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Operator Ideals and the Principle of Local Reflexivity

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Received 10 May 1992

0.1. Introduction

Our aim is, to give necessary and sufficient conditions which allow us to transform the local reflexivity principle of Lindenstrauss and Rosenthal [Li—Rt] from the canonical operator norm \( \| \cdot \| \) to \( p \)-Banach ideal norms \( \| \cdot \|_p \), where \((\mathcal{A}, \| \cdot \|_p)\) is a given \( p \)-Banach ideal \((0 < p \leq 1)\).

We will recognize two important facts:

- By a natural generalization of the \textit{weak} \( \mathcal{A} \)-\textit{local reflexivity principle} (introduced in [Oe1] and [Oe2]), we can omit the assumed maximality of the \( p \)-Banach ideal \((\mathcal{A}, \| \cdot \|_p)\) in theorem 2.9. of [Oe2]. Moreover we are allowed to consider all \( 0 < p \leq 1 \) and not only the case \( p = 1 \).

- There are interesting relations between the above mentioned generalization of weak local reflexivity and structural properties of the ideal \((\mathcal{A}, \| \cdot \|_p)\) such as \textit{accessibility} (introduced in [D] and [D—F]). Hence, tensor norms are involved (cf. [Oe2]).

0.2. Notation and terminology

We shall use the common notations of Banach-space-theory; in particular \( B_E \) denotes the closed unit ball of a normed space \( E \) (over \( K = \mathbb{R} \) or \( \mathbb{C} \)), \( E' \) the dual space of \( E \) and \( \mathcal{L}(E, F) \) is the class of all (continuous) operators between the normed spaces \( E \) and \( F \). Given \( T \in \mathcal{L}(E, F) \), the dual operator of \( T \) is denoted by \( T' \). \( NORM, \) \( BAN \) and \( FIN \) denotes the class of all normed spaces, Banach spaces and finite dimensional spaces respectively. \( FIN(E) \) is the class of all finite dimensional subspaces of a normed space \( E \) and \( COFIN(E) \) is the class of all finite codimensional subspaces of \( E \). Concerning operator ideals we follow Pietsch's book ([P]). If \((\mathcal{A}, \| \cdot \|_p)\) and \((\mathcal{B}, \| \cdot \|_q)\) are both normed operator ideals, we sometimes use the abbreviation \( \mathcal{A} = \mathcal{B} \) to indicate the equality \((\mathcal{A}, \| \cdot \|_p) = (\mathcal{B}, \| \cdot \|_q)\) and we write \( \mathcal{A}^d \) instead of \( \mathcal{A}^{\text{dual}} \). If \( T : E \to F \) is an operator, we indicate that it is a metric injection \( (\|Tx\| = \)  

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= \|x\|) by writing
\[ T : E \overset{1}{\to} F \]
and that it is a metric surjection \((F\text{ has the quotient norm of }E\text{ via }T)\) by
\[ T : E \overset{1}{\to} F. \]

If there exists an isometric isomorphism between the spaces \(E\) and \(F\), we write \(E \cong F\). For \(G \in \text{FIN}(E)\), \(J^E_G : G \overset{1}{\to} E\) denotes the canonical metric injection and for \(G \in \text{COFIN}(E)\), \(G\) closed, \(Q^E_G : E \overset{1}{\to} E|G\) denotes the canonical metric surjection.

We assume the reader to be familiar with the basics of the general theory of tensor norms as they are presented in \([Gr]\), \([D]\) and \([D-F]\). Another important tool to describe local properties of ideal components is given by the \textit{trace} on a normed space \(E\) which is the linearization of the duality bracket

\[ E' \times E \to K \]
\[ (a, x) \mapsto \langle x, a \rangle, \]
whence
\[ tr : E' \otimes E \to K \]
\[ \sum_{i=1}^{n} a_i \otimes x_i \mapsto \sum_{i=1}^{n} \langle x_i, a_i \rangle. \]

We recall that a Banach space \(E\) has the \textit{metric approximation property} (short: m.a.p.) if for all compact sets \(K \subseteq E\) and for all \(\varepsilon > 0\) there is a finite dimensional operator \(L \in \mathcal{F}(E, E)\) with \(\|L\| \leq 1\) such that \(\|Lx - x\| \leq \varepsilon\) for all \(x \in K\).

Finally we remember the important

\textbf{Principle of local reflexivity:} Let \(M\) and \(F\) be Banach spaces, \(M\) finite dimensional and \(T \in \mathcal{L}(M, F^*)\). Then for every \(\varepsilon > 0\) and \(N \in \text{FIN}(F^*)\) there is an \(R \in \mathcal{L}(M, F)\) such that

(i) \(\|R\| \leq (1 + \varepsilon) \|T\|\)
(ii) \(\langle Rx, b \rangle = \langle b, Tx \rangle \forall (x, b) \in M \times N\)
(iii) \(j_F Rx = Tx \forall x \in M\) with \(Tx \in j_F(F)\).

\textbf{1. The weak \((\mathcal{A})\)-local reflexivity principle}

In the following, \((\mathcal{A}, \|\cdot\|_\mathcal{A})\) always denotes a \(p\)-Banach ideal with \(0 < p \leq 1\) fixed. Recall, that the \textit{adjoint ideal} \((\mathcal{A}^*, \|\cdot\|_{\mathcal{A}^*})\) is given by all operators \(T \in \mathcal{L}(E, F)\) \((E, F \in \text{BAN})\) for which there exists a number \(q \geq 0\) such that for all \(E_0, F_0 \in \text{BAN}\) and for all \(A \in \mathcal{F}(F, F_0), S \in \mathcal{A}(F_0, E_0), B \in \mathcal{F}(E_0, E)\)

\[ |tr(TBSA)| \leq q \cdot \|B\| \cdot \|S\|_{\mathcal{A}} \cdot \|A\|. \]

By definition, \(\|T\|_{\mathcal{A}^*} := \inf(q)\) where the infimum is formed by all such \(q \geq 0\) ([P]).

According to \([G-L-R]\) the \textit{conjugate ideal} \((\mathcal{A}^\Lambda, \|\cdot\|_{\mathcal{A}^\Lambda})\) is given by all operators
for which there exists a number \( q \geq 0 \) such that for all \( L \in \mathcal{F}(F, E) \)
\[ |\text{tr}(TL)| \leq q \cdot \|L\|_\mathcal{A}. \]
By definition, \( \|T\|_\mathcal{A} := \inf(q) \) where the infimum is formed by all such \( q \geq 0 \).

1.1. Definition: Let \( \varepsilon > 0, F \) be a Banach space, \( M \in \text{FIN} \) and \( N \in \text{FIN}(F') \). We are talking about the weak \((\mathcal{A})\)-local reflexivity principle (short: \((w)\)-(\(\mathcal{A}\))-l.r.p.) if for every \( T \in \mathcal{L}(M, F') \) there is an \( S \in \mathcal{L}(M, F) \) such that
\[ \langle b, Tx \rangle = \langle Sx, y \rangle \forall (x, b) \in M \times N \]
and
\[ \|S\|_\mathcal{A} \leq (1 + \varepsilon) \|T\|_{\mathcal{A}^{**}}. \]
Obviously the \((w)\)-(\(\mathcal{A}\))-l.r.p. always implies the \((w)\) \(\mathcal{A}\)-l.r.p., and if \((\mathcal{A}, \|\cdot\|_\mathcal{A})\) is a maximal Banach ideal \((p = 1)\), then the \((w)\) \(\mathcal{A}\)-l.r.p. implies the \((w)\)-(\(\mathcal{A}\))-l.r.p.

To prove our main theorem 1.5., we need the following

1.1. Lemma: Let \( L \in \mathcal{F}(E, F), A \in \mathcal{L}(N, E') \) and \( \varepsilon > 0 \), where \( E, F \) are arbitrary Banach spaces and \( \dim N < \infty \). Let \((\mathcal{A}, \|\cdot\|_\mathcal{A})\) be a \( p \)-Banach ideal \((0 < p \leq 1)\) such that the \((w)-(\mathcal{A})\)-l.r.p. holds. Then there is an operator \( B \in \mathcal{L}(N, E) \) such that \( \|B\|_\mathcal{A} \leq (1 + \varepsilon) \|A\|_{\mathcal{A}^{**}} \) and \( L'' = L''j_{E}B = j_{F}LB \).

Proof: Since the range of \( L' \) is a finite dimensional subspace of \( E' \), there is an operator \( B \in \mathcal{L}(N, E) \) such that \( \langle L'b, Ay \rangle = \langle By, L'b \rangle \forall b \in F', y \in N \) and \( \|B\|_\mathcal{A} \leq (1 + \varepsilon) \|A\|_{\mathcal{A}^{**}} \). Hence, for all \( b \in F', y \in N \) we have \( \langle b, L''Ay \rangle = \langle LBy, b \rangle = \langle b, j_{F}LB y \rangle \). Easy to prove, but nevertheless of importance is the following

1.1. Lemma: Let \((\mathcal{A}, \|\cdot\|_\mathcal{A})\) be a \( p \)-Banach ideal \((0 < p \leq 1)\). Let \( M \in \text{FIN} \) and \( F \in \text{BAN} \). Then
\[ \mathcal{A}^\Delta(F, M) \cong \mathcal{A}(M, F)' \]
where the isometric isomorphism is given by canonical trace duality.

Remember, that \((\mathcal{A}, \|\cdot\|_\mathcal{A})\) is called left-accessible if for all \((E, N) \in \text{BAN} \times \text{FIN} \), \( T \in \mathcal{L}(E, N) \) and \( \varepsilon > 0 \) there are \( L \in \text{COFIN}(E) \). \( S \in \mathcal{L}(E[L, N]) \) such that \( T = SQ_L^E \) and \( \|S\|_\mathcal{A} \leq (1 + \varepsilon) \|T\|_{\mathcal{A}^\Delta([D], [D-F])} \). By using tensor norm techniques (!) the following non-trivial result can be shown:

1.4. Proposition: Let \((\mathcal{B}, \|\cdot\|_\mathcal{B})\) be a Banach ideal and \( E, F \in \text{BAN} \) such that \( E' \) or \( F \) has the m.a.p. Then
\[ \mathcal{B}^{\min}(E, F) \subseteq (\mathcal{B}^\Delta)^{dd}(E, F) \]
In particular \((\mathcal{A}^\Delta)^{dd}, \|\cdot\|_{(\mathcal{A}^\Delta)^{dd}}\) is left-accessible for each maximal Banach ideal \((\mathcal{A}, \|\cdot\|_\mathcal{A})\).

Prof: cf. [Oe1] and [Oe2].

117
Now we have all prepared to prove our main

1.5. Theorem: Let \((\mathcal{A}, \| \cdot \|^A)\) be a \(p\)-Banach ideal \((0 < p \leq 1)\). TFAE:

1. \((\mathcal{A}^A, \| \cdot \|^{A^A})\) is left-accessible
2. \(\mathcal{A}^{**}(M, F^\prime) \cong \mathcal{A}(M, F)^\prime \quad \forall (F, M) \in \text{BAN} \times \text{FIN} \)
3. The \((w)-(\mathcal{A})\)-l.r.p. holds.

Proof: (1) \(\Rightarrow\) (2): Let \((\mathcal{A}^A, \| \cdot \|^{A^A})\) be left-accessible and \((F, M) \in \text{BAN} \times \text{FIN}\). By \([D-F, 25.2]\) it follows that \(\mathcal{A} \circ \mathcal{A} = (\mathcal{A}^*)^\min\) and so \(1.3.\) implies that \(\mathcal{A}(M, F)^\prime \cong (\mathcal{A}^*)^\min(F, M)\). Hence dualization yields \((D-F, 22.6.)\) \(\mathcal{A}(M, F)^{\prime\prime} \cong (\mathcal{A}^{**})\).

(2) \(\Rightarrow\) (3): This implication follows by using Helly’s lemma \([P]\) and the canonical trace duality \(1.3.;\) namely, observe that by assumption (2)

\[ \mathcal{A}(M, F)^{\prime\prime} \cong \mathcal{A}^{**}(M, F^\prime) \]

is an isometric isomorphism, where \(\langle b, T_\xi x \rangle := \langle \text{tr}(b \otimes x \cdot) , \xi \rangle \quad (x \in M, b \in F^\prime)\).

Let \(\varepsilon > 0, N \in \text{FIN}(F^\prime)\) and \(T \in \mathcal{L}(M, F^\prime)\). Let \(\{x_1, \ldots, x_n\}\) be a basis of \(M\) and \(\{b_1, \ldots, b_m\}\) be a basis of \(N \subseteq F^\prime\). Let \(L_{ij} := b_j \otimes x_i\). By \(1.3.,\) the linear span of \(\{\text{tr}(L_{ij}) : 1 \leq i \leq m, 1 \leq j \leq n\}\) is a finite dimensional subspace of \(\mathcal{A}(M, F)^\prime\). By assumption, there is a \(\xi_0 \in \mathcal{A}(M, F)^\prime\), such that \(\langle b_i, T x_j \rangle = \langle \text{tr}(L_{ij}), \xi_0 \rangle \forall i \in \{1, \ldots, m\} \forall j \in \{1, \ldots, n\}\). By Helly, there exists an \(S \in \mathcal{A}(M, F)\) with \(\|S\|^A \leq (1 + \varepsilon) \|\xi_0\| = (1 + \varepsilon) \|T\|^A\) and with

\[ \langle \text{tr}(L_{ij}), \xi_0 \rangle = \langle S, \text{tr}(L_{ij}) \rangle = \text{tr}(L_{ij} S) = \langle S x_j, b_i \rangle \]

for all \(i\) and \(j\). Hence -- by linearity of \(T\) -- the claim follows.

(3) \(\Rightarrow\) (1): Let \(\mathcal{B} := \mathcal{A}^{**}\). Since \((\mathcal{B}^A)^d, \| \cdot \|_{(\mathcal{B}^A)^d}\) is left-accessible (by \(1.4.,\)) it suffices to show that for all \((E, N) \in \text{BAN} \times \text{FIN}\) and for all \(L \in \mathcal{L}(E, N)\) we have

\[ \|L\|_{\mathcal{A}^A} = \|L\|_{\mathcal{A}^A}. \]

Obviously, \(\|L\|_{\mathcal{A}^A} \leq \|L\|_{\mathcal{A}^A} \leq \|L\|_{\mathcal{A}^A}\). To prove the other inequality we use lemma 1.2.. Let \(A \in \mathcal{F}(N^\prime, E^\prime)\) be given. By assumption we can choose an operator \(B \in \mathcal{L}(N^\prime, E)\) as in lemma 1.2.. It follows that

\[ \|\text{tr}(L^\prime)\| = \|\text{tr}(j_B L^\prime)\| \leq \|L\|_{\mathcal{A}^A} \|B\|_{\mathcal{A}^A} \leq (1 + \varepsilon) \|L\|_{\mathcal{A}^A} \|A\|_{\mathcal{A}^{**}}. \]

Hence \(\|L\|_{\mathcal{A}^A} \leq \|L\|_{\mathcal{A}^A}\) and \((*)\) is proven.

Until now we do not know, if there exists a \textit{maximal} Banach ideal \((\mathcal{A}, \| \cdot \|^A)\) such that the \((w)-(\mathcal{A})\)-l.r.p. does not hold. In this case \((\mathcal{A}^A, \| \cdot \|^{A^A})\) is not left-accessible, and especially \((\mathcal{A}^{**}, \| \cdot \|^A)\) would be another candidate for a non (left-) accessible maximal Banach ideal (since \((\mathcal{A}^A, \| \cdot \|^A) \not\subseteq (\mathcal{A}^{**}, \| \cdot \|^A)\)). Indeed, the hard problem of constructing such a candidate was open for a long time and had been recently solved by Pisier on the Oberwolfach meeting in September 1991, using a factorization over his own Pisier space \(P\) (cf. \([D-F, 31.6.]\)). Therefore it seems also very
non-trivial to construct a maximal Banach ideal \((\mathcal{A}, \| \cdot \|_{\mathcal{A}})\), for which the \((w)-(\mathcal{A})\)-l.r.p. does not hold.

1.5. Remark: There exists a minimal Banach ideal \((\mathcal{A}, \| \cdot \|_{\mathcal{A}})\) such that the \((w)-(\mathcal{A})\)-l.r.p. does not hold.

Proof: Let \(\mathcal{A} := \mathcal{A}^{min}_{0}\), where \(\mathcal{A}^{*}_{0}\) is Pisier's counterexample of a non left-accessible, maximal Banach ideal. Since in general \((\mathcal{B}^{min}, \| \cdot \|_{\mathcal{A}^{min}}) \subseteq ((\mathcal{A}^{*})^{\Delta}, \| \cdot \|_{(\mathcal{A}^{*})^{\Delta}})\), it follows for arbitrary \(L \in \mathcal{F}(E, F) \in \mathcal{F}(E, F)\) that
\[
\|L\|_{\mathcal{A}^{*}} \leq \|L\|_{(\mathcal{A}^{min})^{\Delta}} \leq \|L\|_{((\mathcal{A}^{*})^{\Delta})^{\Delta}} \leq \|L\|_{\mathcal{A}^{*}}.
\]
Therefore the left-accessibility of \(\mathcal{A}^{\Delta}\) would imply the left-accessibility of \(\mathcal{A}^{*}_{0}\), which is a contradiction. \(\square\)

1. The local reflexivity principle for operator ideals

Let \((\mathcal{A}, \| \cdot \|_{\mathcal{A}})\) be a \(p\)-Banach ideal \((0 < p \leq 1)\) such that the \((w)-(\mathcal{A})\)-l.r.p. holds. Then it is possible to transfer the principle of local reflexivity in the following sense:

2.1. Theorem: Let \((\mathcal{A}, \| \cdot \|_{\mathcal{A}})\) be a \(p\)-Banach ideal \((0 < p \leq 1)\) such that the \((w)-(\mathcal{A})\)-l.r.p. holds. Let \(\epsilon > 0\), \(M \in \text{FIN}, F \in \text{BAN}, N \in \text{FIN}(F')\) and \(S \in \mathcal{L}(M, F')\).

Then there exists an operator \(R \in \mathcal{L}(M, F)\) such that
\[
\begin{align*}
(i) & \quad \|R\|_{\mathcal{A}} \leq (1 + \epsilon) \|S\|_{\mathcal{A}^{**}} \\
(ii) & \quad \langle Rx, b \rangle = \langle b, Sx \rangle \quad \forall (x, b) \in M \times N \\
(iii) & \quad j_{F}Rx = Sx \quad \forall x \in M \quad \text{with} \quad Sx \in j_{F}(F).
\end{align*}
\]

Proof: Let \(M_{0} := \{x \in M : Sx \in j_{F}(F)\}\) and \(J : M_{0} \downarrow M\) the canonical embedding. Let \(S_{0} : M_{0} \rightarrow F, x \mapsto j_{F}^{-1}(Sx)\). Let \(N \subseteq L \subseteq F'\) with \(\text{dim } L < \infty\) and \(\epsilon > 0\).

By assumption there exists an \(R_{L} \in \mathcal{F}(M, F)\) such that \(\|R_{L}\|_{\mathcal{A}} \leq (1 + \epsilon) \|S\|_{\mathcal{A}^{**}}\) and \(\langle R_{L}x, b \rangle = \langle b, Sx \rangle\) for all \((x, b) \in M \times L\). Hence
\[
\langle R_{L}Jx, b \rangle = \langle b, j_{F}S_{0}x \rangle = \langle S_{0}x, b \rangle \quad \forall (x, b) \in M_{0} \times L.
\]

Let \(\Phi := \{L \in \text{FIN}(F') : N \subseteq L\}\). By canonical set inclusion, \(\Phi\) is a partially ordered set. Let \(A = \sum_{i=1}^{n} b_{i} \otimes x_{i} \in \mathcal{L}(F, M_{0})\) be arbitrary given \((b_{1}, \ldots, b_{n}) \in F'\) and \(x_{1}, \ldots, x_{n} \in M_{0}\). Choose \(L_{0} \in \Phi\) such that \(b_{1}, \ldots, b_{n} \in L_{0}\). Hence, by (*) we obtain for all \(L \in \Phi\) with \(L \supseteq L_{0}\):
\[
\text{tr}(R_{L}JA) = \sum_{i=1}^{n} \langle R_{L}Jx_{i}, b_{i} \rangle = \sum_{i=1}^{n} \langle S_{0}x_{i}, b_{i} \rangle = \text{tr}(S_{0}A) .
\]

By the canonical trace duality 1.3., it follows that \(S_{0}\) is the \(\sigma(\mathcal{A}(M_{0}, F), \mathcal{A}(M_{0}, F)^{\prime})\)-limit of the net \((R_{L}J)_{L \in \Phi}\). Now consider the set \(C\), consisting of all operators \(UJ\) where \(U \in \mathcal{L}(M, F)\), \(\|U\|_{\mathcal{A}} \leq (1 + \epsilon) \|S\|_{\mathcal{A}^{**}}\) and \(\langle Ux, b \rangle = \langle b, Sx \rangle\) for all \((x, b) \in M \times N\). Since \(R_{L}J \in C\) for all \(L \in \Phi\), \(S_{0}\) is an element of the \(\sigma(\mathcal{A}(M_{0}, F), \mathcal{A}(M_{0}, F^{\prime})).\)
\( \mathcal{A}(M_0, F)' \)-closure of the convex set \( C \), hence \( S_0 \) is an element of the \( \| \cdot \|_{\mathcal{A}} \)-closure of \( C \). Therefore to each \( \delta > 0 \) there exists an \( U_0 J \in C \) such that \( \| S_0 - U_0 J \|_{\mathcal{A}} < \delta \).

Let \( Q: M \to M_0 \) an arbitrary projection. Then \( \| Q \| \leq 1 \) and evidently the statements (ii) and (iii) are valid for the operator \( R := (S_0 - U_0 J) Q + U_0 \in \mathcal{L}(M, F) \). Since
\[
\| R \|_{\mathcal{A}} \leq \| S_0 - U_0 J \|_{\mathcal{A}} + \| U_0 \|_{\mathcal{A}} < \delta + (1 + \varepsilon) \| S \|_{\mathcal{A}^{**}},
\]
statement (i) follows, and the theorem is proven.

\[ \square \]

**Literature**


