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Representation of Linear Functionals by a Trace on Algebras of Unbounded Operators Defined on Dualmetrizable Domains

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A representation of linear continuous functionals on certain algebras of unbounded operators in Hilbert spaces as trace functionals is given for the case, where the domain of the algebra is a dualmetric space. This is complementary to the metric case, which was investigated some years ago.

Introduction

Algebras of unbounded operators in a Hilbert space are essentially used as models in the quantum mechanics. One of the most important questions of this theory is the representability of the normal states (positive linear functionals) by trace operators. This would generalize the duality $\mathscr{K}(H)' = \mathscr{N}(H)$ between the set of compact operators and the nuclear operators, which is true in the bounded case, to the situation of unbounded operators. In this case, we have to consider instead of H a dense linear subspace D of H and instead of the C^*-algebra $\mathscr{L}(H)$ the maximal $*$-algebra $\mathscr{L}^+(D)$ of all (possibly unbounded) linear operators defined on D and having D as an invariant subspace (For a more precise definition see the next section.). We will show that the algebra $\mathscr{L}^+(D)$ defines a natural topology t on D. The case, where D becomes a metrizable space with respect to this topology was studied extensively in past by several authors and the problem concerning trace functionals could be solved affirmatively ([3], [7], [8], [9]). Some of these results were presented on the 9th Winter School of Abstract Analysis [2]. But the methods used there fail completely for dualmetrizable domains, i.e. for domains which are dual spaces of metrizable spaces. Such domains are not less important in the theory of algebras of unbounded operators, as it was shown in ([4]). However, using a complete different technique we can overcome the difficulties in the dualmetric case and we can give a representation theorem for functionals as one of the main results of this paper. Roughly spoken, this new technique is based on the finite dimensional characterizations of Hilbert spaces and of nuclear operators on Hilbert spaces.

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1. Notations and basic results

In all the following let H be any fixed Hilbert space and let D be any dense linear subspace of H. For any closable operator A in H we denote by $\bar{A}$, A^* and $D(A)$ its closure, adjoint and domain, respectively. The restriction of A^* to D will be denoted by A^+. Following [8], the maximal op*-algebra associated with D is defined by

$$\mathscr{L}^+(D) := \{A \in \mathrm{End}\,(D)\colon\ A^* \text{ exists},\ D \subseteqq D(A^*),\ A^*(D) \subseteqq D\}\,.$$

Obviously, $\mathscr{L}^+(D)$ is a $*$-algebra. The graph topology t on D is defined by the system of all seminorms

$$p_A(d) := \|Ad\|\,, \quad d \in D\,, \quad A \in \mathscr{L}^+(D)\,.$$

This topology coincides with the projective topology on D defined by the mappings $A\colon D \to H$ for $A \in \mathscr{L}^+(D)$. Since the identity 1_H belongs to $\mathscr{L}^+(D)$, the canonical embedding $J\colon D \to H$ is $t - \|\cdot\|$-continuous. Moreover, any operator $A \in \mathscr{L}^+(D)$ is $t - t$-continuous as a map from D into itself. From now on we restrict ourselves to closed domains, i.e., we suppose

$$D = \bigcap\{D(\bar{A})\colon A \in \mathscr{L}^+(D)\} = \bigcap\{D(A^{**})\colon A \in \mathscr{L}^+(D)\}\,.$$

Since the graph topology t is even generated by the system of the energetic norms

$$p_A^{erg}(d) = (\|Ad\|^2 + \|d\|^2)^{1/2}\,, \quad A \in \mathscr{L}^+(D)\,,$$

and since the domains $D(\bar{A})$ are Hilbert spaces with respect to the energetic norms, D is even the projective limit of Hilbert spaces. In particular, (D, t) is semireflxive and complete. For the purpose of the next chapter we construct a Gelfand tripel. Let D_b' be the strong dual space of (D, t). To avoid antilinear mappings we use the complex conjugate space D^+ of D' by replacing the original scalar multiplication in D' by the new scalar multiplication $(\lambda, x) \to \bar{\lambda}x$. Since any vector $h \in H$ defines a continuous linear functional f_h on D by $\langle d, f_h\rangle = (d, h)_H$, we get linear continuous embeddings

$$D \to H \to D_b^+\,.$$

In this sense we consider D as a linear subspace of D^+. Using the bipolar theorem it is easy to show that D is $\sigma(D', D)$ dense in D'. Since D is semireflexive, D is even $\sigma(D', D'')$ dense in D'. But then Mazur's theorem shows that D is also dense in D_b' and D_b^+ with respect to the strong topology.

2. Bounded subsets and bounding operators

To express the notion of continuity of linear functionals on algebras of operators, we need a suitable topology on $\mathscr{L}^+(D)$. Such a topology was introduced by

LASSNER in [8] and it is called the uniform topology τ_D. This topology is given by the system of all seminorms

$$p_M(A) := \sup \{|(Ad_1, d_2)|: d_1, d_2 \in M\}, \quad A \in \mathscr{L}^+(D),$$

where M runs over a basis of the absolutely convex and t-bounded subsets of D. In the context of the embedding $D \subseteq H \subseteq D_b^+$ the topology τ_D is the restriction of the bounded open topology on $\mathscr{L}(D, D_b^+)$ to $\mathscr{L}^+(D)$.

Now, it is very important that there is a close connection between the following subjects:

- the structure of τ_D
- the system of the bounded subsets of D
- the system of the bounded ellipsoids in D
- some sort of bounding operators
- the representation of linear continuous functionals by a trace.

Of course the connection between the first two questions is evident. Let us now introduce the notion of bounding operators. An operator $T \in \mathscr{L}^+(D)$ is called to be *bounding*, if the product XTY is a bounded operator in H for all operators $X, Y \in \mathscr{L}^+(D)$. The set of all bounding operators in $\mathscr{L}^+(D)$ is denoted by $\mathscr{B}(D)$. Each bounding operator T maps even H into D in a continuous way and the image $T(S_H)$ of the unit ball of H is a bounded subset of D. Moreover, we have the following external characterization of bounding operators.

Proposition ([4]). *An operator $T \in \mathscr{L}^+(D)$ is bounding if and only if there is a linear continuous extension $\tilde{T}: D_b^+ \to D$ of T.*

For the proof of this proposition once more the assumption of the closedness of D in the defined above sense is needed.

A bounded absolutely convex and closed subset M of D is called to be a *bounded Hilbert ball*, if the gauge functional

$$p_M(d) := \inf \{\varrho > 0: d \in \varrho M\}$$

satisfies the parallelogram equation

$$p_M(u + v)^2 + p_M(u - v)^2 = 2p_M(u)^2 + 2p_M(v)^2$$

for all $u, v \in$ *span* M. In this case, there is a scalar product $(\cdot, \cdot)_M$ associated to M. Now, the image $T(S_H)$ of the unit ball of H is a bounded ellipsoid for each bounding operator T. Conversely, using a theorem of KATO [6] it can be shown that the bounded ellipsoids are exactly the sets $M = T(S_H)$ for some positive bounding operator $T \in \mathscr{L}^+(D)$.

Recall that a locally convex space E is a *DF-space* (or a dualmetric space), if it has a countable fundamental system of the bounded subsets and if the intersection of any sequence of closed absolutely convex zero-neighbourhoods is a zero-neighbourhood provided that it absorbs all bounded subsets of E. This class of spaces contains the strong dual spaces of all Fréchet (= complete metrizable) spaces, and the dual of a DF-space is as a Fréchet space. But the structure of F- and DF-spaces is very different, and only Banach spaces can have both properties simultaneously. Now we come to the main result of this section.

Theorem. *If D is a metrizable (F-) or a dualmetrizable (DF-) space with respect to t, then each bounded subsets of D is contained in a bounded Hilbert ball.*

For metrizable spaces this was shown by KÜRSTEN [7] in 1986. The much more difficult DF-case was treated by the author in 1989 using the local characterization of ellipsoids. For details we refer to [4]. The description of the bounded subsets enables us to characterize the topology τ_D by bounding operators:

Corollary. *Under the assumptions of the theorem the uniform topology τ_D on $\mathscr{L}^+(D)$ is given by the system of all seminorms*

$$p_T(X) := \|TXT\| , \quad 0 \leqq T \in \mathscr{B}(D) .$$

Let $0 \leqq T \in \mathscr{B}(D)$ be given. If (E_λ) denotes its spectral measure, then a family of bounded projectors is defined by

$$P_\varepsilon = \int_\varepsilon^\infty dE_\lambda , \quad \varepsilon > 0 .$$

Now, it is easy to see, that for each $X \in \mathscr{L}^+(D)$ the convergence $p_T(X - P_\varepsilon X P_\varepsilon) = \|TXT - P_\varepsilon TXTP_\varepsilon\| \to 0$ holds true. But $P_\varepsilon X P_\varepsilon$ belongs to $\mathscr{B}(D)$. This yields:

Corollary. *Under the assumptions of the theorem the subspace $\mathscr{B}(D)$ is τ_D-dense in $\mathscr{L}^+(D)$.*

3. Trace functionals

In this section we will give a generalization of the formulae $\mathscr{K}(H)' = \mathscr{N}(H)$ to the case of unbounded operators. An operator $T \in \mathscr{B}(D)$ is called to be *nuclear bounding*, if the closure of the product XTY is a nuclear operator on H for all $X, Y \in \mathscr{L}^+(D)$. The set of all nuclear bounding operators is denoted by $\mathscr{N}(D)$.

Theorem. *If D is a DF-space then the following characterization of bounding nuclear operators holds true:*

$$\mathscr{N}(D) = \mathscr{B}(D) \, . \, \mathscr{N}(H) \, . \, \mathscr{B}(D) .$$

Here, $\mathscr{N}(H)$ denotes the nuclear operators on H.

The proof of the theorem is rather difficult and it will appear elsewhere. Remember that the space $\mathscr{K}(H)$ of the compact operators on H is the norm closure of the set $\mathscr{F}(H)$ of all operators on H having a finite rank. Therefore, we should introduce on D the set

$$\mathscr{F}(D) = \{F \in \mathscr{L}^+(D): rank\,(F) < \infty\}\,.$$

Corollary. *If D is a DF-space, then the equation*

$$(\mathscr{F}(D), \tau_D)' \cong \mathscr{N}(D)$$

holds algebraically, where the isomorphism is given by

$$\langle F, \varphi\rangle = tr\, FT_\varphi \quad \text{for some} \quad T_\varphi \in \mathscr{N}(D)\,.$$

Proof. Let us show that every operator $T \in \mathscr{N}(D)$ generates a continuous functional. In fact, T admits a representation $T = S_1NS_2$ for some $N \in \mathscr{N}(H)$ and $S_1, S_2 \in \mathscr{B}(D)$. Hence for every $F \in \mathscr{F}(D)$ we have the following estimation, where ν denotes the nuclear norm:

$$|tr\, FT| = |tr\, FS_1NS_2| = |tr\, S_2FS_1N| \leqq \|S_2FS_1\|\, \nu(N)\,.$$

But $p(F) = \|S_2FS_1\|$ is a τ_D-continuous seminorm on $\mathscr{L}^+(D)$ by the Corollary in section 2. Conversely, let any τ_D-continuous functional $\varphi \in \mathscr{F}(D)'$ be given. Then there is some $S \in \mathscr{B}(D)$ such that

$$|\varphi(F)| \leqq p_S(F) = \|SFS\|\,, \quad F \in \mathscr{F}(D)\,,$$

holds true. We consider the subspace $\mathscr{F}_S := S\,\mathscr{F}(D)\,S$ of $\mathscr{F}(H)$. For every element $F_0 = SFS \in \mathscr{F}_S$ we have

$$|\varphi_0(F_0)| := |\varphi(F)| \leqq \|SFS\| = \|F_0\|\,.$$

Hence, φ_0 is a norm-continuous functional on $\mathscr{F}_S$. By the Hahn-Banach-Theorem there is an extension of φ_0 to $\overline{\mathscr{F}(H)} = \mathscr{K}(H)$. But the functionals on $\mathscr{K}(H)$ admit a representation

$$\varphi_0(A) = tr\, AT_0$$

for some $T_0 \in \mathscr{N}(H)$ and all $A \in \mathscr{K}(H)$. Then the operator $T = ST_0S \in \mathscr{N}(D)$ gives the desired representation for φ. In fact, we have

$$\varphi(F) = \varphi_0(SFS) = tr\, SFST_0 = tr\, FST_0S = tr\, FT$$

for all $F \in \mathscr{F}(D)$.

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