César Ruiz
On a class of universal Orlicz function spaces


Persistent URL: [http://dml.cz/dmlcz/701959](http://dml.cz/dmlcz/701959)

Terms of use:

© Univerzita Karlova v Praze, 1990

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library [http://project.dml.cz](http://project.dml.cz)
On a Class of Universal Orlicz Function Spaces

C. Ruiz

Spain

Received 11 March 1990

In [L-T I], J. Lindenstrauss and L. Tzafriri have given, for every $1 \leq c < d < \infty$, examples of universal Orlicz sequence spaces $l^p$ for every Orlicz sequence space $l^q$ with $\varphi$ an Orlicz function $c$-convex and $d$-concave. Moreover it was proved that every $l^q$ is isomorphic to a complemented subspace of $l^p$.

The aim of this paper is to show a class of universal Orlicz function spaces $L^p[0, 1]$, which are universal for a prefixed class of Orlicz sequence spaces $l^p$. In our case we also get complemented subspaces.

As a consequence we deduce that every separable Nakano sequence space $l^{(pn)}$ can be isomorphically represented as a weighted Orlicz sequence space $l^p(w)$ for a suitable Orlicz function $\Psi$ and some weight sequence $w = (w_n)$ with finite sum, $\sum_{n=1}^{\infty} w_n < \infty$.

Let us start recalling some topics about Orlicz spaces. Let $\varphi$ be an Orlicz function (i.e. $\varphi: [0, \infty) \to [0, \infty)$ is a non-decreasing convex continuous function such that $\varphi(0) = 0$, $\varphi(x) > 0$ if $x > 0$, $\varphi(1) = 1$ and $\lim_{x \to \infty} \varphi(x) = \infty$). By $\varphi^-$ and $\varphi^+$ we mean the left and the right derivative of the function $\varphi$ respectively. Recall that a non-decreasing convex continuous function $\varphi$ verify that: $0 \leq \varphi^-(x) \leq \varphi^+(x)$ for every $x \geq 0$, (Lemma 1.1 [K-R]).

Two Orlicz functions $\varphi$ and $\psi$ are equivalent at $\infty$, we write $\varphi \sim_{\infty} \psi$, (resp. at 0, $\varphi \sim_{0} \psi$) if there exist $K > 1$ and $x_0 > 0$ such that: $K^{-1} \varphi(x) \leq \psi(x) \leq K \varphi(x)$ for every $x \geq x_0$, (resp. $x \leq x_0$). $\varphi$ and $\psi$ are equivalent if they are equivalent at $\infty$ and 0.

Next Definition and Proposition can be seen in [M] and [Wo II].

Definition. Let $1 \leq c < d < \infty$. A Orlicz function $\varphi$ is said to be between $c$
and if $\varphi(x)/x^c$ is non-decreasing on $(0, \infty)$, and $\varphi(x)/x^d$ is non-increasing on $(0, \infty)$.

By $\mathcal{A}(c, d)$ it is denoted the set of all Orlicz function between $c$ and $d$.

It easy to prove that $\varphi \in \mathcal{A}(c, d)$ if and only if

$$c \leq \frac{x \varphi^-(x)}{\varphi(x)} \leq \frac{x \varphi^+(x)}{\varphi(x)} \leq d \quad \text{for every } x > 0.$$

**Proposition.** Let $1 \leq c < d < \infty$. For every Orlicz function $\varphi \in \mathcal{A}(c, d)$ there exists an Orlicz function $\psi \in \mathcal{A}(c, d)$ with continuous derivative, and a constant $K > 1$ which only depends on $c$ and $d$, such that:

(1) $K^{-1} \varphi(x) \leq \psi(x) \leq K \varphi(x)$ for every $x \geq 0$.

Associated to an Orlicz function $\varphi \in \mathcal{A}(1, d)$, for some $d < \infty$, it is defined the following compact subsets of $C[0, \infty)$ (the space of all continuous functions on $[0, \infty)$ equipped with the compact-open topology):

$$E_{\varphi, \lambda}^\infty = \left\{ \frac{\varphi(sx)}{\varphi(s)} : s \leq \lambda \right\}; \quad E_{\varphi, \lambda}^0 = \bigcap_{\lambda > 0} E_{\varphi, \lambda}^0$$

If we consider the space $C[1, \infty)$ instead of $C[0, \infty)$, the following sets are also compact in $C[1, \infty)$:

$$F_{\varphi, \lambda}^\infty = \left\{ \frac{\varphi(sx)}{\varphi(s)} : s \geq \lambda \right\}; \quad F_{\varphi, \lambda}^0 = \bigcap_{\lambda > 0} F_{\varphi, \lambda}^0$$

It is easy to prove if $K^{-1} \varphi(x) \leq \psi(x) \leq K \varphi(x)$ for every $x \geq 0$, then for every $\phi \in E_{\varphi, \lambda}^\infty$ (or $E_{\varphi, \lambda}^0$, or $E_{\varphi, \lambda}^0$, or $F_{\varphi, \lambda}^\infty$, or ...) there exists $\xi \in E_{\psi, \lambda}^\infty$ (or $E_{\psi, \lambda}^0$, or $E_{\psi, \lambda}^0$, or $F_{\psi, \lambda}^\infty$, or ...) such that

(2) $K^{-2} \phi(x) \leq \xi(x) \leq K^2 \phi(x)$ for every $x \geq 0$

For further information about these sets see [L-T I] and [H-P I and II].

Let $(\Omega, \mu)$ be a measure space. The Orlicz space $L^\varphi(\Omega)$ is the set of all real $\mu$-measurable functions $f$ on $\Omega$ such that:

$$I_\varphi(f/u) = \int_{\Omega} \varphi(|f(t)||u|) \, d\mu(t) < \infty \quad \text{for some } u > 0,$$

equipped with the Luxemburg norm, $\|f\|_\varphi = \inf \{u > 0 : I_\varphi(f/u) \leq 1\}$. Our attention is concentrated in three cases: when $(\Omega, \mu) = ([0, 1], \mu)$ and $\mu$ is the Lebesgue measure, or $(\Omega, \mu) = (\mathbb{N}, \mu)$ and $\mu$ is the cardinal measure, or $(\Omega, \mu) = (\mathbb{N}, \mu)$ and $\mu(n) = w_n$, for an arbitrary sequence $(w_n)$ of positive numbers. So we get the Orlicz spaces $L^\varphi[0, 1]$, $I^\varphi$, and $I^\varphi(w)$ respectively. Moreover let us recall that $\varphi \sim 0 \psi$ (resp. $\varphi \sim \infty \psi$), if and only if $I^\varphi = I^\psi$ (resp. $L^\varphi[0, 1] = L^\psi[0, 1]$) and the identity is an isomorphism.
From the existence of the averaging projection, it is known that if \( \varphi \) is equivalent at 0 to \( \phi \in E^0_\varphi \) (resp. \( \phi \in E^0_\varphi \)), then \( l^\varphi = l^\phi \) is isomorphic to a complemented subspace of \( L^\varphi[0, 1] \).

\[
(3) \quad l^\varphi = l^\phi \subseteq L^\varphi[0, 1]
\]

(resp. \( l^\varphi = l^\phi \subseteq l^\varphi \)) (see \([L-T I]\) and \([H-P II]\)).

Lindenstrauss and Tzafriri proved the following result in \([L-T II]\) (Theorem 4.b.12):

**Theorem.** For every \( 1 \leq c < d < \infty \) there exists an Orlicz function \( \Psi = \Psi_{c,d} \) such that:

i) \( c \leq x \Psi'(x)/\Psi(x) \leq d \) for all \( x \in [0, 1] \),

ii) for every Orlicz function \( \varphi \) with \( c \leq x \varphi'(x)/\varphi(x) \leq d \) for all \( x \in [0, 1] \), there exists a function in \( E^0_\varphi \) equivalent at 0 to \( \varphi \). Hence, \( l^\varphi \subseteq l^\varphi \).

We are going to use of the argument of Lindenstrauss and Tzafriri to build a new Orlicz functions \( \Psi \) near \( \infty \) such that the Orlicz spaces \( L^\Psi[0, 1] \) are universal spaces for a prefixed class of Orlicz sequence spaces \( l^\varphi \):

**Proposition 1.** Let \( 1 \leq c < d < \infty \). There exists an Orlicz function \( \Psi = \Psi_{c,d} \) with continuous derivative such that:

i) \( \Psi \in \mathcal{K}(c, d), \)

ii) there exists a constant \( K = K_{c,d} > 1 \) such that for all Orlicz functions \( \varphi \in \mathcal{K}(c, d) \) there exists another Orlicz function \( \phi \in E^0_\varphi \) verifying that:

\[
K^{-1} M(x) \leq \phi(x) \leq K M(x) \quad \text{for every} \quad x \geq 1.
\]

**Proof.** Assume first that \( c > 1 \). We consider the subset of \( C[1, \infty) \): \( \mathcal{K} = \{ \psi \in \mathcal{K}(c, d) : \psi \) has continuous derivative and \( \psi'(1) = d \} \). If \( \psi \in \mathcal{K} \), then \( x^c \leq \psi(x) \leq \lambda^d \) for every \( x \geq 1 \). Moreover \( K \) is an equicontinuous set of \( C[1, \infty) \) because:

\[
|\psi(y) - \psi(x)| \leq \psi'(\lambda) \lambda|y - x| \leq \lambda^d|y - x|
\]

for every \( \psi \in \mathcal{K} \) and \( 1 \leq x \leq y \leq \lambda \). Since \( \mathcal{K} \) is relative-compact set of \( C[1, \infty) \), we can find a sequence \( (\psi_n) \subseteq \mathcal{K} \) dense in \( \mathcal{K} \).

Put \( \tau_n = 2^{2^{n-1}} n = 1, 2, \ldots \) and define

\[
(4) \quad \Psi(x) = \begin{cases} 
\psi_1(x) & \text{if } 1 \leq x \leq \tau_1 \\
\psi_n(x/\tau_n) \Psi(\tau_n) & \text{if } \tau_n \leq x \leq \tau_{n+1} \quad n = 1, 2, \ldots
\end{cases}
\]

We have, \( \Psi'^{+}(\tau_n) = (d/\tau_n) \Psi(\tau_n) \) \( n = 1, 2, \ldots \), and for \( n \geq 2 \)

\[
\Psi'(\tau_n) = \psi_n^{-1}(\tau_n/\tau_{n-1}) (1/\tau_{n-1}) \Psi(\tau_{n-1}) = \psi_{n-1}(\tau_n/\tau_{n-1}) (1/\tau_{n-1}) \Psi(\tau_n) \leq d(1/\tau_n) \Psi(\tau_n) = \Psi^{+}(\tau_n) .
\]

By (4) and (5) \( \Psi \) is an Orlicz function and \( \Psi \in \mathcal{K}(c, d) \). From (1) and (2), we may
assume that $\Psi$ has a continuous derivative. Moreover, for all $n \in \mathbb{N}$, $(\Psi(\tau_n x))/\Psi(\tau_n)) = \psi_n(x)$ for every $1 \leq x \leq \tau_n$ what implies that the set $F^\omega_\Psi$ contains all functions which belong to $\mathcal{X}$.

Let $M$ be now an Orlicz function belonging to $\mathcal{X}(c, d)$. From (1) we may assume that $M$ has a continuous derivative. Choose $x_1 = x_{c,d}$ such that

$$x_1 \frac{d(c - 1)}{c(d - 1)} = 2$$

and $x_2$ such that

$$d(M(x_1) + M'(x_1)(x_2 - x_1)) = M'(x_1)x_2.$$  

It is easy to verify that

$$2 = x_1 \frac{d(c - 1)}{c(d - 1)} \leq x_2 \leq x_1.$$  

If put:

$$\phi_0(x) = \begin{cases} 
M(x) & \text{if } x_1 \leq x \\
M(x_1) + M'(x_1)(x - x_1) & \text{if } x_2 \leq x \leq x_1 \\
(M(x_1) + M'(x_1)(x_2 - x_1))(x/x_2)^d & \text{if } 1 \leq x \leq x_2,
\end{cases}$$

then the Orlicz function $\phi(x) = \phi_0(x)/\phi_0(1) \in \mathcal{X}$ and therefore $\phi \in F^\omega_\Psi$. Moreover, taking

$$K = \max \{x_1^d, (d/c) x_1^{d - c}, (x_1^d + x_1^d dx_1)(1/2)^d\}$$

we get that $K M(x) \leq \phi(x) \leq K M(x)$ for every $x \geq 1$.

The case $c = 1$ has an easy solution in view of the following facts:

$$\mathcal{X}(1, d) = \{\psi(x)/x : \psi \in \mathcal{X}(2, d + 1)\} \quad \text{and} \quad \Psi(x) = \frac{\Psi_{2,d+1}(x)}{x},$$

then $F^\omega_\Psi = \{\phi(x)/x : \phi \in F^\omega_{x_{2,d+1}}\}$. $\blacksquare$

**Theorem 2.** Let $1 \leq c < d < \infty$. There exists an Orlicz function $\Psi = \Psi_{c,d}$ with continuous derivative such that

i) $\Psi \in \mathcal{X}(c, d)$,

ii) for every Orlicz function $M \in \mathcal{X}(c, d)$ and for all $\varphi \in E^\omega_M$ it holds that $l^p \subseteq L^p[0, 1]$.

**Proof.** Take $\Psi$ the Orlicz function of the above Proposition. By (3) we need only to prove that there exists $\phi \in E^\omega_\Psi$ such that $\phi \sim \varphi$. By Proposition 1 and (2) we may assume that $M \in F^\omega_\Psi$. So for some scalar sequence $(s_n)$ convergent to $\infty$ we have:

$$M(x) = \lim_{n \to \infty} \frac{\Psi(s_n x)}{\Psi(s_n)} \quad \text{for every } x \geq 1.$$  

If $\varphi \in E^\omega_M$, then for some scalar sequence $(t_m)$ convergent to $\infty$:  

$$98$$
\[ \varphi(x) = \lim_{m \to \infty} \frac{M(t_m x)}{M(t_m)} \quad \text{for every } x \geq 0. \]

Therefore

\[ \varphi(x) = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\Psi(t_m s_n x)}{\Psi(t_m s_n)} \]

for every \( x \geq 0 \). Now, as \( E^0_\varphi \) is a compact set we deduce that \( \varphi \in E^0_\varphi \). \( \blacksquare \)

We need two definitions before to present a remarkable consequence of the above Theorem. An Orlicz function \( \varphi \) is minimal at \( \infty \) (resp. at 0) if for function \( \psi \in E^0_\varphi \) (resp. \( \psi \in E^0_\varphi \)), then \( E^0_{\varphi, \psi} = E^0_\varphi \) (resp. \( E^0_{\varphi, \psi} = E^0_\varphi \)). This concept of minimality was introduced by Hernández and Peirats in [H-P I] extending the one given by Lindenstrauss and Tzafriri, [L-T I]. Basic properties of minimal functions at \( \infty \) or at 0 are the following: \( E^0_{\varphi, \psi} = E^0_\varphi = E^0_\psi = E^0_{\varphi, \psi} \) and \( \varphi \in E^0_\varphi \). The functions \( x^p \) are minimal Orlicz functions, (for further information see [H-P I and II]).

Let \( N = (\varphi_n) \) be a sequence of Orlicz functions. The vector space

\[ l^N = \{(x_n): \exists u > 0 \sum_{n=1}^{\infty} \varphi_n(|x_n|/u) < \infty\} \]

is what is called modular sequence space (or also Musielak-Orlicz sequence space). If \( N = (x^p_n) \), where \( (p_n) \) is a positive scalar sequence, then the space \( l^{(p_n)} \) is called Nakano sequence space.

Recall that if \( \varphi \in \mathcal{K}(c, d) \) (resp. \( \varphi_n \in \mathcal{K}(c, d) \) for every \( n \in N \)) for some \( 1 \leq c < d < \infty \), then the unit vectors sequence, \( (e_n) \), is base of \( l^p \) (resp. \( l^N \), where \( N = (\varphi_n) \)).

**Corollary 3.** Let \( \Psi = \Psi_{c,d} \) be the Orlicz function of the above Theorem.

i) For every minimal function \( \varphi \in \mathcal{K}(c, d) \) \( l^p \) is isomorphic to a complemented subspace of \( L^\Psi [0, 1] \). In particular \( l^p \subset l^\Psi [0, 1] \) for all \( p \in [c, d] \).

ii) For every sequence of minimal Orlicz functions \( N = (\varphi_n) \) with \( \varphi_n \in \mathcal{K}(c, d) \) for all \( n \in N \), there exists a weight sequence of finite sum \( w = (w_n) \) such that \( l^N \approx l^\Psi (w) \subset L^\Psi [0, 1] \). In particular \( l^{(p_n)} \subset l^\Psi [0, 1] \), for every Nakano separable space \( l^{(p_n)} \) with \( p_n \in [c, d] \) for all \( n \in N \).

**Proof.** ii) From Proposition 1 and the proof of Theorem 2 there exist \( K > 1 \) and \( (\phi_n) \in E^0_{\varphi, \phi} \) such that \( K^{-1} \phi_n(x) \leq \phi_n(x) \leq K \phi_n(x) \) for every \( x \geq 0 \) and \( n \in N \). Hence, \( l^{(\phi_n)} = l^{(\phi_n)} \) and the identity is an isomorphism (see [Wo I]). We can take an increasing scalar sequence \( c = (c_n) \) with \( \sum_{n=1}^{\infty} (1/\Psi(c_n)) < 1 \) such that

\[ \left| \frac{\Psi(c_n x)}{\Psi(c_n)} - \phi_n(x) \right| \leq 1/2^n \quad \text{for all } x \in [0, 1] \quad \text{and } n \in N. \]
If \( w = (w_n = [1/\psi(c_n)]) \), then the canonical basis of \( l^{(\phi_n)} \) is equivalent to the basis 
\((c_n e_n) \) of \( l^{\psi}(w) \), which implies that \( l^{(\phi_n)} \) is isomorphic to \( l^{\psi}(w) \). Let \((A_n)\) be a sequence of measurable sets of \([0,1]\), mutually disjoint, such that \( \mu(A_n) = w_n \) for every \( n \in \mathbb{N} \). Then the complemented subspace of \( L^{\psi}[0,1] \) spanned by the sequence of characteristic functions of the sets \( A_n \) is isometric to \( l^{\psi}(w) \). So \( l^N \cong l^{(\phi_n)} \cong l^{\psi}(w) \in C L^{\psi}[0,1] \).

**Remark.** In [H-Ru], has been proved that every modular separable space \( l^N \) can be isomorphically represented as a weighted Orlicz sequence space \( l^{\psi}(w) \) for some Orlicz function \( \psi \) and \( w = (w_n) \) with \( w_n \to \infty \). Notice that Corollary 3 part ii) gives a result of this kind for a weight sequence \( w = (w_n) \) with \( \sum_{n=1}^{\infty} w_n < \infty \).

**Remark.** We do not know whether there exists an Orlicz function \( \psi \in \mathcal{K}(c, d) \) such that for every function \( \varphi \in \mathcal{K}(c, d) \) \( \varphi \in E^{*}_{\psi} \), and so \( l^{\psi} \in C L^{\psi}[0, 1] \). Of course, if this kind of Orlicz function spaces exists, then the function \( \psi \) as in Proposition 1 will be one of them.

This paper is part of the author's Doctoral Thesis prepared under the supervision of F. L. Hernández.

**References**


