(T) \) to \( f^{-1}(V) \). We get a functor \( O : \text{Top} \to \text{Loc} \), where \( \text{Top} \) is the category of topological spaces and continuous maps. \( O \) has a right adjoint \( P : \text{Loc} \to \text{Top} \) assigning to a locale \( L \) the topological space \( P(L) \) of prime (i.e. \( \land \)-irreducible and \( \neq 1 \)) elements of \( L \). Open sets of \( P(L) \) are \( \mathfrak{x} = \{a \in P(L) : x \leq a\} \), where \( x \in L \).

From the topological point of view, we will formulate results in the category \( \text{Loc} \), but proofs, which are mostly carried out in lattice-theoretic terms, in the category \( \text{Frm} \).

Let \( L \) be a locale. \( L \) is regular ([3]) if \( a = \bigvee \{x \in L : x \preceq a\} \) for all \( a \in L \), where \( x \preceq a \) means \( x^* \lor a = 1 \) (where \( x^* \) is the pseudocomplement of \( x \)). \( L \) is Hausdorff ([6]) if \( a, b \in L, 1 \neq a \not\preceq b \) implies that there exists \( c \in L \) such that \( c^* \not\preceq a, c \not\preceq b \). It was proved in [6] that \( L \) is a Hausdorff locale iff \( a = \bigvee \Box a \) for each \( a \in L \setminus \{1\} \), where \( \Box a = \{x \in L : x \leq a, x^* \leq a\} \). \( L \) is a \( T_2 \)-locale ([10]) if, for each \( a \in L \setminus \{1\} \), there exists an ideal \( A \subseteq \Box a \) such that \( a = \bigvee A \). \( L \) is conjunctive if for each two elements \( a, b \in L \) with \( a \not\preceq b \) there is an element \( c \in L \) such that \( a \lor c = 1 \) and \( b \lor c \neq 1 \). We put \( \Box 1 = L \).

We say that an element \( a \in L, a \neq 1 \) of a locale \( L \) is prime (semiprime, resp.) if \( x \land y \leq a \Rightarrow x \leq a \) or \( y \leq a \) (or \( x \land y = 0 \Rightarrow x \leq a \) or \( y \leq a \), resp.) holds, for each \( x, y \in L \). If we denote \( D(L) \) (\( P(L) \) resp., \( S(L) \) resp.) the set of all dual atoms (prime elements resp., semiprime elements resp.) in \( L \) then \( D(L) \subseteq P(L) \subseteq S(L) \). We say that \( L \) is a \( T_1 \)-locale (an \( S \)-locale resp.) if \( P(L) = D(L) \) (\( S(L) = D(L) \) resp.)—see [9]. Spatial Hausdorff locales (or \( T_2 \)-locales or \( S \)-locales) are topologies of usual Hausdorff topological spaces. A locale \( L \) is dually atomic if for any \( 1 \neq a \in L \) there is a dual atom \( d \in D(L) \) such that \( d \geq a \).

Recall that sublocales of \( L \) correspond to nuclei on \( L \), i.e., to maps \( j : L \to L \) such that \( a \leq j(a), jj(a) = j(a) \) and \( j(a \land b) = j(a) \land j(b) \) for all \( a, b \in L \). A surjective homomorphism \( f : K \to L \) of frames is closed if \( f(a) = f(b) \Rightarrow a \lor f(0) = b \lor
\( \vee f^\circ(0) \) for each \( a, b \in K \), where \( f^\circ(0) = \mathcal{V}(x \in K: f(x) = 0) \). We denote \( L_r = \{ l \in L: l = l^{**} \} \).

1. Weakly locally compact locales

Let us recall that a locale \( L \) is almost compact if each covering of \( L \) has a finite dense subset. For a locale \( L \) we will denote \( S_L = \{ l \in L: l^* = 0 \} \). Then the following are equivalent:

1. \( L \) is not almost compact.
2. An ideal \( Q \) in \( L \) exists such that \( Q \subseteq S_L \), \( \forall Q = 1 \).
3. A proper filter \( F \) in \( L \) exists such that \( \mathcal{V}(a^*: a \in F) = 1 \).

Such a filter is called an \( \alpha \)-filter.

Some properties of almost compact locales are in [10]. Recall that a topological space \( T \) is locally compact iff for each \( x \in T \) there exists an open set \( U \) such that \( x \in U, \overline{U} \) is compact. If \( L \) is a locale then we put \( F_C = \{ a \in L: \uparrow a \text{ is compact} \} \).

1.1. Proposition. Let \( T \) be a topological space, \( O(T) \) be the locale of all open sets of \( T \). Then \( T \) is locally compact iff \( \mathcal{V}(a^*: a \in F_C) = 1 \).

Proof. \( \Rightarrow \): If \( x \in T \) then an open set \( U \) exists such that \( x \in U, \overline{U} \) is compact, i.e., \( T \setminus \overline{U} \) is open, \( T \setminus \overline{U} \in F_C \). Clearly, \( x \in U \subseteq (T \setminus \overline{U})^* \), i.e. \( \mathcal{V}(a^*: a \in F_C) = 1 \).

\( \Rightarrow \): If \( x \in T \) then \( a \in F_C \) exists such that \( x \in a^* \). Clearly, \( T \setminus a \) is compact and closed. Now, we have \( a^* \subseteq T \setminus a \), i.e., \( \overline{a^*} \subseteq T \setminus a \). Evidently, \( \overline{a^*} \) is compact.

Motivated by 1.1, we adopt the following

Definition. Let \( L \) be a locale. We say that \( L \) is weakly locally compact or wl-compact if \( \mathcal{V}(a^*: a \in F_C) = 1 \).

Clearly, compact locales are wl-compact. Namely, if \( L \) is compact then \( 0 \in F_C \), i.e., \( 1 = 0^* = \mathcal{V}(a^*: a \in F_C) \).

1.2. Proposition. Let \( L \) be a locale which is not compact. Then \( L \) is wl-compact iff \( F_C \) is an \( \alpha \)-filter.

Proof. \( \Rightarrow \): Since \( \mathcal{V}(a^*: a \in F_C) = 1 \) we have to show that \( F_C \) is a filter. Evidently, \( 0 \notin F_C \) and \( b \geq a, a \in F_C \Rightarrow b \in F_C \). Let \( a, b \in F_C \), \( \mathcal{V}_{i \in I} x_i = 1, x_i \geq a \land b \) for any \( i \in I \). Since \( \uparrow a, \uparrow b \) are compact we have \( \mathcal{V}(x_i: i \in K) \lor a = 1 = \mathcal{V}(x_i: i \in K) \lor b \) for some finite \( K \subseteq I \). Now, we have \( 1 = [\mathcal{V}(x_i: i \in K) \lor a] \land [\mathcal{V}(x_i: i \in K) \lor b] = \mathcal{V}(x_i: i \in K) \lor (a \land b), \) i.e., \( a \land b \in F_C \). The rest of the proof is obvious.

As an application of 1.2 we have the following characterization of compact locales.
1.3. **Theorem.** A locale $L$ is compact iff $L$ is wl-compact and almost compact.

**Proof.** $\Rightarrow$: It is evident.
$\Leftarrow$: This results immediately from 1.2 by the fact that a frame $L$ is not almost compact iff there exists an $\alpha$-filter in $L$ (see [10]).

1.4. **Lemma.** Let $L$ be a locale, $a \in L$. If $\uparrow x$ is compact in $L$ then $\uparrow(x \vee a)$ is compact in $\uparrow a$.

1.5. **Proposition.** Every closed sublocale of a wl-compact locale is a wl-compact locale.

**Proof.** Let $L$ be a frame, $a \in L$. Now, we have $1 = \bigvee(x^*: \uparrow x$ is compact in $L) = \bigvee(x^* \vee a: \uparrow(x \vee a)$ is compact in $\uparrow a) \leq \bigvee(y^\circ \geq a: \uparrow y$ is compact in $\uparrow a)$, where $y^\circ$ is the pseudocomplement in $\uparrow a$. In all we obtain that $\uparrow a$ is wl-compact.

1.6. **Proposition.** Let $L$ be a wl-compact locale. Then for each $1 \neq a \in F_C$ there exists $d \in D(L)$ such that $d \geq a$. Moreover, $L$ is dually atomic.

**Proof.** If $1 \neq a \in F_C$ then $\uparrow a$ is dually atomic because $\uparrow a$ is compact. Clearly, $D(\uparrow a) \subseteq D(L)$. Namely, if $d$ is a dual atom in $\uparrow a$ and $x > a$, $x \in L$ then $x \in \uparrow a$, i.e., $x = 1$. The rest follows from the fact that there exists $a \in F_C, a \neq 1$. Evidently, if $F_C \setminus \{1\} = \emptyset$ then $1 = \bigvee(a^*: a \in F_C) = \bigvee(a^*: a \in F_C \setminus \{1\}) = 0$, a contradiction. If $1 \neq b \in L$ then $\uparrow b$ is wl-compact, i.e., there is an element $m \in D(\uparrow b) \subseteq D(L)$.

1.7. **Proposition.** Let $L$ be a frame, $a, b \in L$ such that $\uparrow a, \uparrow b$ be wl-compact. Then $\uparrow(a \land b)$ is wl-compact.

**Proof.** If $\uparrow x$ is compact in $\uparrow a$, $\uparrow y$ is compact in $\uparrow b$ then $\uparrow(x \land y)$ is compact in $\uparrow(a \land b)$. Now, we have $\bigvee(x^{\circ 1}: \uparrow x$ is compact in $\uparrow a) = 1 = \bigvee(y^{\circ 2}: \uparrow y$ is compact in $\uparrow b)$, where $x^{\circ 1}, (y^{\circ 2})$ is the pseudocomplement in $\uparrow a (\uparrow b)$. Clearly, $x^{\circ 1} \land y^{\circ 2} \leq (x \land y)^\circ$, where $(x \land y)^\circ$ is the pseudocomplement in $\uparrow(a \land b)$. Evidently, $1 = \bigvee(x^{\circ 1} \land y^{\circ 2}: \uparrow x$ is compact in $\uparrow a, \uparrow y$ is compact in $\uparrow b) \leq \bigvee(z^{\circ}: \uparrow z$ is compact in $\uparrow(a \land b))$, i.e., $\uparrow(a \land b)$ is wl-compact.

1.8. **Remark.** It is interesting to note that wl-compact Hausdorff spaces are regular but there exists a wl-compact Hausdorff locale which is not regular (see [10], Prop. 2.4).

1.9. **Proposition.** If $L$ is a wl-compact regular locale then $a = \bigvee(x \ll a: x^* \in F_C)$ for each $a \in L$. 

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Proof. Let \( a \in L \). Now, we have \( a = \bigvee \{ x : x \ll a \} \), \( 1 = \bigvee \{ y : y^* \in F_c \} \). Clearly, \( a = \bigvee \{ x \land y : x \ll a, y^* \in F_c \} = \bigvee \{ z : z \ll a, z^* \in F_c \} \).

This suggests the following

1.10. Lemma. Let \( L \) be a locale. Then it holds:

(i) \( x \ll a, x^* \in F_c \Rightarrow x \ll a \) (\( x \) is way below \( a \) — see [4]).

(ii) If \( L \) is a regular \( w \)-compact locale then \( x \ll a \iff x \ll a, x^* \in F_c \).

Proof. (i) Let \( x \ll a, x^* \in F_c \) and \( S \subseteq L \) be a directed set such that \( a \leq \bigvee S \). Then \( x^* \lor \bigvee S = 1 \), i.e., there is \( s \in S \) such that \( x^* \lor s = 1 \) and we have \( x \leq s \).

(ii) Since \( L \) is a regular \( w \)-compact frame we have from 1.9 and 1.10 (i) that \( L \) is continuous, i.e., the space \( (P(L), O(P(L))) \) is a locally compact Hausdorff space. Now, let \( x \ll a \). Then there exists by [5], 4.2 a compact set \( K \subseteq P(L) \) such that \( x \subseteq K \subseteq a \). Clearly, it is easy to check that \( P(L) \setminus K \in F_c \) and we have \( P(L) \setminus K \subseteq x \), i.e., \( x \ll a, x^* \in F_c \).

1.11. Corollary. Let \( L \) be a regular locale. Then \( L \) is continuous iff \( L \) is a \( w \)-compact locale.

Proof. It follows from 1.10 and 1.9.

1.12. Lemma. If \( L \) is a \( w \)-compact locale then for each \( a \in F_c \) there exists \( x \in F_c \) such that \( x \ll a \).

Proof. Evidently, \( \bigvee \{ x^* : x \in F_c \} = 1 \). Since \( \uparrow a \) is compact in \( L \) then there exists \( x \in F_c \) such that \( x^* \lor a = 1 \), i.e., \( x \ll a, x \in F_c \).

We call the attention to the fact that the proofs are in the category \( Frm \) of frames.

1.13. Proposition. If \( L \) is a locale then \( L \cong L \times 2 \), where 2 denotes the dyadic locale which has precisely two elements 0 and 1.

Proof. If \( i_1 : L \rightarrow L + 2 \), \( i_2 : 2 \rightarrow L + 2 \) are the canonical injections then each element in \( L + 2 \) has the form \( i_1(x) \) for some \( x \in L \). Namely, if \( \bar{x} \in L + 2 \) then \( \bar{x} = \bigvee_j (i_1(x_j) \land i_2(y_j), x_j \in L, y_j \in 2 \). Now, we have \( \bar{x} = \bigvee (i_1(x_j) \land i_2(y_j) : y_j = 0) \lor \bigvee (i_1(x_j) \land i_2(y_j) : y_j = 1) = \bigvee i_1(x_j) = i_1(\bigvee x_j) = i_1(x) \) for some \( x \in L \). The rest is obvious.

1.14. Proposition. A finite product of \( w \)-compact locales is \( w \)-compact.

Proof. It is enough to prove that a sum of two \( w \)-compact frames is \( w \)-compact.

The rest follows by an obvious induction.
Let \( L, K \) be wl-compact frames, \( i_1: L \to L + K, i_2: K \to L + K \) be the canonical injections. Let \( x \in L, y \in K, \uparrow x \) be compact in \( L, \uparrow y \) be compact in \( K \). Now, we have \( \uparrow x + \uparrow y \cong \uparrow (i_1(x) + i_2(y)), i.e., \( \uparrow (i_1(x) \vee i_2(y)) \) is compact because a sum of compact frames is compact. Evidently, \( \land (i_1(x) \lor i_2(y)) \land (i_1(x) \lor i_2(y)) \land \uparrow x \) is compact in \( L, \uparrow y \) is compact in \( K \) since \( \land (i_1(x) \lor i_2(y)) \land \uparrow y \) is compact in \( L, \uparrow y \) is compact in \( K \) = \( i_1(\land (i_1(x) \lor i_2(y)) \land \uparrow x) \) \( \uparrow y = \uparrow y \) is compact in \( L \). Hence \( \land (i_1(x) \lor i_2(y)) \land \uparrow y \) is compact in \( K \). Thus, \( \land (i_1(x) \lor i_2(y)) \land \uparrow y \) is compact in \( K \) = 1 because \( L \) and \( K \) are wl-compact.

1.15. Theorem. Let \( L_y, \gamma \in \Gamma \) be locales. Then the product \( \prod_L \gamma \) is compact in \( K \) = 1 because all \( L_y \) are wl-compact and \( \gamma \) are compact for all but finitely many \( \gamma \in \Gamma \).

Proof. \( \Rightarrow \): a) Let \( y_0 \in \Gamma \). Since \( \Sigma L_y \) is compact then there exists a dual atom \( D \) in \( \Sigma L_y \) which has the form \( D = \land (i(x); d_i \) is a dual atom in \( L_y, \gamma \in \Gamma \)). If we put \( x = i_0(0) \lor \land (i(x); \gamma \neq \gamma_0) \) then \( \uparrow x \) is compact (see 1.6), \( \uparrow x = L_{y_0} + 2, \) where \( \sum \gamma \neq \gamma_0 \).

b) Let \( D \) be the dual atom from the part a). Since \( \Sigma L_y \) is compact we have \( 1 = \land (a^*; \uparrow a \) is compact in \( \Sigma L_y \)). Now, there exists some \( a \in \Sigma L_y, \uparrow a \) is compact in \( \Sigma L_y \) such that \( a^* = D \), i.e., there exist indices \( \gamma_1, ..., \gamma_n \in \Gamma \) and elements \( x_i \in L_{\gamma_i} (i = 1, ..., n) \) such that \( i_{\gamma_1}(x_1) \wedge ... \wedge i_{\gamma_n}(x_n) \equiv d, \) \( i_{\gamma_1}(x_1) \wedge ... \wedge i_{\gamma_n}(x_n) \equiv d^* \).

Clearly, \( \sum i_{\gamma_1}(x_1) \wedge ... \wedge i_{\gamma_n}(x_n) \equiv i_{\gamma_1}(x_1) \wedge ... \wedge i_{\gamma_n}(x_n) \equiv b \equiv a, \) i.e., \( \uparrow b \) is compact in \( \Sigma L_y \).

Let \( \gamma \neq \gamma_i \) \( (i = 1, ..., n) \). We show that \( L_y \) is compact. If \( j \in L_y, \land y_j = 1 \) then \( \land y_j = 1, i.e., \land y_j = 1 \). Now, we have \( 1 = \land x_1(\land y_j) \lor x_1(x_1) \lor ... \lor x_n(x_n) \equiv b \equiv 1, \) \( b \equiv a, \) i.e., \( \uparrow b \) is compact in \( \Sigma L_y \).

\(-\leq:\) Let each \( L_y \) be compact. We denote \( \Gamma_0 \) the set of indices of all non-compact \( L_y \). Clearly, \( \Gamma_0 \) is finite and we have \( \sum_{\gamma \in \Gamma \setminus \Gamma_0} L_y \equiv \sum_{\gamma \in \Gamma_0} L_y + \sum_{\gamma \in \Gamma_0} L_y \). From 1.14 we know that \( \sum_{\gamma \in \Gamma \setminus \Gamma_0} L_y \) is compact and from Tychonoff theorem we have that \( \sum_{\gamma \in \Gamma \setminus \Gamma_0} L_y \) is compact and hence \( \Sigma L_y \) is again compact.

1.16. Theorem. Let \( L_y (\gamma \in \Gamma \) be locales. Then the sum \( \Sigma L_y \) is compact in \( \sum \gamma \) for all \( \gamma \in \Gamma \).

Proof. \( \Rightarrow \): Let \( \pi_y: \Pi L \to L_y \) be the canonical projections (in the category \( Frm \)) and let us put \( x_{\gamma_0} = \land (y \in L_\gamma; \pi_{\gamma_0}(y) = 0) \) for each \( \gamma_0 \in \Gamma \). Then \( \uparrow x_{\gamma_0} \equiv L_{\gamma_0} \) and \( \uparrow x_{\gamma_0} \) is compact (see 1.6).

\(-\leq:\) Let each \( L_y \) be compact and \( \uparrow y \gamma \) be compact in \( L_y \). Then \( \land y \gamma = \land (y \in L_\gamma; \pi_{\gamma}(y) = y_{\gamma}) \) is such that \( \land y \gamma \) is compact in \( \Pi L_y \) which can be easily verified. Now, we have \( \pi_{\beta}(\land y \gamma) = 0 \) for \( \beta \equiv \gamma \), \( \pi_{\gamma}(\land y \gamma) = y_{\gamma}. \) Evidently, \( \land y \gamma \) is compact in \( \Pi L_y \) = \( \land (y_{\gamma}; \uparrow y_{\gamma} \) is compact in \( L_y \)) = 1 because all \( L_y \) are wl-compact.
2. A note on almost compact locales

2.1. Lemma. If \( L \) is a locale and \( Q \subseteq L \) is an ideal maximal with respect to the property \( Q \iota = S_L \), then

(i) \( x \in Q \Rightarrow x^{**} \in Q \),
(ii) \( Q \) is prime in \( Id(L) \), i.e., \( x \land y \in Q \Rightarrow x \in Q \) or \( y \in Q \).

Proof. (i) If \( x \in Q \), \( x^{**} \notin Q \) then \( y \in Q \) exists such that \( 0 = (x^{**} \lor y)^* = x^* \land y^* = (x \lor y)^* \) a contradiction with the fact that \( x \lor y \in Q \subseteq S_L \).

(ii) If \( x \land y \in Q \), \( x \in L \setminus Q \), \( y \in L \setminus Q \) then \( x_1, y_1 \in Q \) exist such that \( (x \lor x_1)^* = 0 = (y \lor y_1)^* \). Now, we have \( 0 = (x^* \land x_1^*) \lor (y^* \land y_1^*) \geq (x^* \lor y^*) \land (x_1^* \lor y_1^*). \) If we put \( z_1 = x_1 \lor y_1 \) then \( z_1 \in Q \), \( z_1^* = x_1^* \land y_1^* \). Clearly, \( x^* \lor y^* \leq z_1^{**} \in Q \), i.e., \( x^* \lor y^* \in Q \). Now, we have that \( a = (x \lor y)^* \lor x^* \lor y^* \in Q \) and \( a^* = (x ^* \land y^*) \land (x \lor y)^* = 0 \), a contradiction with with \( a \in \subseteq S_L \).

2.2. Theorem. Let \( L_\gamma (\gamma \in \Gamma) \) be locales. Then the product \( \Pi L_\gamma \) is almost compact iff \( L_\gamma \) are almost compact for all \( \gamma \in \Gamma \).

Proof. \( \Rightarrow \): Let \( i_\gamma : L_\gamma \rightarrow \Sigma L_\gamma \) be the canonical injections, \( \gamma_0 \in \Gamma \) and \( S_{\gamma_0} \subseteq L_{\gamma_0} \) be such that \( \bigvee S_{\gamma_0} = 1 \).

We put \( S = \{i_{\gamma_0}(s) : s \in S_{\gamma_0}\} \). Clearly, \( S \subseteq \Sigma L_\gamma \), \( \bigvee S = 1 \) and by almost compactness there exists a finite set \( F \subseteq S \) such that \( \bigvee(F)^* = 0 \). Now, we have that there exists a finite set \( F_{\gamma_0} \subseteq S_{\gamma_0} \) such that \( 0 = \left[ \left[ \bigvee(i_{\gamma_0}(s) : s \in F_{\gamma_0}) \right]^* = \left[ i_{\gamma_0}(\bigvee(s : s \in F_{\gamma_0})) \right]^* = i_{\gamma_0}(\bigvee(s : s \in F_{\gamma_0}))^* \right] = i_{\gamma_0}(\bigvee(s : s \in F_{\gamma_0}))^* \). Since \( i_{\gamma_0} \) is dense then there exists a finite dense subset \( F_{\gamma_0} \subseteq S_{\gamma_0} \), i.e., \( L_{\gamma_0} \) is almost compact.

\( \Leftarrow \): If \( L_\gamma (\gamma \in \Gamma) \) are almost compact frames and if \( \Sigma L_\gamma \) is not almost compact then there exists a maximal ideal \( Q \) with regard to the property \( Q \subseteq S_{\Sigma L_\gamma} \) such that \( \bigvee Q = 1 \). Let \( Q_\gamma = \{x_\gamma \in L_\gamma : i_\gamma(x_\gamma) \in Q\} \). Since \( Q \) is an ideal, each \( Q_\gamma \) is an ideal, \( Q_\gamma \subseteq S_{L_\gamma} \). We put \( q_\gamma = \bigvee Q_\gamma \). Clearly, \( q_\gamma \neq 1 \) because \( L_\gamma \) is almost compact. If \( X = \bigvee(i_{\gamma}(q_\gamma) : \gamma \in \Gamma) \) then \( X \downarrow 1 \), \( Q \subseteq \downarrow X \). Namely, if \( i_{\gamma_1}(x_1) \land \ldots \land i_{\gamma_n}(x_n) \in Q \) then \( x_{\gamma_1} \) exists such that \( i_{x_{\gamma_1}}(x_1) \in Q \) because \( Q \) is prime. Now, we have \( i_{x_{\gamma_1}}(x_1) \leq \leq i_{x_{\gamma_1}}(q_{\gamma_1}) \), i.e., \( i_{x_{\gamma_1}}(x_1) \land \ldots \land i_{x_{\gamma_n}}(x_n) \in [X] \). On the other hand, \( 1 = \bigvee Q \subseteq \downarrow X = X \), a contradiction. Finally, \( \Sigma L_\gamma \) is almost compact.

2.3. Proposition. If \( L \) is an almost compact locale, \( a \in L \), then the closed sublocale \( \uparrow a \) is almost compact.

Proof. If \( x_i \in \uparrow a, \ \bigvee_{j=1}^n x_{ij} = 1 \) then \( (\bigvee x_{ij})^{**} = 1 \) for some finite set of \( x_{ij}, 1 \leq j \leq n \).

If \( z \land \bigvee_{j=1}^n x_{ij} \leq a \) then \( a^* \leq (z \land \bigvee_{j=1}^n x_{ij})^{**} = [z^{**} \land (\bigvee_{j=1}^n x_{ij})^{**}]^* = z^* \), i.e., \( z \leq z^{**} \leq a^{**} = a \). Now, we have \( (\bigvee x_{ij})^{**} = 1 \), where \( \otimes \) denotes the pseudo-complement in \( \uparrow a \).
2.4. Proposition. If $L$ is a locale, $j_i: L \rightarrow L_{j_i}$, $i \in \{1, \ldots, n\}$ are nuclei on $L$ such that the locales $L_{j_i}$ are almost compact then the locale $L_j$ is almost compact, where $j = \bigwedge_{i=1}^{n} j_i$.

Proof will be done for $n = 2$. Let $(j_1 \wedge j_2) (V(a_k: k \in I)) = 1$, $a_k \in L$. Since $L_{j_1}$ and $L_{j_2}$ are almost compact then a finite set $K \subseteq I$ exists such that $j_i(x) \wedge j_j(V(a_k: k \in K)) = j_i(0)$ implies $j_i(x) = j_i(0)$ for each $x \in L$, $i = 1, 2$.

If $(j_1 \wedge j_2)(x) \wedge (j_1 \vee j_2)(V(a_k: k \in K)) = (j_1 \wedge j_2)(0)$ then $j_i(x) \wedge j_j(V(a_k: k \in K)) = j_i(0)$, i.e., $j_i(x) = j_i(0)$ for $i = 1, 2$. Now, we have that $(j_1 \wedge j_2)(x) = (j_1 \wedge j_2)(0)$ and $L_{j_1 \wedge j_2}$ is almost compact.

2.5. Lemma. ([5]). If $L$ is a locale, $j \leq k$ are nuclei of $L$, $a, b \in L$ then

- (i) $k(a) \neq k(b) \Rightarrow j(a) \neq j(b)$,
- (ii) $k(a) > k(0) \Rightarrow j(a) > j(0)$ hold.

Proof. $j(a) = j(b) \Rightarrow k(a) = k(j(a)) = k(j(b)) = k(b)$.

Now we introduce a generalization of [8] on locales.

2.6. Proposition. Let $L$ be a locale, $A$ be a chain of nuclei of $L$ such that each nuclei $j \in A$ is not 1 and $L_j$ is almost compact. Then the set $G = \{g \in L: j(g) \text{ is dense in } L_j \text{ for some } j \in A\}$ has the finite intersection property.

Proof. Let $g_1, \ldots, g_n \in G$, $j_i(g_i)$ is dense in $L_{j_i}$, $1 \leq i \leq n$, $j_1 \leq j_2 \leq \ldots \leq j_n$. Then $j_i(g_n) > j_i(0)$ and from lemma 2.5 we have $j_{i-1}(g_n) > j_{i-1}(0)$. Since $j_i(g_{n-1})$ is dense in $L_{j_{n-1}}$, we have $j_{i-1}(g_{n-1}) > j_{i-1}(0)$. Consequently, $j_{n-2}(g_{n-1} \wedge g_n) > j_{n-2}(0)$. Now, we have $j_{n-2}(g_{n-2} \wedge g_{n-1} \wedge g_n) > j_{n-2}(0)$. Finally, we obtain $j_1(g_1 \wedge \ldots \wedge g_n) > j_1(0)$, i.e., $g_1 \wedge \ldots \wedge g_n \neq 0$.

2.7. Lemma. If $L$ is a Hausdorff locale, $1 \neq \alpha \in L$ such that $\uparrow \alpha$ is almost compact then for each dual atom $d \in D(L)$ such that $d \vee \alpha = 1$ there exists $h \in L$ with $d \vee h^* = 1$, $\alpha \vee h$ is dense in $\uparrow \alpha$.

Proof. Clearly, $1 = a \vee d = a \vee V(x: x \not\leq d)$, i.e., there exists $h \not\leq d$ such that $a \vee h$ is dense in $\uparrow \alpha$.

2.8. Lemma. If $L$ is a dually atomic almost compact Hausdorff locale and $A \subseteq L$ is a chain such that $a \in A$ implies $1 \neq a$, $\uparrow a$ is almost compact, then $\bigvee A \neq 1$.

Proof. From 2.6 we know that $G = \{g \in L: a \vee g \text{ is dense in } \uparrow a \text{ for some } a \in A\}$ has the finite intersection property, i.e., $\bigvee (g^*: g \in G) \neq 1$. Now, there exists a dual atom $d \in D(L)$ such that $d \geq g^*$ for all $g \in G$.
Let \( 1 = \bigvee A \). Then \( a \in A \) exists with \( a \lor d = 1 \), i.e., \( h \in L \) exists such that \( d \lor h^* = 1 \). \( a \lor h \) is dense in \( \uparrow a \). Evidently, \( h \in G \), i.e., \( d \geq h^* \), a contradiction.

Recall that a locale \( L \) is compact iff for each chain \( \{a_i\}_{i \in I} \), \( a_i \neq 1 \) for each \( i \in I \), is \( \bigvee a_i \neq 1 \).

2.9. Theorem. Let \( L \) be a Hausdorff locale. Then \( L \) is compact iff \( \uparrow a \) is almost compact for each \( a \in L \).

Proof. \( \Rightarrow \): It is evident.
\( \Leftarrow \): Clearly, \( L \) is almost compact and dually atomic. Namely, \( L = \uparrow 0 \) and \( \uparrow a \) is almost compact for each \( 1 \neq a \in L \), i.e., there exists an element \( d \) such that \( a \leq d \in D(\uparrow a) \subseteq D(L) \) (see [10], 2.13). The rest follows from 2.8.

Recall that a topological space \( T \) is locally almost compact if for each \( x \in T \) there exists a neighbourhood \( U(x) \) of \( x \) such that \( U(x) \) is almost compact. Equivalently, \( T \) is locally almost compact iff for each \( x \in T \) there exists an open set \( U \) such that \( x \in U \), \( \overline{U} \) is almost compact.

Let \( L \) be a locale. We put \( F_a = \{ x \in L : \uparrow x^* \) is almost compact\}. Clearly, \( D(L) \subseteq \subseteq F_a \) and each dense element lies in \( F_a \).

2.10. Proposition. Let \( T \) be a topological space. Then \( T \) is locally almost compact iff \( \bigvee (x^* : x \in F_a) = 1 \).

Proof is similar as for wl-compact spaces.

Definition. We say that a locale \( L \) is locally almost compact if \( \bigvee (x^* : x \in F_a) = 1 \).

Clearly, each wl-compact locale is locally almost compact and each almost compact locale is locally almost compact.

2.11. Lemma. Let \( L \) be a locale, \( l \in L_r \). Then \( \uparrow l \) is almost compact iff for each \( S \subseteq L \) such that \( \bigvee S = 1 \) there exists \( S' \subseteq S \), \( S' \) finite such that \( (l \lor \bigvee S')^* = 0 \).

Proof. \( \Rightarrow \): If \( S \subseteq L \), \( \bigvee S = 1 \) then there is \( S' \subseteq S \), \( S' \) finite such that \( (l \lor \bigvee S') \) is dense in \( \uparrow l \), i.e., \( y \land (l \lor \bigvee S') \leq l \) implies \( y \leq l \). If \( y \land (l \lor \bigvee S') = 0 \) then \( y \leq (l \lor \bigvee S') = l^* \lor \bigvee (S')^* \). Now, we have \( y = y \land l \leq l \lor l^* \lor (\bigvee S')^* = 0 \).
\( \Leftarrow \): If \( S \subseteq L \), \( \bigvee S = 1 \) then there exists \( S' \subseteq S \), \( S' \) finite such that \( (l \lor \bigvee S')^* = 0 \) = 0. If \( y \land (l \lor \bigvee S') \leq l \) then \( l^* \leq (y^* \lor (l \land \bigvee S')^*)^* = y^* \), i.e., \( y \leq y^* \leq l^{**} = l \).

2.12. Proposition. Let \( L \) be a locale which is not almost compact. Then \( L \) is locally almost compact iff \( F_a \) is an \( \alpha \)-filter.

Proof follows from 2.11.

2.13. Lemma. Let \( L \) be a regular locale, \( l \in L_r \). Then \( l \in F_c \) iff \( l \in F_a \).
Proof. \( F_c \subseteq F_a \). If \( l \in F_a \) then \( \uparrow l \) is almost compact and regular, i.e., \( \uparrow l \) is compact (see [10], 2.7). Now, we have that \( l \in F_c \).

2.14. Proposition. If \( L \) is a regular locally almost compact locale then \( L \) is \( \text{wl-compact} \).

Proof. Evidently, \( 1 = \bigvee(x^*: x \in F_a) = \bigvee(x^*: x^{**} \in F_a) = \bigvee(x^*: x^{**} \in F_c) \).

2.15. Proposition. If \( L \) is a locally almost compact locale then \( L \) has at least one semiprime element. Moreover, for each \( 1 \neq x \in L_r \), \( x \in F_a \) there exists \( p \in S(L) \) such that \( x \leq p \).

Proof. The Proposition can be proved similarly as 1.4.

3. The one-point extensions

Definition. (i) Let \( K \) be a locale and \( L \) be a dense sublocale in \( K \). Then we say that \( K \) is an extension of \( L \).

(ii) Let \( L \) be a locale, \( F \subseteq L \) be a filter on \( L \). The sublocale \( L_F \subseteq L + 2 \), generated by the set \( \{(l, 0): l \in L\} \cup \{(a, 1): a \in F\} \) is called a one-point extension of \( L \).

This construction is a special case of the “Artin glueing” construction for locales (see [12]).

Evidently, \( L \) is a dense sublocale of \( L_F \). We shall denote \( e_a = \bigvee(\varepsilon: (a, \varepsilon) \in L_F) \) for each \( a \in L \).

3.1. Lemma. If \( L \) is a locale then \( (a, \varepsilon)^* = (a^*, e_{a^*}) \) holds in \( L_F \).

Proof. We have \((a, \varepsilon) \wedge (a^*, e_{a^*}) = (0, 0)\) because \( 0 \notin F \). If \((a, \varepsilon)^* = (b, \beta)\) then \( b \leq a^* \) and \( \beta \leq \varepsilon_b \leq e_{a^*} \).

Now, we give an explicit description of the sets \( P(L_F) \) and \( D(L_F) \).

3.2. Proposition. Let \( L \) be a locale, \( F \subseteq L \) be a filter and \((a, \varepsilon) \in L_F \). Then the following propositions hold:

1. \((a, \varepsilon) \in P(L_F)\) iff \( a = 1, \varepsilon = 0 \) or \( a \in P(L), \varepsilon = e_a \).
2. \((a, \varepsilon) \in D(L_F)\) iff \( a = 1, \varepsilon = 0 \) or \( a \in D(L), \varepsilon = 1 \).

Proof. \( 1. \Rightarrow \) If \((a, \varepsilon) \in P(L_F)\) then \( a \in P(L) \cup \{1\} \). Namely, if \( a \neq 1, a \notin P(L) \) then \( x, y \in L \) exist such that \( x \wedge y \leq a, x \not\leq a, y \not\leq a \). Clearly, \((x, 0) \wedge (y, 0) \leq (a, \varepsilon), (x, 0) \not\leq (a, \varepsilon), (y, 0) \not\leq (a, \varepsilon), a \) contradiction.

If \( a = 1 \) then \( \varepsilon = 0 \). If \( a \neq 1, a \in P(L) \) then \((1, 0) \wedge (a, e_a) \leq (a, \varepsilon)\), i.e., \( e_a \leq \varepsilon \leq e_a \).
\[\Rightarrow: \text{Evidently, } (1, 0) \in D(L_F) \subseteq P(L_F). \text{ Consider } (a, \varepsilon_a) \text{ for some } a \in P(L). \text{ If } (x, \beta) \land (y, \gamma) \leq (b, \varepsilon_b) \text{ then } x \leq a \text{ or } y \leq a, \text{ i.e., } \beta \leq \varepsilon_a \text{ or } \gamma \leq \varepsilon_a. \text{ Now, we have } (a, \varepsilon_a) \in P(L_F).\]

2. The proof is similar.

3.3. Corollary. Let \( L \) be a locale, \( F \subseteq L \) be a filter of \( L \). Then \( L_F \) is a \( T_1 \)-locale if and only if \( L \) is a \( T_1 \)-locale and \( D(L) \subseteq F \).

\textbf{Proof.} \( \Leftarrow: \) Clearly, \( L \) is a \( T_1 \)-frame. If \( d \in D(L) \) then \( (d, \varepsilon_d) \in P(L_F) = D(L_F) \), i.e., \( \varepsilon_d = 1 \). We have \( d \in F \).

\( \Rightarrow: \) Let \( (a, \varepsilon) \in P(L_F) \). Clearly, \( (1, 0) \in D(L_F) \) and if \( a \neq 1, a \in P(L), \varepsilon = \varepsilon_a \) then \( a \in D(L) \subseteq F \), i.e., \( (a, \varepsilon) \in D(L_F) \).

3.4. Corollary. Let \( L \) be a locale. Then \( L_F \) is dually atomic if and only if for each \( 1 = f \in F \) there exists \( d \in D(L) \) such that \( f \leq d \).

\textbf{Proof.} \( \Rightarrow: \) If \( 1 = f \in F \) then \( (f, 1) \in L_F \) and \( (d, 1) \in D(L_F) \) exists such that \( (f, 1) \leq (d, 1) \), i.e., \( f \leq d, d \in D(L) \).

\( \Leftarrow: \) Let \( (a, \varepsilon) \neq (1, 1), (a, \varepsilon) \in L_F \). If \( \varepsilon = 0 \) then \( (a, \varepsilon) \leq (1, 0) \in D(L_F) \). If \( \varepsilon = 1 \), \( 1 = a \in F \) then \( d \in D(L) \) exists such that \( a \leq d \), i.e., \( (a, \varepsilon) \leq (d, 1) \in D(L_F) \).

3.5. Proposition. Let \( L \) be a locale, \( F \) be a filter of \( L \) and \( (a, \varepsilon) \in L_F \) then the following propositions hold:
1. \( a \in S(L) \Rightarrow (a, \varepsilon_a) \in S(L_F) \).
2. \( (a, \varepsilon) \in S(L_F) \Rightarrow a \in S(L) \lor \{1\} \).
3. \( (a, \varepsilon) \in S(L_F), F \) is an \( \alpha \)-filter of \( L \Rightarrow a \in S(L), \varepsilon = 1 \) or \( a = 1, \varepsilon = 0 \).

\textbf{Proof.} 1., 2. are evident.

3. Let \( (a, \varepsilon) \in S(L_F) \). If \( a = 1 \) then \( \varepsilon = 0 \). If \( a \neq 1, a \in S(L) \) then \( x \in F \) exists such that \( x^* \leq a \). We have \( (x, 1) \land (x^*, 0) \leq (0, 0) \), i.e., \( (x, 1) \leq (a, \varepsilon) \) and \( \varepsilon = 1 \).

3.6. Corollary. Let \( F \) be an \( \alpha \)-filter on a locale \( L \). Then \( L_F \) is an \( S \)-locale iff \( L \) is an \( S \)-locale.

\textbf{Proof.} \( \Rightarrow: \) \( L \) is a homomorphic image \( L_F \), i.e., \( L \) is an \( S \)-frame. \( \Leftarrow: \) If \( (a, \varepsilon) \in S(L_F) \) and \( a \neq 1 \) then \( a \in S(L) = D(L), \varepsilon = 1 \), i.e., \( (a, \varepsilon) \in D(L_F) \).

3.7. Proposition. \( L_F \) is spatial iff \( L \) is spatial.

\textbf{Proof.} \( \Rightarrow: \) If \( 1 = a \in L \) then \( (a, 0) = (1, 0) \land \land \{(p, \varepsilon_p) \geq (a, 0) : p \in P(L)\} \), i.e., \( a = \land \{p \geq a : p \in P(L)\} \).
\(\Leftarrow\): If \((a, \varepsilon) \neq (1, 1), (a, \varepsilon) \in L_{F}\) then \((a, \varepsilon) = \bigwedge\{(p, \varepsilon_p) \geq (a, \varepsilon) : (p, \varepsilon_p) \in P(L_{F})\}\) because \(a = \bigwedge\{p \geq a : p \in P(L)\}\).

3.8. **Proposition.** \(L_{F}\) is conjunctive iff for arbitrary two elements \(a, b \in L\) such that \(1 \neq a \leq b\) there exists \(c \in F\) such that \(a \vee c = 1, b \vee c = 1\) and \(F \setminus \{1\}\) is cofinal in \(L \setminus \{1\}\).

**Proof.** \(\Rightarrow\): If \(1 \neq a \leq b, a, b \in L\) then \((1, 1) \neq (a, 0) \leq (b, 0), \text{ i.e., } (c, \varepsilon) \in L_{F}\) exists such that \((a, 0) \vee (c, \varepsilon) = (1, 1), (1, 1) \neq (b, 0) \vee (c, \varepsilon)\). We have \(\varepsilon = 1, a \vee c = 1, b \vee c = 1\) and \(c \in F\).

If \(1 \neq b \in L\) then \((1, 1) \neq (1, 0) \leq (b, 0), \text{ i.e., } (c, \varepsilon) \in L_{F}\) exists with \((1, 0) \vee (c, \varepsilon) = (1, 1), (1, 1) \neq (b, 0) \vee (c, \varepsilon)\) and we have \(\varepsilon = 1, b \leq c \vee b = 1, c \vee \vee b \in F\).

\(\Leftarrow\): If \((a, \varepsilon), (b, \beta) \in L_{F}, (1, 1) \neq (a, \varepsilon) \leq (b, \beta)\) then we have the following cases:

a) If \(1 \neq a \leq b\) then \(c \in F\) exists such that \(a \vee c = 1, b \vee c = 1\), i.e., \((a, \varepsilon) \vee (c, 1) = (1, 1), (b, \beta) \vee (c, 1) = (1, 1)\).

b) If \(1 = a \leq b\) and \(1 \neq c \in F\) exists such that \(b \leq c\). We have \((1, 0) \vee (c, 1) = (1, 1), (b, \beta) \vee (c, 1) = (c, 1) \neq (1, 1)\).

c) If \(1 \neq a \leq b\) then \(\varepsilon = 1, \beta = 0\) and we have \((a, \varepsilon) \vee (1, 0) = (1, 1), (b, \beta) \vee (1, 0) = (1, 0) \neq (1, 1)\).

Finally, \(L_{F}\) is conjunctive.

3.9. **Lemma.** If \(L\) is a locale, \(F \subseteq L\) is a filter of \(L, x \in L_{F}\) then \(x \in F \iff (x^*, 0) \ll (1, 0)\).

**Proof.** \(\Rightarrow\): If \(x \in F\) then \((x, 1) \vee (1, 0) = (1, 1), (x^*, 0) \leq (1, 0), \text{ i.e., } (x^*, 0) \ll (1, 0)\).

\(\Leftarrow\): If \((x^*, 0) \ll (1, 0)\) then \((x, \varepsilon_x) \vee (1, 0) = (1, 1), \text{ i.e., } \varepsilon_x = 1\). We have \(x \in F\).

3.10. **Corollary.** \(F\) is an \(\alpha\)-filter iff \((1, 0) = \bigvee(z \in L_{F} : z \ll (1, 0))\).

Proof follows from 3.9.

3.11. **Theorem.** If \(L\) is a locale and \(F\) is a filter of \(L\) then the following propositions are equivalent:

1. \(L_{F}\) is a Hausdorff locale.
2. \(L\) is a Hausdorff locale and \(F\) is an \(\alpha\)-filter.
3. (i) \(a = \bigvee(x \sqcap a : x^* \in F)\) for each \(a \in L\),
   (ii) For each \(1 \neq a \in F\) there exists \(x \in F\) such that \(x \sqcap a\).

**Proof.** \(1 \Rightarrow 2\): Clearly, \(L\) is a Hausdorff frame and \((1, 0)\) is a dual atom in \(L_{F}\). Since \((1, 0) = \bigvee(z : z \ll (1, 0))\) we have that \(F\) is an \(\alpha\)-filter.
2 ⇒ 3: (i) If \( a \in L \) then \( a = \bigvee(x \in L : x \sqcap a) = \bigvee(y \land x : y* \in F, x \sqcap a) = \bigvee(z \in L : z \sqcap a, z* \in F) \).

(ii) If \( 1 \neq a \in F \) then \( x \leq a \) exists with \( x* \in F \). If we put \( z = a \land x* \) then \( z \leq a, z* \leq a \) because \( a* \lor x** \leq a \), i.e., \( z \in F, z \sqcap a \).

3 ⇒ 1: Let \( (1,1) \neq (a,\varepsilon) \in L_F \). If \( \varepsilon = 0 \) then \( (a,0) = \bigvee((x,\varepsilon) : x \sqcap a, x* \in F) = \bigvee((x,0) : (x*,1) \leq (a,0)) \). If \( \varepsilon = 1 \) then \( z \in F \) exists with \( z \sqcap a \). Clearly \( (a,1) = (a,0) \lor (z,1) = \bigvee((x,\beta) : (x,\beta) \sqcap (a,1)) \), i.e., \( L_F \) is a Hausdorff frame.

3.12. Theorem. If \( L \) is a locale, \( F \) is a filter of \( L \) then the following are equivalent:
1. \( L_F \) is regular.
2. (i) \( a = \bigvee(x \ll a : x* \in F) \) for each \( a \in L \).
   (ii) For each \( a \in F \) there exists \( x \in F \) such that \( x \ll a \).

Proof. 1 ⇒ 2: (i) If \( a \in L \) then \( (a,0) = \bigvee((x,\varepsilon) : (x,\varepsilon) \ll (a,0)) = \bigvee((x,\varepsilon) : (x*,\varepsilon*) \lor (a,0) = (1,1)) = \bigvee((x,\varepsilon) : x \ll a, x* \in F) \). Now, we have \( a = \bigvee(x : x \ll a, x* \in F) \).

(ii) If \( a \in F \) then \( (a,1) = \bigvee((x,\varepsilon) : (x*,\varepsilon*) \lor (a,1) = (1,1)) \). Clearly, \( (x,1) \leq (a,1) \) exists such that \( x* \lor a = 1 \), i.e., \( x \in F \) exists with \( x \ll a \).

2 ⇒ 1: Let \( (a,\varepsilon) \in L_F \). If \( \varepsilon = 0 \) then \( (a,0) = \bigvee((x,0) : x \ll a, x* \in F) = \bigvee((x,0) : (x,0) \ll (a,0)) \). If \( \varepsilon = 1 \) then \( x \in F \) exists with \( x \ll a \). We have \( (a,1) = (a,0) \lor (x,1) = \bigvee((y,\varepsilon) : (y,\varepsilon) \ll (a,1)) \).

4. The one-point compactifications

4.1. Proposition. If \( L \) is a non-compact locale then the locale \( L_{FC} \) is compact.

Proof. If \( \bigvee((x_i,\varepsilon_i) : i \in I) = (1,1) \) then there exists \( i_0 \in I \) with \( \varepsilon_{i_0} = 1 \), i.e., \( x_{i_0} \in F_C \). Clearly, \( x \) finite set \( K \subseteq I \) exists such that \( \bigvee((x_i : i \in K) \lor x_{i_0} = 1 \), i.e. \( \bigvee((x_i,\varepsilon_i) : i \in K) \lor (x_{i_0},1) = (1,1) \).

Definition. Let \( L \) be a non-compact locale. We say that \( L_{FC} \) is the one-point compactification of \( L \).

Evidently, if \( L \) is spatial then \( L_{FC} \) is the Alexandroff extension of \( L \).

4.2. Proposition. Let \( L \) be a non-compact locale. Then \( L_{FC} \) is a \( T_1 \)-locale iff \( L \) is a \( T_1 \)-locale.

Proof follows from 3.3 because \( D(L) \subseteq F_C \).

The following is a locale analogy of the Alexandroff compactification for Hausdorff spaces.
4.3. **Proposition.** Let $L$ be a non-compact locale. Then $L_{Fc}$ is a Hausdorff locale iff $L$ is a $wl$-compact Hausdorff locale.

Proof follows from 3.11.

4.4. **Corollary.** A $wl$-compact Hausdorff locale is a $T'_2$-locale.

**Proof.** Clearly, $L_{Fc}$ is a compact Hausdorff frame, i.e., $L_{Fc}$ is a $T'_2$-frame (see [10], 1.4) because $L_{Fc}$ is dually atomic. Since $L$ is a homomorphic image of $L_{Fc}$ we have that $L$ is a $T'_2$-frame.

4.5. **Proposition.** Let $L$ be a non-compact locale. Then $L_{Fc}$ is regular iff $L$ is $wl$-compact and regular.

**Proof.** $\Rightarrow$: It follows from 4.3 and from the fact that homomorphic images of regular frames are regular.

$\Leftarrow$: It follows from 1.9, 1.12 and 3.1.

4.6. **Corollary.** A $wl$-compact regular locale $L$ is spatial. Moreover, $L$ is completely regular.

**Proof.** If $L$ is non-compact then $L_{Fc}$ is spatial and completely regular, i.e., $L$ is spatial and completely regular.

4.7. **Proposition.** If $L$ is a locale which is not almost compact then $thr$ locale $L_{F_a}$ is almost compact.

**Proof.** If $\bigvee((x_i, \varepsilon_i): i \in I) = (1, 1)$ then $i_0 \in I$ exists with $\varepsilon_{i_0} = 1$, i.e., $x_{i_0}^* \in F_a$. Further, a finite set $K \subseteq I$ exists such that $[\bigvee(x_i: i \in K)]^* \wedge x_{i_0} = 0$, i.e., $[\bigvee((x_i, \varepsilon_i): i \in K) \vee (x_{i_0}, 1)]^* = \Lambda(x_i^*, \varepsilon_{x_i}) \wedge (x_{i_0}^*, 0) = (0, 0)$.

**Definition.** Let $L$ be a locale which is not almost compact. We say that $L_{F_a}$ is the one-point almost compactification of $L$.

4.8. **Proposition.** Let $L$ be a locale which is not almost compact. Then it holds:

1. $L_{F_a}$ is a $T_1$-locale iff $L$ is a $T_1$-locale.
2. $L_{F_a}$ is a Hausdorff locale iff $L$ is a Hausdorff locale which is locally almost compact.

**Proof.** 1. It follows from 3.3 because $D(L) \subseteq F_a$. 2. It follows from 3.11.

The proposition 4.8.2 is well known for spaces (see [8]).

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References