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On Approximation of Multifunctions with respect to the Vietoris Topology

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We consider pointwise approximation of semicontinuous multifunctions by semicontinuous step multifunctions with respect to the Vietoris topology.

1. Introduction

Approximation by step multifunctions has been studied by Beer [1] and Spakowski [7, 8]. In these papers the convergence is considered with respect to the Hausdorff metric, or more generally, with respect to the Hausdorff uniformity. Our aim is to examine the problem of approximation with respect to the Vietoris topology. In papers [3, 4] the authors have shown that one way to obtain measurable selectors in the nonseparable case is to consider Bochner type measurable multifunctions, i.e. multifunctions that arise as certain limits of sequences of multifunctions of a simple type. This is another reason for our study.

Let X be a set, Y a topological space and F a multifunction from X to Y, i.e. a map from X to the family of all subsets of Y. We say that a net $\{F_t, t \in T\}$ of multifunctions from X to Y approximates F (with respect to the Vietoris topology) if for every $x \in X$ $F_t(x)$ converges to $F(x)$ both in the upper and lower Vietoris topology, i.e. for every open subset U of Y there exists $t_0 \in T$ such that for all $t > t_0$ $F_t(x) \subset U$ provided $F(x) \subset U$, and $F_t(x) \cap U \neq \emptyset$ provided $F(x) \cap U \neq \emptyset$, respectively.

If X and Y are both topological spaces F is said to be lower (upper) semicontinuous at $x_0 \in X$ if for every open subset U of Y satisfying $F(x_0) \cap U \neq \emptyset$ ($F(x_0) \subset U$) there exists a neighbourhood V of x_0 such that for all $x \in V$ $F(x) \cap U \neq \emptyset$ ($F(x) \subset U$). F is said to be continuous at x_0 if F is both lower and upper semicontinuous at x_0. F is said to be (lower semi-, upper semi-) continuous if F is (lower semi-, upper semi-) continuous at each point of X. It is known and easy to prove that the sum $F_1 \cup F_2$ of two lower (upper) semicontinuous multifunctions F_1 and F_2 is lower (upper) semicontinuous too.

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We say that F is a step multifunction if the set of values of F is finite. For multifunctions defined on a topological T_1-space the problem of approximation by a net of step multifunctions is quite trivial. Namely, we will show that for every multifunction F from a T_1-space X to a topological space Y there exist a net of lower semicontinuous step multifunctions and a net of upper semicontinuous step multifunctions both approximating F. Indeed, let F be a multifunction as above and define T to be the family of all finite subsets of X upward directed by the inclusion. For $t \in T$ we define $F_t(x) = F(x)$ if $x \in t$ and $F_t(x) = Y$ otherwise in X, and $G_t(x) = F(x)$ if $x \in t$ and $G_t(x) = \emptyset$ otherwise in X. The F_t's are lower semicontinuous step multifunctions and the G_t's are upper semicontinuous step multifunctions. To prove the convergence note only that $F_t(x) = F(x)$ and $G_t(x) = F(x)$ whenever $t > t_0 = \{x\}$.

2. Results for upper semicontinuous multifunctions

Throughout this section X is a metrizable and separable space. In this case, we may and do assume that X is metrizable by a totally bounded metric (see [2], p. 335). Moreover, let $A_1 \subset A_2 \subset \ldots$ be a sequence of finite subsets of X such that A_n is $1/n$ – dense in X for every n.

Theorem 1. Let F be a continuous multifunction from X to a topological space Y. Then there exists a sequence of upper semicontinuous step multifunctions that approximates F.

Proof. For every n and $s \in A_n$ we define $F_{s,n}(x) = F(s)$ if $x \in \bar{B}(s, 1/n)$ and $F_{s,n}(x) = \emptyset$ otherwise in X, where $\bar{B}(s, 1/n)$ is the closed ball in X with center s and radius $1/n$, and

$$F_n(x) = \bigcup_{s \in A_n} F_{s,n}(x) \quad \text{for} \quad x \in X .$$

The F_n's are upper semicontinuous step multifunctions. It remains to prove that the sequence $\{F_n, n \in \mathbb{N}\}$ approximate F.

Let U be an open subset of Y and x_0 a point of X. Assume first that $F(x_0) \cap U = \emptyset$. There exists n_0 such that $F(x) \cap U \neq \emptyset$ for all x in $\bar{B}(x_0, 1/n_0)$, for F is lower semicontinuous. Pick any s_0 in $\bar{B}(x_0, 1/n_0) \cap A_{n_0}$. Then for all $n > n_0$

$$F_n(x_0) \supset F_{s_0,n}(x_0) = F(s_0) .$$

Hence $F_n(x_0) \cap U \neq \emptyset$ for $n > n_0$. It means that $F_n(x_0)$ converges to $F(x_0)$ in the lower Vietoris topology.

Now, assume that $F(x_0) \subset U$. There exists n_0 such that $F(x) \subset U$ for all x in $\bar{B}(x_0, 1/n_0)$, for F is upper semicontinuous. For all $n > n_0$ and s in $\bar{B}(x_0, 1/n) \cap A_n$ we have

$$F_{s,n}(x_0) = F(s) \subset U .$$

Hence $F_n(x_0) \subset U$ for $n > n_0$. It means that $F_n(x_0)$ converges to $F(x_0)$ in the upper Vietoris topology.

Remark. If, in the above proof, we replace the closed balls by the open ones we get a sequence of lower semicontinuous step multifunctions that approximate F again. Moreover, if the multifunction F has closed values then the values of the approximating multifunctions are closed, too.

Theorem 2. Let F be an upper semicontinuous multifunction from X to a topological space Y. Then there exists a sequence of lower semicontinuous step multifunctions that approximates F.

Proof. For every n and $s \in A_n$ we define

$$F_{s,n}(x) = \bigcup_{z \in B(s,1/n)} F(z) \quad \text{if} \quad x \in B(s, 1/n)\,,$$
$$= \emptyset \qquad\qquad \text{otherwise in } X\,,$$

where $B(s, 1/n)$ is the open ball in X with center s and radius $1/n$, and

$$F_n(x) = \bigcup_{s \in A_n} F_{s,n}(x) \quad \text{for} \quad x \in X\,.$$

The F_n's are lower semicontinuous step multifunctions. The property $F_n(x) \supset F(x)$ implies that $F_n(x)$ converges to $F(x)$ in the lower Vietoris topology. The upper semicontinuity of F implies that $F_n(x)$ converges to $F(x)$ in the upper Vietoris topology. Indeed, if $F(x) \subset U$ for some open set U in Y then for sufficiently large n and s in $B(x, 1/n)$ we have $F_{s,n}(x) \subset U$. This completes the proof.

Theorem 3. Let F be an upper semicontinuous multifunction from X to a topological normal space Y. Then there exists a sequence of upper semicontinuous step multifunctions that approximate F.

Proof. For every n and s in A_n we define

$$F_{s,n}(x) = \bigcup_{z \in B(s,1/n)} F(z) \quad \text{if} \quad x \in B(s, 1/n)\,,$$
$$= y \qquad\qquad \text{otherwise in } X$$

and

$$F_n(x) = \bigcap_{s \in A_n} F_{s,n}(x) \quad \text{for} \quad x \in X\,.$$

The $F_{s,n}$'s are upper semicontinuous step multifunctions. Consequently (see [5], Theorem 1, p. 179) the F_n's are upper semicontinuous step multifunctions, too. The property $F_n(x) \supset F(x)$ implies that $F_n(x)$ converges to $F(x)$ in the lower Vietoris topology. The upper semicontinuity of F implies that $F_n(x)$ converges to $F(x)$ in the upper Vietoris topology. This completes the proof.

3. Results for lower semicontinuous multifunctions

In this section we see that the approximation of lower semicontinuous multifunctions is rather complicated.

Let X and $A_1, A_2, \ldots$ be as in Section 2.

Theorem 4. Let F be a lower semicontinuous multifunction from X to a normed linear space Y. If the values of F are closed, convex and totally bounded subsets of Y with nonempty interior then there exists a sequence of lower semicontinuous step multifunctions that approximate F.

Proof. For every n and s in A_n we define

$$\begin{aligned} F_{s,n}(x) &= \bigcap_{z \in B(s,1/n)} F(z) \quad \text{if} \quad x \in B(s, 1/n)\,, \\ &= \emptyset \qquad\qquad\qquad \text{otherwise in } X\,, \end{aligned}$$

and

$$F_n(x) = \bigcup_{s \in A_n} F_{s,n}(x) \quad \text{for} \quad x \in X\,.$$

First, let us observe that

$$\bigcap_{z \in B(s,1/n)} F(z) \neq \emptyset \quad \text{for sufficiently large } n$$

(see [7], Lemma 4.1 and p.364). The F_n's are lower semicontinuous step multifunctions. The property $F_n(x) \subset F(x)$ implies that $F_n(x)$ converges to $F(x)$ in the upper Vietoris topology. Now, let $F(x) \cap U \neq \emptyset$ for some open set U in Y. There exists a point $p \in U \cap \mathit{int}\, F(x)$. From the proof of Lemma 4.1 cited above we infer that there exists a neighbourhood V of x such that

$$p \in \bigcap_{z \in V} F(z)\,.$$

Therefore for sufficiently large n, namely for n such that $V \supset B(s, 1/n)$, we have $p \in U \cap F(x)$. This proves that $F_n(x)$ converges to $F(x)$ in the lower Vietoris topology. The proof is finished.

Our last theorem shows that one can remove the assumption that the values of F have nonempty interior provided the space Y is finite dimensional.

Theorem 5. Let F be a lower semicontinuous multifunction from X to a finite dimensional space Y. If the values of F are convex and compact, then there exists a sequence of lower semicontinuous compact valued step multifunctions that approximate F.

Proof. For every n, k and s in A_n we define

$$F_{s,n,k}(x) = \bigcap_{z \in B(s,1/n)} \overline{B(F(z), 1/k)} \quad \text{if} \quad x \in B(s, 1/n)$$

$$= \emptyset \qquad \text{otherwise in } X\,,$$

where $B(F(z), 1/k)$ is the sum of all $B(y, 1/k)$ over y in $F(z)$, and define

$$F_{n,k}(x) = \bigcup_{s \in A_n} F_{s,n,k}(x) \quad \text{for} \quad x \in X\,.$$

The $F_{n,k}'$s are lower semicontinuous compact valued step multifunctions. It remains to prove that the sequence $\{F_{n,k}, (n, k) \in \mathbb{N} \times \mathbb{N}\}$ approximates F.

Let U be an open set in Y and $F(x) \subset U$. The compactness of $F(x)$ implies that

$$\bigcap_{z \in B(s,1/n)} \overline{B(F(z), 1/k)} \subset \overline{B(F(x), 1/k)} \subset U$$

for sufficiently large k and every n. From this we infer that $F_{n,k}(x)$ converges to $F(x)$ in the upper Vietoris topology.

Now, let $F(x) \cap U \neq \emptyset$. Then $B(F(x), 1/k) \cap U \neq \emptyset$. Let $r > 0$ and let us consider the following multifunction

$$G_r(x) = \overline{B(F(x), r)} \quad \text{for} \quad x \in X\,.$$

The values of G_r are compact convex sets with nonempty interior. Moreover it is a lower semicontinuous multifunction (see [6], Proposition 2.5 and 2.3). Note that $G_r(x) \cap U \neq \emptyset$. Hence there exists a point $p \in U \cap \mathit{int}\, G_r(x)$. Such as in the proof of Theorem 4, we infer that there exists a neighbourhood V of x such that

$$p \in \bigcap_{z \in V} G_r(z) \cap U\,.$$

Therefore for sufficiently large n and some $s \in A_n$

$$p \in \bigcap_{z \in B(s,1/n)} \overline{B(F(z), r)} \cap U\,.$$

For $r = 1/k$ we get

$$p \in \bigcap_{z \in B(s,1/n)} \overline{B(F(z), 1/k)} \cap U\,.$$

Hence $F_{n,k}(x) \cap U \neq \emptyset$ for sufficiently large n and every k. This proves that $F_{n,k}(x)$ converges to $F(x)$ in the lower Vietoris topology. The proof is finished.

The example below shows that the completeness assumption in Theorem 5 can not be omitted.

Example. For $a > 0$ let $F(a)$ be the halfline $\{(x, y) \in \mathbb{R}^2\colon y = ax \text{ and } x > 0\}$. Then F is a lower semicontinuous (not upper semicontinuous) multifunction. We prove that F has no approximation by any sequence of step multifunctions.

Indeed, suppose that $\{F_n, n \in \mathbb{N}\}$ is a sequence of step multifunctions that approximate F. Let us take any $a > 0$. Then $F_n(a)$ converges to $F(a)$ in the upper Vietoris topology. On the other hand, let us observe that:

(*) there exists k such that for every $n > k$ there exists $r > 0$ such that $F_n(a) \cap \cap (\mathbb{R}^2 \setminus B((0, 0), r)) \subset F(a)$.

Otherwise, for every k there exist $n > k$ and a point $p_n \in F_n(a) \setminus F(a)$ such that $d((0, 0), p_n) > n$, where d is the metric of $\mathbb{R}^2$. From this we infer that there exist $n_1 < n_2 < \dots$ such that

$$p_{n_k} \in F_{n_k}(a) \setminus F(a)$$

and $d((0, 0), p_{n_k}) > n_k$. Let U be any open set in $\mathbb{R}^2$ such that

$$F(a) \subset U \quad \text{and} \quad p_{n_k} \notin U \quad \text{for all} \quad k\,.$$

Hence $F_{n_k}(a) \not\subset U$ for all k. We have a contradiction because $F_n(a)$ converges to $F(a)$ in the upper Vietoris topology.

Now, the property (*) implies that the set $\{F_n(a)\colon n \in \mathbb{N} \text{ and } a > 0\}$ is uncountable, a contradiction because the F_n's are step multifunctions.

References

[1] Beer G., The approximation of upper semicontinuous multifunctions by step multifunctions, Pacif. J. Math., 87 (1980), 11—19.

[2] Engelking R., General topology, PWN, Warszawa 1977.

[3] Hansell R. W., Extended Bochner measurable selectors, Math. Ann., 277 (1987), 79—94.

[4] Himmelberg C. J., Van Vleck F. S. and Prikry K., The Hausdorff metric and measurable selections, Topology and its Appl., 20 (1985), 121—133.

[5] Kuratowski K., Topology I, Academic Press and PWN, New York 1966.

[6] Michael E., Continuous selections I, Ann. of Math., 63 (1956), 361—382.

[7] Spakowski A., On approximation by step multifunctions, Comment. Math., 25 (1985), 363—371.

[8] Spakowski A., On approximation by step multifunctions without compactness conditions, to appear.