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# On a Class of Universal Orlicz Function Spaces

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In [L-T I], J. Lindenstrauss and L. Tzafriri have given, for every $1 \leqq c < d < \infty$, examples of universal Orlicz sequence spaces $l^{\Psi}$ for every Orlicz sequence space $l^{\varphi}$ with $\varphi$ an Orlicz function $c$-convex and $d$-concave. Moreover it was proved that every $l^{\varphi}$ is isomorphic to a complemented subspace of $l^{\Psi}$.

The aim of this paper is to show a class of universal Orlicz function spaces $L^{\Psi}[0, 1]$, which are universal for a prefixed class of Orlicz sequence spaces $l^{\varphi}$. In our case we also get complemented subspaces.

As a consequence we deduce that every separable Nakano sequence space $l^{(p_n)}$ can be isomorphically represented as a weighted Orlicz sequence space $l^{\Psi}(w)$ for a suitable Orlicz function $\Psi$ and some weight sequence $w = (w_n)$ with finite sum, $\sum_{n=1}^{\infty} w_n < \infty$.

Let us start recalling some topics about Orlicz spaces. Let $\varphi$ be an Orlicz function (i.e. $\varphi: [0, \infty) \to [0, \infty)$ is a non-decreasing convex continuous function such that $\varphi(0) = 0$, $\varphi(x) > 0$ if $x > 0$, $\varphi(1) = 1$ and $\lim_{x\to\infty} \varphi(x) = \infty$). By $\varphi'^-$ and $\varphi'^+$ we mean the left and the right derivative of the function $\varphi$ respectively. Recall that a non-decreasing convex continuous function $\varphi$ verify that: $0 \leqq \varphi'^-(x) \leqq \varphi'^+(x)$ for every $x \geqq 0$, (Lemma 1.1 [K-R]).

Two Orlicz functions $\varphi$ and $\psi$ are equivalent at $\infty$, we write $\varphi \sim^{\infty} \psi$, (resp. at 0, $\varphi \sim^{0} \psi$) if there exist $K > 1$ and $x_0 > 0$ such that: $K^{-1}\varphi(x) \leqq \psi(x) \leqq K\varphi(x)$ for every $x \geqq x_0$, (resp. $x \leqq x_0$). $\varphi$ and $\psi$ are equivalent if they are equivalent at $\infty$ and 0.

Next Definition and Proposition can be seen in [M] and [Wo II].

**Definition.** *Let $1 \leqq c < d < \infty$. A Orlicz function $\varphi$ is said to be between $c$*

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*and d if $\varphi(x)/x^c$ is non-decreasing on $(0, \infty)$, and $\varphi(x)/x^d$ is non-increasing on $(0, \infty)$. By $\mathscr{K}(c, d)$ it is denoted the set of all Orlicz function between c and d.*

It easy to prove that $\varphi \in \mathscr{K}(c, d)$ if and only if

$$c \leqq \frac{x\varphi'^{-}(x)}{\varphi(x)} \leqq \frac{x\varphi'^{+}(x)}{\varphi(x)} \leqq d \quad \text{for every} \quad x > 0 .$$

**Proposition.** *Let $1 \leqq c < d < \infty$. For every Orlicz function $\varphi \in \mathscr{K}(c, d)$ there exists an Orlicz function $\psi \in \mathscr{K}(c, d)$ with continuous derivative, and a constant $K > 1$ which only depends of c and d, such that:*

$$K^{-1}\varphi(x) \leqq \psi(x) \leqq K\varphi(x) \quad \textit{for every} \quad x \geqq 0 . \tag{1}$$

Associated to an Orlicz function $\varphi \in \mathscr{K}(1, d)$, for some $d < \infty$, it is defined the following compact subsets of $C[0, \infty)$ (the space of all continuous functions on $[0, \infty)$ equipped with the compact-open topology):

$$E^0_{\varphi,\lambda} = \overline{\left\{\frac{\varphi(sx)}{\varphi(s)} : s \leqq \lambda\right\}} ; \quad E^0_{\varphi} = \bigcap_{\lambda>0} E^0_{\varphi,\lambda}$$

$$E^\infty_{\varphi,\lambda} = \overline{\left\{\frac{\varphi(sx)}{\varphi(s)} : s \geqq \lambda\right\}} ; \quad E^\infty_{\varphi} = \bigcap_{\lambda>0} E^\infty_{\varphi,\lambda} \cdot$$

If we consider the space $C[1, \infty)$ instead of $C[0, \infty)$, the following sets are also compact in $C[1, \infty)$:

$$F^\infty_{\varphi,\lambda} = \overline{\left\{\frac{\varphi(sx)}{\varphi(s)} : s \geqq \lambda\right\}} ; \quad F^\infty_{\varphi} = \bigcap_{\lambda>0} F^\infty_{\varphi,\lambda} \cdot$$

It is easy to prove if $K^{-1}\varphi(x) \leqq \psi(x) \leqq K\varphi(x)$ for every $x \geqq 0$, then for every $\phi \in E^\infty_{\varphi,\lambda}$ (or $E^\infty_{\varphi}$, or $E^0_{\varphi,\lambda}$, or $F^\infty_{\varphi,\lambda}$, or ...) there exists $\xi \in E^\infty_{\psi,\lambda}$ (or $E^\infty_{\psi}$, or $E^0_{\psi,\lambda}$, or $F^\infty_{\psi,\lambda}$, or ...) such that

$$K^{-2}\phi(x) \leqq \xi(x) \leqq K^2\phi(x) \quad \text{for every} \quad x \geqq 0 \tag{2}$$

For further information about these sets see [L-T I] and [H-P I and II].

Let $(\Omega, \mu)$ be a measure space. The Orlicz space $L^\varphi(\Omega)$ is the set of all real $\mu$-measurable functions $f$ on $\Omega$ such that:

$$I_\Omega(f/u) = \int_\Omega \varphi(|f(t)|/u)\, d\mu(t) < \infty \quad \text{for some} \quad u > 0 ,$$

equipped with the Luxemburg norm, $\|f\|_\varphi = \inf\{u > 0 : I_\Omega(f/u) \leqq 1\}$. Our attention is concentred in three cases: when $(\Omega, \mu) = ([0, 1], \mu)$ and $\mu$ is the Lebesgue measure, or $(\Omega, \mu) = (\mathbb{N}, \mu)$ and $\mu$ is the cardinal measure, or $(\Omega, \mu) = (\mathbb{N}, \mu)$ and $\mu(n) = w_n$, for an arbitrary sequence $(w_n)$ of positive numbers. So we get the Orlicz spaces $L^\varphi[0, 1]$, $l^\varphi$, and $l^\varphi(w)$ respectively. Moreover let us recall that $\varphi \sim^0 \psi$ (resp. $\varphi \sim^\infty \psi$), if and only if $l^\varphi = l^\psi$ (resp. $L^\varphi[0, 1] = L^\psi[0, 1]$) and the identity is an isomorphism.

From the existence of the averaging projection, it is known that if $\varphi$ is equivalent at 0 to $\phi \in E_\psi^\infty$ (resp. $\phi \in E_\psi^0$), then $l^\varphi = l^\phi$ is isomorphic to a complemented subspace of $L^\psi[0, 1]$,

$$l^\varphi = l^\phi \underset{c}{\lesssim} L^\psi[0, 1] \tag{3}$$

(resp. $l^\varphi = l^\phi \underset{c}{\lesssim} l^\psi$) (see [L-T I] and [H-P II]).

Lindenstrauss and Tzafriri proved the following result in [L-T II] (Theorem 4.b.12):

**Theorem.** *For every $1 \leqq c < d < \infty$ there exists an Orlicz function $\Psi = \Psi_{c,d}$ such that:*

i) $c \leqq x\, \Psi'(x)/\Psi(x) \leqq d$ *for all* $x \in [0, 1]$,

ii) *for every Orlicz function $\varphi$ with $c \leqq x\, \varphi'(x)/\varphi(x) \leqq d$ for all $x \in [0, 1]$, there exists a function in $E_\psi^0$ equivalent at 0 to $\varphi$. Hence, $l^\varphi \underset{c}{\lesssim} l^\Psi$.*

We are going to use of the argument of Lindenstrauss and Tzafriri to build a now Orlicz functions $\Psi$ near $\infty$ such that the Orlicz spaces $L^\Psi[0, 1]$ are universal spaces for a prefixed class of Orlicz sequence spaces $l^\varphi$:

**Proposition 1.** *Let $1 \leqq c < d < \infty$. There exists an Orlicz function $\Psi = \Psi_{c,d}$ with continuous derivative such that:*

i) $\Psi \in \mathscr{K}(c, d)$,

ii) *there exists a constant $K = K_{c,d} > 1$ such that for all Orlicz functions $M \in \mathscr{K}(c, d)$ there exists another Orlicz function $\phi \in F_\Psi^0$ verifying that:*

$$K^{-1} M(x) \leqq \phi(x) \leqq K\, M(x) \quad \textit{for every} \quad x \geqq 1\,.$$

**Proof.** Assume first that $c > 1$. We consider the subset of $C[1, \infty)$: $\mathscr{K} = \{\psi \in \mathscr{K}(c, d)$: $\psi$ has continuous derivative and $\psi'(1) = d\}$. If $\psi \in \mathscr{K}$, then $x^c \leqq \psi(x) \leqq x^d$ for every $x \geqq 1$. Moreover K is an equicontinuous set of $C[1, \infty)$ because:

$$|\psi(y) - \psi(x)| \leqq \psi'(\lambda)\, \lambda |y - x| \leqq \lambda^d d |y - x|$$

for every $\psi \in \mathscr{K}$ and $1 \leqq x \leqq y \leqq \lambda$. Since $\mathscr{K}$ is relative-compact set of $C[1, \infty)$, we can find a sequence $(\psi_n) \subseteq \mathscr{K}$ dense in $\overline{\mathscr{K}}$.

Put $\tau_n = 2^{2^{n-1}}$ $n = 1, 2, \ldots$ and define

$$\Psi(x) = \begin{cases} \psi_1(x) & \text{if } \; 1 \leqq x \leqq \tau_1 \\ \psi_n(x/\tau_n)\, \Psi(\tau_n) & \text{if } \; \tau_n \leqq x \leqq \tau_{n+1} \quad n = 1, 2, \ldots \end{cases} \tag{4}$$

We have, $\Psi'^{+}(\tau_n) = (d/\tau_n)\, \Psi(\tau_n)$ $n = 1, 2, \ldots$, and for $n \geqq 2$

$$\Psi'^{-}(\tau_n) = \psi'_{n-1}(\tau_n/\tau_{n-1})\, (1/\tau_{n-1})\, \Psi(\tau_{n-1}) = \tag{5}$$

$$= \frac{\psi'_{n-1}(\tau_n/\tau_{n-1})\, (\tau_n/\tau_{n-1})}{\psi_{n-1}(\tau_n/\tau_{n-1})}\, (1/\tau_n)\, \Psi(\tau_n) \leqq d(1/\tau_n)\, \Psi(\tau_n) = \Psi'^{+}(\tau_n)\,.$$

By (4) and (5) $\Psi$ is an Orlicz function and $\Psi \in \mathscr{K}(c, d)$. From (1) and (2), we may assume that $\Psi$ has a continuous derivative. Moreover, for all $n \in N$, $(\Psi(\tau_n x))/\Psi(\tau_n)) = = \psi_n(x)$ for every $1 \leqq x \leqq \tau_n$ what implies that the set $F_\Psi^\infty$ contains all functions which belong to $\mathscr{K}$.

Let M be now an Orlicz function belonging to $\mathscr{K}(c, d)$. From (1) we may assume that M has a continuous derivative. Choose $x_1 = x_{c,d}$ such that

$$x_1 \frac{d(c-1)}{c(d-1)} = 2$$

and $x_2$ such that

$$d(M(x_1) + M'(x_1)(x_2 - x_1)) = M'(x_1)\, x_2 .$$

It is easy to verify that

$$2 = x_1 \frac{d(c-1)}{c(d-1)} \leqq x_2 \leqq x_1 .$$

If put:

$$\phi_0(x) = \begin{cases} M(x) & \text{if } \; x_1 \leqq x \\ M(x_1) + M'(x_1)(x - x_1) & \text{if } \; x_2 \leqq x \leqq x_1 \\ (M(x_1) + M'(x_1)(x_2 - x_1))(x/x_2)^d & \text{if } \; 1 \leqq x \leqq x_2 , \end{cases}$$

then the Orlicz function $\phi(x) = \phi_0(x)/\phi_0(1) \in \mathscr{K}$ and therefore $\phi \in F_\Psi^\infty$. Moreover, taking

$$K = \max \{x_1^d, (d/c)\, x_1^{d-c}, (x_1^d + x_1^d\, dx_1)(1/2)^d\}$$

we get that $K^{-1}M(x) \leqq \phi(x) \leqq KM(x)$ for every $x \geqq 1$.

The case $c = 1$ has an easy solution in view of the following facts:

$$\mathscr{K}(1, d) = \{\psi(x)/x : \psi \in \mathscr{K}(2, d+1)\} \quad \text{and if} \quad \Psi(x) = \frac{\Psi_{2,d+1}(x)}{x},$$

then $F_\Psi^\infty = \{\phi(x)/x : \phi \in F_{\Psi_{2,d+1}}^\infty\}$. ■

**Theorem 2.** *Let* $1 \leqq c < d < \infty$. *There exists an Orlicz function* $\Psi = \Psi_{c,d}$ *with continuous derivative such that*

i) $\Psi \in \mathscr{K}(c, d)$,

ii) *for every Orlicz function* $M \in \mathscr{K}(c, d)$ *and for all* $\varphi \in E_M^\infty$ *it holds that* $l^\varphi \underset{c}{\subsetneq} L^\psi[0, 1]$.

**Proof.** Take $\Psi$ the Orlicz function of the above Proposition. By (3) we need only to prove that there exists $\phi \in E_\Psi^\infty$ such that $\phi \sim^0 \varphi$. By Proposition 1 and (2) we may assume that $M \in F_\Psi^\infty$. So for some scalar sequence $(s_n)$ convergent to $\infty$ we have:

$$M(x) = \lim_{n\to\infty} \frac{\Psi(s_n x)}{\Psi(s_n)} \quad \text{for every} \quad x \geqq 1 .$$

If $\varphi \in E_M^\infty$, then for some scalar sequence $(t_m)$ convergent to $\infty$:

$$\varphi(x) = \lim_{m\to\infty} \frac{M(t_m x)}{M(t_m)} \quad \text{for every} \quad x \geqq 0\,.$$

Therefore

$$\varphi(x) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{\Psi(t_m s_n x)}{\Psi(t_m s_n)}$$

for every $x \geqq 0$. Now, as $E_\Psi^\infty$ is a compact set we deduce that $\varphi \in E_\Psi^\infty$. ∎

We need two definitions before to present a remarkable consequence of the above Theorem. An Orlicz function $\varphi$ is *minimal* at $\infty$ (resp. at 0) if for function $\psi \in E_{\varphi,1}^\infty$ (resp. $\psi \in E_{\varphi,1}^0$), then $E_{\varphi,1}^\infty = E_{\psi,1}^\infty$ (resp. $E_{\varphi,1}^0 = E_{\psi,1}^0$). This concept of minimality was introduced by Hernández and Peirats in [H-P I] extending the one given by Lindenstrauss and Tzafriri, [L-T I]. Basic properties of minimal functions at $\infty$ or at 0 are the following: $E_{\varphi,1}^\infty = E_\varphi^\infty = E_\varphi^0 = E_{\varphi,1}^0$ and $\varphi \in E_\varphi^\infty$. The functions $x^p$ are minimal Orlicz functions, (for further information see [H-P I and II]).

Let $N = (\varphi_n)$ be a sequence of Orlicz functions. The vector space

$$l^N = \{(x_n)\colon \exists u > 0 \ \sum_{n=1}^{\infty} \varphi_n(|x_n|/u) < \infty\}$$

equipped with the norm

$$\|(x_n)\|_N = \inf\{u > 0\colon \sum_{n=1}^{\infty} \varphi_n(|x_n|/u) \leqq 1\}$$

is what is called *modular sequence space* (or also Musielak-Orlicz sequence space). If $N = (x^{p_n})$, where $(p_n)$ is a positive scalar sequence, then the space $l^{(p_n)}$ is called Nakano sequence space.

Recall that if $\varphi \in \mathcal{K}(c, d)$ (resp. $\varphi_n \in \mathcal{K}(c, d)$ for every $n \in \mathbb{N}$) for some $1 \leqq c < d < \infty$, then the unit vectors sequence, $(e_n)$, is base of $l^\varphi$ (resp. $l^N$, where $N = (\varphi_n)$).

**Corollary 3.** *Let $\Psi = \Psi_{c,d}$ be the Orlicz function of the above Theorem.*

i) *For every minimal function $\varphi \in \mathcal{K}(c, d)$ $l^\varphi$ is isomorphic to a complemented subspace of $L^\Psi[0, 1]$. In particular $l_p \underset{c}{\subsetsim} L^\Psi[0, 1]$ for all $p \in [c, d]$.*

ii) *For every sequence of minimal Orlicz functions $N = (\varphi_n)$ with $\varphi_n \in \mathcal{K}(c, d)$ for all $n \in \mathbb{N}$, there exists a weight sequence of finite sum $w = (w_n)$ such that $l^N \approx l^\Psi(w) \underset{c}{\subsetsim} L^\Psi[0, 1]$. In particular $l^{(p_n)} \underset{c}{\subsetsim} L^\Psi[0, 1]$, for every Nakano separable space $l^{(p_n)}$ with $p_n \in [c, d]$ for all $n \in \mathbb{N}$.*

**Proof.** ii) From Proposition 1 and the proof of Theorem 2 there exist $K > 1$ and $(\phi_n) \in E_\Psi^\infty$ such that $K^{-1}\phi_n(x) \leqq \varphi_n(x) \leqq K\phi_n(x)$ for every $x \geqq 0$ and $n \in \mathbb{N}$. Hence, $l^{(\phi_n)} = l^{(\varphi_n)}$ and the identity is an isomorphism (see [Wo I]. We can take an increasing scalar sequence $c = (c_n)$ with $\sum_{n=1}^{\infty} (1/\Psi(c_n)) < 1$ such that

$$\left| \frac{\Psi(c_n x)}{\Psi(c_n)} - \phi_n(x) \right| \leqq 1/2^n \quad \text{for all} \quad x \in [0, 1] \quad \text{and} \quad n \in \mathbb{N}\,.$$

If $w = (w_n = [1/\Psi(c_n)])$, then the canonical basis of $l^{(\phi_n)}$ is equivalent to the basis $(c_n e_n)$ of $l^{\Psi}(w)$, which implies that $l^{(\phi_n)}$ is isomorphic to $l^{\Psi}(w)$. Let $(A_n)$ be a sequence of measurable sets of $[0, 1]$, mutually disjoint, such that $\mu(A_n) = w_n$ for every $n \in \mathbb{N}$. Then the complemented subspace of $L^{\Psi}[0, 1]$ spanned by the sequence of characteristic functions of the sets $A_n$ is isometric to $l^{\Psi}(w)$. So $l^N = l^{(\phi_n)} \approx \approx l^{\Psi}(w) \underset{c\sim}{\subseteq} L^{\Psi}[0, 1]$. ■

**Remark.** In [H-Ru], has been proved that every modular separable space $l^N$ can be isomorphically represeted as a weighted Orlicz sequence space $l^{\Psi}(w)$ for some Orlicz function $\Psi$ and $w = (w_n)$ with $w_n \to \infty$. Notice that Corollary 3 part ii) gives a result of this kind for a weight sequence $w = (w_n)$ with $\sum_{n=1}^{\infty} w_n < \infty$.

**Remark.** We do not know whether there exists an Orlicz function $\Psi \in \mathscr{K}(c, d)$ such that for every function $\varphi \in \mathscr{K}(c, d)$ $\varphi \in E_{\Psi}^{\infty}$, and so $l^{\varphi} \underset{c\sim}{\subseteq} L^{\psi}[0, 1]$. Of course, if this kind of Orlicz function spaces exists, then the function $\Psi$ as in Proposition 1 will be one of them.

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