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A Note on Reconstructing of Finite Trees from Small Subtrees

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Some basic observations on the reconstruction of a finite tree from its subtrees are given. It is conjectured that every tree with n vertices is determined by the collection formed of all its subtrees with the number of vertices $q = 1, 2, \ldots, k$ provided k is greater than $n/2$.

1. Preliminaries

All graphs considered are finite, simple and undirected. If $G = (X, E)$ is a graph and Y is a subset of the set of vertices X then G/Y denotes the induced graph with the set of vertices Y and the set of edges formed of all edges from E contained in Y. We use the symbol $\cong$ to denote the isomorphism of graphs.

For every two graphs H, G we define the frequency $frq(H, G)$ as the number of induced graphs of G isomorphic to H. Four types of similarity can be defined.

Definition. Let $\mathscr{C}$ be a class of graphs and let k be an integer. For two graphs G_1, G_2 we define $G_1 \sim^k G_2$ ($G_1 \sim^{\leq k} G_2$, respectively) iff $frq(H, G_1) = frq(H, G_2)$ for every graph H on k (on $\leqq k$, respectively) vertices, $G_1 \sim^k_{\mathscr{C}} G_2$ ($G_1 \sim^{\leq k}_{\mathscr{C}} G_2$, respectively) iff $frq(H, G_1) = frq(H, G_2)$ for every graph H from $\mathscr{C}$ having k (having $\leqq k$, respectively) vertices.

2. Reconstructing trees

In the reconstruction theory we are interested in the implication "$G_1 \sim G_2 \Rightarrow \Rightarrow G_1 \cong G_2$" where $\sim$ denotes some of the similarity types. There are some positive answers in the case of trees (the class of all trees will be denoted by $\mathscr{T}$).

Kelly [4] proved in 1957 that the implication holds for trees G_1, G_2 on n vertices in the case of similarity $\sim^{n-1}$, i.e. that trees are reconstructible from one vertex deleted subgraphs. Further, Harary and Palmer [3] proved in 1966 that the implica-

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tion holds for trees G_1, G_2 on n vertices in the case of similarity $\sim_{\mathscr{T}}^{n-1}$, i.e. that trees are reconstructible from endvertex deleted subtrees. In 1976, Giles [1] showed that the implication holds when G_1, G_2 are trees on n vertices ($n > 4$) and the type of similarity is $\sim_{\mathscr{T}}^{n-2}$, i.e. that trees are reconstructible from two-vertex deleted subtrees (with a trivial exception). He also studied k-vertex deleted subtrees reconstruction.

The reconstruction of a graph from "small" subgraphs was first investigated by Manvel [5]. The following theorem we regard as basic.

Theorem 1. *Let k, n be integers, T_1, T_2 trees on n vertices. The following three properties are equivalent*

(i) $T_1 \sim^k T_2$,
(ii) $T_1 \sim^{\geqq k} T_2$,
(iii) $T_1 \sim_{\mathscr{T}}^{\leqq k} T_2$.

Proof. Apply Theorem 1.7 from [8] to the class of trees.

One may try to add the fourth property to Theorem 1, namely the property (iv) $T_1 \sim_{\mathscr{T}}^{k} T_2$ We are going to show (in Theorem 2) that it is not possible.

Let $n_1, \ldots, n_s$ be integers and n their sum. We define the "star" graph $St(n_1, \ldots, n_s)$ as a tree consisting of s paths of lengths $n_1, \ldots, n_s$ all "emanating" from one common vertex. Obviously, this tree has $n + 1$ vertices.

Theorem 2. *Let $k \geqq 5$ be an integer. Then for every $n \geqq 2k - 2$ there exist two trees T_1, T_2 on n vertices such that $T_1 \sim_{\mathscr{T}}^{k} T_2$ but not $T_1 \sim_{\mathscr{T}}^{\leqq k} T_2$.*

Proof. Take $q = n - k + 1$, $T = St(1, k - 4, q)$ and suppose $T = (X, E)$ where $X = \{x_1, \ldots, x_{n-2}\} \cup \{a\}$, $E = \{\{x_i, x_{i+1}\}; i = 1, \ldots, n - 3\} \cup \{a, x_{q+1}\}$. Now, for $x, y \notin X$ let $X_1 = X \cup \{x\}$, $X_2 = X \cup \{y\}$, $E_1 = E \cup \{x, x_2\}$, $E_2 = E \cup \{x_{n-2}, y\}$ and finally $T_j = (X_j, E_j)$ for $j = 1, 2$. The trees T, T_1, T_2 are shown in Figure 1 for the case $k = 5$, $n = 8$.

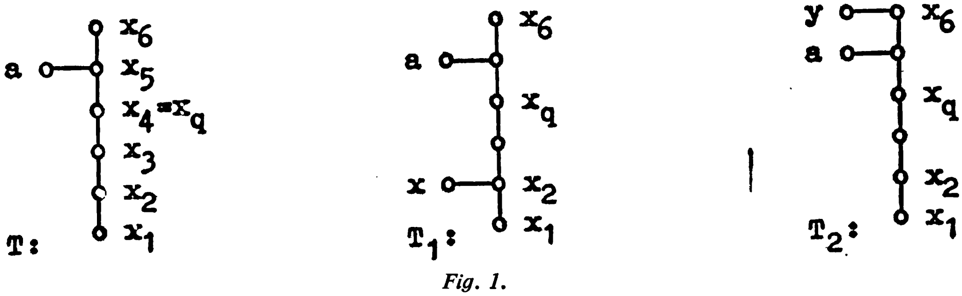

Fig. 1.

To prove the required properties we put $M_j = \{K \subset X_j;\ T_j/K$ is a tree and $\operatorname{card} K = k\}$ and we define $f: M_1 \to M_2$ as follows:

(1) if $x \notin K$ then $f(K) = K$,
(2) if $x \in K$ et $x_1 \notin K$ then $f(K) = \{x_{q-1}, \ldots, x_{n-2}\} \cup \{y\}$,
(3) if $x \in K$ et $x_1 \in K$ then $f(K) = \{x_q, \ldots, x_{n-2}\} \cup \{y, a\}$.

It can be easily shown that f is a bijection and for every $K \in M_1$ it is true that $T_1/K \cong \cong T_2/f(K)$. Thus, we have $T_1 \sim_{\mathscr{T}}^{k} T_2$. On the other hand, since T_1 has just two vertices of degree 3 while T_2 has only one such a vertex, we have $frq(St(1, 1, 1), T_1) = 2$ but $frq(St(1, 1, 1), T_2) = 1$ and thus $T_1 \sim_{\mathscr{T}}^{\leqq k} T_2$ does not hold.

The following theorem shows that the diameter of a tree cannot be reconstructed from "small" subtrees.

Theorem 3. *For every* $k \geqq 1$ *there exist two trees* T_1, T_2 *on* $3k + 1$ *vertices such that* $T_1 \sim_{\mathscr{T}}^{\leqq k} T_2$ *and* diam $T_1 \neq$ diam T_2.

Proof. Let $T = St(k - 1, k, k) = (X, E)$ where $X = \{x_1, \ldots, x_{2k}\} \cup \{y_1, \ldots, y_k\}$ and $E = \{\{x_i, x_{i+1}\}; i = 1, \ldots, 2k - 1\} \cup \{\{y_i, y_{i+1}\}; i = 1, \ldots, k - 1\} \cup \{x_k, y_1\}$. Now, let $T_1 = St(k, k, k) = (X_1, E_1)$ and $T_2 = St(k - 1, k, k + 1) = (X_2, E_2)$ where $X_1 = X \cup \{x\}$, $X_2 = X \cup \{w\}$, $E_1 = E \cup \{x, x_1\}$, $E_2 = E \cup \{x_{2k}, w\}$.

We take $M_j = \{K \subset X_j;$ card $K \leqq k$ and T_j/K is a tree$\}$ and we define f: $M_1 \to M_2$ as follows:

(1) if $x \notin K$ then $f(K) = K$,

(2) if $x \in K$ then $K = \{x_1, \ldots, x_r\} \cup \{x\}$, where $r < k$, and then we put $f(K) = \{x_{2k-r+1}, \ldots, x_{2k}\} \cup \{w\}$.

It is clear that for every $K \in M_1$ we have $T_1/K \cong T_2/f(K)$ but, on the other hand, diam $T_1 = 2k \neq 2k + 1 =$ diam T_2.

3. Bounds of reconstructability

To analyse the problem in more details we define the concept of a function rec.

Definition. Let $\mathscr{F}$ be a class of graphs. The integer valued function $rec_{\mathscr{F}}$ is defined as $rec_{\mathscr{F}}(n) = \min \{k;$ for every two graphs G_1, G_2 on n vertices from the class $\mathscr{F}$ it is true that $G_1 \sim^{\leq k} G_2 \Rightarrow G_1 \cong G_2\}$.

Müller [6] proved that there exists a class $\mathscr{H}$ containing asymptotically the most of graphs with n vertices (in the sense of limit) such that for every $r > 1/2$ there exists n_r such that for $n > n_r$ the inequality $rec_{\mathscr{H}}(n) < r \,.\, n$ holds. On the other hand, we have proved (see [7]) that for the class $\mathscr{G}$ of all graphs and for every $n \geqq 15$ $rec_{\mathscr{G}}(n) > (2n/3) - 6$.

Since Müllers class $\mathscr{H}$ does not contain the class $\mathscr{T}$ of all trees nothing can be derived for $rec_{\mathscr{T}}$.

Theorem 4. For every $n \geqq 6$ $rec_{\mathscr{T}}(n) > n/2$.

Proof. It was shown in [7] that for $T_1 = St(k - 1, k - 1, 1)$, $T_2 = St(k - 2, k, 1)$ with $2k$ vertices $(k \geqq 3)$ it is true that $T_1 \sim^k T_2$. Obviously T_1, T_2 are nonisomorphic.

Checking all the cases of trees with $\leqq 10$ vertices (according [2]) we obtained the exact values of $rec_{\mathscr{T}}(n)$ (see Table 1). Taking into acount this result and supposing that our construction from Theorem 4 is "the best possible" we formulate:

Table 1

($t(n)$ denotes the number of nonisomorphic trees on n vertices)

n	4	5	6	7	8	9	10
$t(n)$	2	3	6	11	23	47	106
$rec_{\mathcal{T}}(n)$	3	3	4	4	5	5	6

Conjecture. For every $n \geqq 4$ $rec_{\mathcal{T}}(n) = [n/2] + 1$, where $[\ ]$ denotes the integral part of a number.

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