Acta Universitatis Carolinae. Mathematica et Physica

Herrmann Haase

Dimension of measures

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 31 (1990), No. 2, 29--34

Persistent URL: http://dml.cz/dmlcz/701949

Terms of use:

© Univerzita Karlova v Praze, 1990

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Dimension of Measures

H. HAASE*)

DDR

Received 11 March 1990

This paper summarizes a talk given at the 18th Winter school on Abstract Analysis — Section of Analysis. Some new inequalities are derived for the dimension of product measures, of the convolution of measures and for projection measures.

1. Definitions. Let (X, ϱ) be a metric space, $E \subseteq X$ and $d(E) = \sup\{\varrho(x, y); x, y \in E\}$. If $\mathscr{R}$ is any countable family of bounded subsets then define $D(\mathscr{R}) = \sup\{d(E); E \in \mathscr{R}\}$. Let $\mathscr{B}$ be the family of all closed balls $B(x, s)$ $(x \in X, s > 0)$. For real $r > 0$ let

$$A(E, r) = \{\mathscr{R};\ D(\mathscr{R}) \leqq r,\ E \subseteq \cup\mathscr{R}\}$$

$$B(E, r) = \{\mathscr{R};\ \mathscr{R} \subset \mathscr{B},\ D(\mathscr{R}) \leqq r,\ B_1, B_2 \in \mathscr{R} \Rightarrow B_1 \cap B_2 = \emptyset$$

$$\text{and if } x \text{ is the centre of } B \in \mathscr{R} \text{ then } x \in E\}\ .$$

For a Hausdorff function h (i.e. $h(0) = 0$, $q > 0 \Rightarrow h(q) > 0$, $q_1 \leqq q_2 \Rightarrow h(q_1) \leqq h(q_2)$, h continuous at 0) let

$$h(\mathscr{R}) = \sum_{E \in \mathscr{R}} h(d(E))\ .$$

Especially, if $\Lambda = \mathrm{id}_{R^+}$ and $a > 0$ then the power function Λ^a is a Hausdorff function.

The Hausdorff measure $h\text{–}m(E)$ for a set $E \subseteq X$ is defined as

$$h\text{–}m(E) = \lim_{r \to 0} h\text{–}m(E, r)$$

where

$$h\text{–}m(E, r) = \inf\{h(\mathscr{R});\ \mathscr{R} \in A(E, r)\}\ .$$

The Hausdorff dimension $\dim(E)$ is

$$\dim(E) = \inf\{a > 0;\ \Lambda^a\text{–}m(E) = 0\}\ .$$

Now let

$$h\text{–}M(E, r) = \sup\{h(\mathscr{R});\ \mathscr{R} \in B(E, r)\}$$

*) Sektion Mathematik, Ernst-Moritz-Arndt-Universität, DDR-2200, Greifswald

and

$$h\text{–}M(E) = \lim_{r \to 0} h\text{–}M(E, r).$$

Since $h\text{–}M$ gives only a premeasure we have to define

$$h\text{–}\hat{M}(E) = \inf\{\sum_i h\text{–}M(E_i);\ E \subseteqq \bigcup_i E_i\}$$

as the packing measure $h\text{–}\hat{M}$. The packing dimension of E is then given by

$$\operatorname{Dim}(E) = \inf\{a > 0;\ \Lambda^a\text{–}\hat{M}(E) = 0\}.$$

This approach as well as the notations are due to Tricot [6]. In the sequel we only consider Borel probability measures μ on x. Their Hausdorff resp. packing dimension is given by

$$\dim(\mu) = \inf\{\dim(E);\ \mu(E) > 0\}$$

and

$$\operatorname{Dim}(\mu) = \inf\{\operatorname{Dim}(E);\ \mu(E) > 0\}.$$

These definitions are well-known.

2. Some remarks concerning a local definition of dimension

For a compact metric space (X, ϱ) Ledrappier [5] defines the dimension δ of a measure μ as

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \delta \quad \mu \text{ a e.}$$

He proves that

(1) $\underline{\delta} \leqq \liminf\limits_{r \to 0} \dfrac{\log \mu(B(x, r))}{\log r}$

$\Rightarrow (\mu(E) > 0 \Rightarrow \dim(E) \geqq \underline{\delta})$

(2) $\limsup\limits_{r \to 0} \dfrac{\log \mu(B(x, r))}{\log r} \leqq \bar{\delta}$

$\Rightarrow$ There exist closed sets E_i with Kolmogoroff-dimension $K(E_i) \leqq \bar{\delta}$ and $\mu(\bigcup_i E_i) = 1$.

The Kolmogoroff-dimension of a set E is defined to be

$$K(E) = \limsup_{r \to 0} \frac{\log N(E, r)}{-\log r}$$

where $N(E, r)$ is the smallest cardinality of a covering of E by balls of radius r. Due to Tricot [6] it is

$$K(E) = \inf\{a > 0;\ \Lambda^a\text{-}M(E) = 0\}$$

and thus we conclude that

$$\mathrm{Dim}\,(\mu) \leqq \bar{\delta}\,.$$

3. Some Examples concerning calculations of $\dim(\mu)$

Example 1. Withers [7].

Let $p, q, r > 0,\ p < 1,\ q + r < 1$ and let

$$g_0, g_1\colon [\![0, 1]\!] \to [\![0, 1]\!]$$

defined to be $g_0(x) = qx$ and $g_1(x) = rx + 1 - r$ for all $x \in [\![0, 1]\!]$. For $S = [\![0, 1]\!] \times [\![0, 1]\!]$ the map $f\colon S \to S$ is defined by

$$f(x, y) = \begin{cases} (g_0(x), y/p) & y \leqq p \\ \left(g_1(x), \dfrac{y - p}{1 - p}\right) & y > p\,. \end{cases}$$

f generates an f-invariant measure μ which is the product measure of some measure ν on the line and the one-dimensional Lebesgue measure m^1. It is proved that

$$\dim(\mu) = 1 + \frac{p \log p + (1 - p) \log(1 - p)}{p \log q + (1 - p) \log r}\,.$$

The next example is more general.

Example 2. The p-balanced measure of Geronimo and Hardin [3]. Let $K \subseteqq R^n$ be a compact subset and $w_i\colon K \to K$, $i = 1, \ldots, N$ are continuous and contractive. Then $\{K, w_i;\ i = 1, \ldots, N\}$ is called to be a hyperbolic iterated function system on K and there exists a compact attractor $A \subseteqq K$ such that

$$A = \bigcup_{i=1}^{N} w_i(A)\,.$$

For given probabilities $p_i > 0$, $\sum_{i=1}^{N} p_i = 1$ there exists a unique measure μ such that for all continuous real valued functions f on A

$$\int_A f\,\mathrm{d}\mu = \sum_{i=1}^{N} p_i \int_A f w_i\,\mathrm{d}\mu\,.$$

Under some restrictive conditions, namely

(i) $w_i(A) \cap w_j(A) = \emptyset$ for $i \neq j$;

(ii) w_i is a similitude ($|w_i(x) - w_i(y)| = s_i|x - y|$);

(iii) $0 < s_1 \leqq s_2 \leqq \ldots \leqq s_N < 1$

it can be proved that

$$\dim(\mu) = \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} = \frac{\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i \log s_i}.$$

We remark at this point that the p-balanced measure μ is singular w.r.t. the measures Λ^r–$\hat{M}$ as well as Λ^r–m if $r \neq \dim(\mu)$ (for r with $\sum_{i=1}^{N} s_i^r = 1$).

Furthermore it seems to be interesting to know whether or not the formula for $\dim(\mu)$ does hold if the w_i's are contractive in the average, i.e. for all $x, y \in R^n$ there is some $r < 1$ such that

$$\prod_{i=1}^{N} s_i^{p_i} \leqq r.$$

Barnsley and Elton [1] have proved that the p-balanced measure still exists.

4. Some dimension inequalities

Let μ and v measures on R^m and R^k. $\mu \otimes v$ denotes the product measure on R^{m+k} Then we obtain

Proposition 1 (Haase [4])

$$\dim(\mu) + \dim(v) \leqq \dim(\mu \otimes v) \leqq \dim(\mu) + \mathrm{Dim}(v) \leqq$$
$$\leqq \mathrm{Dim}(\mu \otimes v) \leqq \mathrm{Dim}(\mu) + \mathrm{Dim}(v).$$

For sets this was proved by Tricot and most of his arguments are straightforward for the measure version except of $\dim(\mu) + \mathrm{Dim}(v) \leqq \mathrm{Dim}(\mu \otimes v)$. The main argument in this case is a variant of [2, Theorem 5.8, p. 72], namely

Proposition 2 *Let E be a plane set and let A be any subset of the x-axis. Suppose that if $x \in A$ then Λ^t–$m(E_x) > c$ (E_x is the x-section of E) for some constant c. Then*

$$\Lambda^{s+t}\text{–}\hat{M}(E) \geqq c\Lambda^s\text{–}\hat{M}(A).$$

This proposition allows us to prove that

$$\Lambda^{s+t}\text{–}\hat{M}(E) \geqq \int \Lambda^t\text{–}m(E_x)\, d\Lambda^s\text{–}\hat{M}(x)$$

holds, which proves the desired inequality. For simplicity let μ and v now be measures on the reals. For a Borel set B let

$$\mu * v(B) = \mu \otimes v(\{(x, y);\ x + y \in B\}).$$

What about the dimension of $\mu * v$?

Let's start with a lemma.

Lemma *Let* $K \subseteqq R$ *be compact with* $\dim(K) = \alpha$ $(\mathrm{Dim}(K) = \alpha)$ *then* $\dim(\{(x, y); x + y \in K\}) = 1 + \alpha$ $(\mathrm{Dim}(\{(x, y); x + y \in K\}) = 1 + \alpha)$.

Proof. Let $E = \{(x, y); 0 \leqq x \leqq 1, x + y \in K\}$. Then it is easy to see that

$$\dim(E) = \dim(\{(x, y);\ x + y \in K\}).$$

Let $F = I \times (-z + K)$ where $z = \min K$ and $I = [\![0, 1]\!]$. Then the map $f: E \to F$ defined by $f(x, y) = (x, y - (z - x))$ is a bijection and because of

$$|f(x_1, y_1) - f(x_2, y_2)| \leqq \sqrt{(3)}\,|(x_1, y_1) - (x_2, y_2)|$$

and

$$|f^{-1}(x_1, y_1) - f^{-1}(x_2, y_2)| \leqq \sqrt{(3)}\,|(x_1, y_1) - (x_2, y_2)|$$

by a direct calculation (where $|\cdot|$ is the Euclidean norm, f^{-1} the inverse map) f is Bi-Lipschitz. Since the Hausdorff dimension as well as the packing dimension are invariant under such maps,

$$\dim(\{(x, y);\ x + y \in K\}) = \dim(F).$$

Since $\dim(F) = \dim(I) + \dim(-z + K) = 1 + \dim(K)$, by an application of version of Proposition 1 the result follows. ∎

Now it is easy to see that the following is true.

Proposition 3

(1) $\dim(\mu * \nu) \geqq \dim(\mu \otimes \nu) - 1$;
(2) $\mathrm{Dim}(\mu * \nu) \geqq \mathrm{Dim}(\mu \otimes \nu) - 1$.

A further result in this direction is

Proposition 4

(1) $\max(\dim(\mu), \dim(\nu)) \leqq \dim(\mu * \nu)$;
(2) $\max(\mathrm{Dim}(\mu), \mathrm{Dim}(\nu)) \leqq \mathrm{Dim}(\mu * \nu)$.

Proof. Let $\varepsilon > 0$ and choose a Borel set $B \subseteqq R$ with $\mu * \nu(B) > 0$ and

$$\dim(B) < \dim(\mu * \nu) + \varepsilon.$$

Since $\mu \otimes \nu(\{(x, y); x + y \in B\}) > 0$ we obtain

$$\mu(\{x; \nu(\{y; x + y \in B\}) > 0\}) > 0$$

by Fubini's theorem. This implies that

$$\mu(\{x; \dim(\{y; x + y \in B\}) \geqq \dim(\nu)\}) > 0.$$

Hence there exists some x with

$$\dim(\{y; x + y \in B\}) \geqq \dim(\nu).$$

Consequently

$$\dim(B) \geqq \dim(\nu)$$

since Hausdorff (packing) measure is translation-invariant. Hence

$$\dim(\nu) < \dim(\mu * \nu) + \varepsilon \quad \text{for all} \quad \varepsilon > 0 .$$

Since μ and ν may be interchanged this yields (1) (resp. (2) by the same arguments). ∎

Let l_α be a line in R^2 with angle α to the x-axis and let proj_α denote the orthogonal projection on the line l_α. For a Borel set $B \subseteqq l_\alpha$ let $\nu_\alpha(B) = \mu(\text{proj}_\alpha^{-1}(B))$ be the projection measure. If $\dim(\mu)$ ($\text{Dim}(\mu)$) is given what can be said about its projection measures ν_α?

Take a Borel set $B \subseteqq l_\alpha$ such that $\nu_\alpha(B) > 0$. Then

$$\mu(\text{Proj}_\alpha^{-1}(B)) > 0 .$$

Obviously, the set $\text{proj}_\alpha^{-1}(B)$ consists of parallel lines l'_α orthogonal to l_α. Hence $\text{proj}_\alpha^{-1}(B)$ is an isometric strip to $B \times R$ and

$$\dim(\mu) \leqq \dim(B \times R) = \dim(B) + 1$$

is true for all such B, hence

Proposition 5

(1) $\dim(\mu) \leqq \dim(\nu_\alpha) + 1$ *for all* α.

(2) $\text{Dim}(\mu) \leqq \text{Dim}(\nu_\alpha) + 1$ *for all* α.

Unfortunately, as I has hoped, the projection theorem for sets (Falconer [2]) does not give news for $\dim(\nu_\alpha)$. The angle α may belong to the exceptional set for $\text{proj}_\alpha^{-1}(B)$ and for angles $\beta \neq \alpha$ $\text{proj}_\beta(\text{proj}^{-1}(B))$ may be the full line l_β.

References

[1] BARNSLEY, M. F., ELTON, J. H., A new class of Markov processes for image encoding, Adv. Appl. Prob. 20 (1988) 19—32.

[2] FALCONER, K. J., The geometry of fractal sets, Cambridge University Press 1985.

[3] GERONIMO, J. S. and HARDIN, D. P., An Exact Formula for the Measure Dimensions Associated with a class of Piecewise linear Maps, Constr. Approx. 5 (1989) 89—98.

[4] HAASE, H., On the dimension of product measures, Mathematika (submitted).

[5] LEDRAPPIER, F., Dimension of invariant measures, Proc. of the conference Ergodic Theory and Related Topics II, Teubner 1987, 116—124.

[6] TRICOT, C., Two definitions of fractional dimension, Math. Proc. Camb. Phil. Soc. 91 (1982) 57—74.

[7] WITHERS, WM. D., Analysis of invariant measures in dynamical systems by Hausdorff measure, Pacific J. of Math. 124 (1987) 385—400.