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Strong Monotonicity and Lipschitz-Continuity of the Duality Mapping

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Introduction

Various kinds of differentiability of the norm and various kinds of smoothness and convexity properties of normed spaces can be described by means of the duality mapping (eg. [1], [3], [5], [7]). In the present paper, characterizations of another two geometric properties of normed spaces in terms of the duality mapping are given.

Definitions and notation

Let $X$ be a real normed linear space, $X^*$ its dual space, $S$ the unit sphere in $X$, $S^*$ the unit sphere in $X^*$. The value of $f \in X^*$ at $x \in X$ is denoted by $f(x)$ or $(f, x)$. By $J$ the duality mapping of $X$ into $2^{X^*}$ is denoted. $J$ is defined by $J(x) = \{ f \in X^* : \| f\| = \| x\|, f(x) = \| x\|^2 \}$. For $x \in X$, by $f_x$ any element of $J(x)$ is denoted. We say that $J$ is Lipschitz-continuous if $J$ is singlevalued and the mapping $x \to f_x$ is Lipschitz-continuous. We say that $J$ is strongly monotone if there exists $b > 0$ such that $(f_x - f_y, x - y) \geq b \| x - y \|^2$ for each $x, y \in X$, $f_x \in J(x)$, $f_y \in J(y)$. By $J^*$ we denote the duality mapping of $X^*$.

According to [2], $X$ is said to satisfy Lindenstrauss convexity condition ($X$ is (LC)), if

$$\exists d > 0 \forall x, y \in S : 2 - \| x + y \| \geq d \| x - y \|^2,$$

and $X$ is said to satisfy Lindenstrauss smoothness condition ($X$ is (LS)), if

$$\exists k > 0 \forall x \in S \forall y \in X : \| x + y \| + \| x - y \| \leq 2 + k \| y \|^2.$$

We say that $X$ satisfies the differentiability condition ($\delta$) ($X$ is ($\delta$)), if

$$\exists c > 0 \forall x \in S \forall y \in X \forall f_x \in J(x) : \| x + y \| - \| x \| - f_x(y) \leq c \| y \|^2.$$

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In [2], [6] the (LC) and (LS) conditions were defined in terms of the modulus of convexity $\delta$ and the modulus of smoothness $\varrho$ as follows: $X$ is (LC) if $\delta(t) \geq \alpha t^2$ for some $\alpha > 0$, and $X$ is (LS) if $\varrho(t) \leq \beta t^2$ for some $\beta > 0$.

**Theorem.** For a normed linear space $X$, the following conditions are equivalent.

(a) $J$ is Lipschitz-continuous,
(b) $X$ is (\(\delta\)),
(c) $X$ is (LS),
(d) $X^*$ is (LC),
(e) $J^*$ is strongly monotone.

Proof. (a) $\Rightarrow$ (b). Let $f, g \in S^*$ and $\lambda \in (0, 1)$, there exists $z \in X$ such that $\|z\| = \lambda |/2k$ and $(f - g, z) \geq \lambda \|f - g\| |z|$. Now $\|f + g\| = \sup\{(f + g, x) : x \in S\} = \sup\{(f, x + z) + (g, x - z) - (f - g, z) : x \in S\} \leq \sup\{\|x + z\| + \|x - z\| : x \in S\} - \lambda |f - g| |z| \leq 2 + k|z|^2 - \lambda |f - g| |z| = 2 - \lambda^2/4k |f - g|^2$. Thus $2 - \|f + g\| \geq 1/4k |f - g|^2$.

(A) $\Rightarrow$ (B). Follows from (a) $\Rightarrow$ (b).

(B) $\Rightarrow$ (C). Follows from (b) $\Rightarrow$ (c).

(C) $\Rightarrow$ (D). Follows from (c) $\Rightarrow$ (d).

(D) $\Rightarrow$ (E). For $x, y \in S$, we have $(f_x - f_y, x - y) = 2 - f_x(x + y) + 2 - f_y(x + y) \geq 2(2 - \|x + y\|^2) \geq 2d \|x - y\|^2$. This proves the strong monotonicity of $J$ on $S$ and yields $f_x(y) + f_y(x) \leq 2(1 - d \|x - y\|^2)$. Now for $\zeta \in [0, 1]$ we have $(\zeta f_x - f_y, \zeta x - y) = \zeta^2 - (\zeta f_x(y) + f_y(x)) + 1 \geq (\zeta - 1)^2 + 2d \|x - y\|^2 \geq 2d$. Thus $\|\zeta x - y\| \geq \zeta \|x - y\| + 1 - \zeta$, since $\zeta \leq 1$ and $d \leq \frac{1}{2}$ (otherwise the inequality in the definition of (LC) does not hold for $y = -x$). Since $\|\zeta x - y\| \geq \zeta \|x - y\| + 1 - \zeta$.
and \( \frac{1}{2}(a + b)^2 \leq a^2 + b^2 \) for any \( a, b \in \mathbb{R} \), we obtain \( \frac{1}{2}\|x - y\|^2 \leq \zeta^2\|x - y\|^2 + (1 - \zeta)^2 \). Therefore \( (\zeta f_x - f_y, \zeta x - y) \geq d\|\zeta x - y\|^2 \). Thus \( (f_x - f_y, x - y) \geq 2\|x - y\|^2 \) holds for any \( x, y \) such that \( \|x\| = 1 \) and \( \|y\| = 1 \), and therefore for every \( x, y \in X \).

(d) \( \Rightarrow \) (e). Follows from (D) \( \Rightarrow \) (E).

(e) \( \Rightarrow \) (a). Since \( f_x \in J(x) \Leftrightarrow \xi \in J^*(f_x) \) (where by \( \xi \) the canonical image of \( x \) in \( X^* \) is denoted), we have \( \|f_x - f_y\| \geq (f_x - f_y, x - y) = (\xi - \xi, f_x - f_y) \geq b\|f_x - f_y\|^2 \).

(E) \( \Rightarrow \) (D). We shall prove two lemmas first.

Lemma 1. Let \((x_n), (y_n) \subseteq S, (\lambda_n) \subseteq (0, \infty)\) be sequences such that \( \lambda_n \to 0 \) and \( 2 - \|x_n + y_n\| \leq \lambda_n\|x_n - y_n\|^2 \). Then (for sufficiently large \( n \))

\[
\|z_n - x_n\| \leq \frac{1}{4}\|y_n - x_n\| \tag{16}
\]

where \( z_n = (x_n + y_n)/\|x_n + y_n\| \) and \( f_{z_n} \in J(z_n) \).

Proof. Since \( \|x_n + y_n\| \to 2 \), \( z_n \) is defined if \( n \) is great enough. If \( \lambda_n \leq \frac{1}{4} \) then

\[
\|z_n - x_n\| = \|(y_n - x_n)/\|x_n + y_n\| + (2/\|x_n + y_n\| - 1) x_n\| \geq \|x_n - y_n\|/\|x_n + y_n\| - 2/\|x_n + y_n\| \geq \|x_n - y_n\|/\|x_n + y_n\| (1 - \lambda_n\|x_n - y_n\|) \geq \frac{1}{4}\|x_n - y_n\| \tag{17}
\]

and \( 1 - f_{z_n}(x_n) \leq 1 - f_{x_n}(x_n) \). Thus \( \|f_{z_n}(y_n) = 2 - \|x_n + y_n\| \leq \lambda_n\|x_n - y_n\|^2 \leq 16\lambda_n\|x_n - z_n\|^2 \).

Lemma 2. Let \( x, y \in S, x \neq -y, z = (x + y)/\|x + y\| \). Then \( \|x + z\| \geq \|x + y\| \).

Proof. Let \( v = (x + z)/\|x + y\| \), then \( v = (x + y + (x + y)/\|x + y\|)/\|x + y\| - y/\|x + y\| = (1 + \|x + y\|)/\|x + y\| \) \( z - 1/\|x + y\| y, \) so \( \|v\| \geq (1 + \|x + y\|)/\|x + y\| - 1/\|x + y\| = 1 \).

Proof of (E) \( \Rightarrow \) (D). Suppose (D) does not hold. Then there exist sequences \((x_n), (y_n), (\lambda_n)\) satisfying conditions of Lemma 1. Therefore

\[
16\|x_n - z_n\|^2 \geq \|x_n - y_n\|^2, \tag{18}
\]

\[
16\lambda_n\|x_n - z_n\|^2 \geq 1 - f_{z_n}(x_n), \tag{19}
\]

where \( z_n \) and \( f_{z_n} \) are as in Lemma 1. Using Lemma 2 and writing \( u_n = (x_n + z_n)/\|x_n + z_n\| \), \( f_{u_n} \in J(u_n) \), we obtain \( 2 - \|x_n + z_n\| \leq 2 - \|x_n + y_n\| \leq \lambda_n\|x_n - y_n\|^2 \leq 16\lambda_n\|x_n - z_n\|^2 \). So the sequence \((x_n), (z_n), (16\lambda_n)\) satisfy conditions of Lemma 1. Thus

\[
16\|u_n - z_n\|^2 \geq \|x_n - z_n\|^2, \tag{20}
\]

\[
16^2\lambda_n\|u_n - z_n\|^2 \geq 1 - f_{u_n}(z_n). \tag{21}
\]
Now, since \( f_{x_n}(u_n) = f_{x_n}(\|x_n + z_n\|/\|x_n + z_n\|) \geq f_{x_n}(x_n + z_n/2) \geq f_{x_n}(x_n) \), we have 
\[
1 - f_{x_n}(u_n) \leq 1 - f_{x_n}(x_n) \leq 16\lambda_n \|x_n - z_n\|^2 \leq 16\lambda_n \|u_n - z_n\|^2.
\]
It follows that 
\[
(f_{x_n} - f_{x_n}, z_n - u_n) = 1 - f_{x_n}(z_n) + 1 - f_{x_n}(u_n) \leq 512\lambda_n \|z_n - u_n\|^2.
\]
\( J \) is not strongly monotone.

(D) \( \Rightarrow \) (A). If \( X \) is (LC), then it is uniformly rotund, and so \( X^* \) is uniformly
smooth. Therefore \( X^* \) is reflexive, which is equivalent to \( \tilde{X} \) being dense in \( X^{**} \).
Since the inequality \( 2 - \|F + G\| \geq d\|F - G\|^2 \) holds for all \( F, G \in \mathcal{S} \), it holds
for all \( F, G \in S^{**} \). So \( X^{**} \) is (LC), too. Now by (D) \( \Rightarrow \) (E), it follows that \( J^{**} \) is
strongly monotone, and by (e) \( \Rightarrow \) (a), \( J^* \) is Lipschitz-continuous.

**Remark.** In fact, the equivalence (c) \( \Leftrightarrow \) (d) has been proved both in [6] and in [2],
where it was formulated and dealt with in terms of moduli of smoothness and convexity (and not using the duality mapping). Moreover, in [2] it was shown that the spaces satisfying (c) [(d)] are just those satisfying the upper [lower] weak parallelogram law. The equivalences of (a) \( \Leftrightarrow \) (b) \( \Leftrightarrow \) (c) (with the mapping \( x \rightarrow f_x \) defined in a slightly different way) were also given in [4].

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