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A Note on Strong Measure Zero Sets

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We say that a metric space (X, ϱ) is strong measure zero if for every sequence of positive numbers $\{\varepsilon_n\}_{n \in \omega}$ there is a sequence of sets $\{L_n\}_{n \in \omega}$ such that $X = \bigcup_n L_n$ and $diam(L_n) < \varepsilon_n$ for each n.

In [GMS] the autors proved a theorem which characterizes strong measure zero sets.

Theorem. *$X \subset \mathbf{R}$ is strong measure zero iff* $\forall_{F \subset \mathbf{R}, meagre} X + F \neq \mathbf{R}$

We strenghten it to the following result.

Theorem 0. *$X \subset R$ is strong measure zero iff*

$$\forall_{D \subset R^2, F_\sigma set} \left(\forall_{x \in R} D_x \text{ is meagre} \Rightarrow \bigcup_{x \in X} D_x \neq R \right)$$

Observe that if F is meagre F_σ-set then $X + F = \bigcup_{x \in X} D_x$ where $D = \bigcup_{x \in \mathbf{R}} \{x\} \times (F + x)$. So we see that $\Rightarrow$ in Theorem 0 implies $\Rightarrow$ in Theorem. It will be shown in Theorem 1. The $\Leftarrow$ in both theorems are very simple but in a case we do not need the algebraic structure in the proof. It will be shown in Theorem 2.

Theorem 1. *Let Y be a σ-compact metric space, Z a locally compact space or completely metrizable space. Then if $X \subset Y$ is a strong measure zero set then*

$$\forall_{D \subset Y \times Z, F_\sigma\text{-set}} \left(\forall_{x \in Y} D_x \text{ is meagre} \Rightarrow \bigcup_{x \in X} D_x \neq Z \right)$$

Proof. The proof is very similar to the proof of Theorem in [M]. Let D be a F_σ-set in $Y \times Z$ with meager vertical sections. Then $D = \bigcup_{n \in \omega} F_n$ are closed with nowhere dense vertical sections. We may assume also that $F_n \subset F_{n+1}$ and $F_n \subset K_n \times Z$, where K_n are compact, $K_n \subset K_{n+1}$ and $Y = \bigcup_{n \in \omega} K_n$.

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Lemma. *Let $C \subset K \times Z$ be a closed set with nowhere dense vertical sections, where $K \subset Y$ is a compact set. Let $B \subset Z$ be a closed ball. Then there exists $\varepsilon > 0$ and a finite family $\mathbf{G}$ of closed balls contained in B such that:*

$$\forall_{L \subset K,\, diam(L) < \varepsilon} \underset{G \in \mathbf{G}}{\exists} (L \times G) \cap C = \emptyset$$

This follow strictly from the compactness of K. Indeed, if x is any point from K then there exist ε_x and a closed ball G_x in Z such that $B(x, \varepsilon_x) \times G \cap C = \emptyset$. We can find a finite family $\{x_1, \ldots, x_k\}$ of elements of K such that $K \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon_{x_i}/3)$. It is easy to see, that the family $\mathbf{G} = \{G_{x_i}: i = 1, \ldots, k\}$ and $\varepsilon = min\{\varepsilon_{x_i}/3: i = 1, \ldots, k\}$ has the required property.

Using Lemma we will construct a finitely branched tree $T \subset \omega^{<\omega}$ and a sequences: $(B_s)_{s \in T}$ of closed balls from Z and $(\varepsilon_s)_{s \in T}$ of positive real numbers with the following properties:

(1) $$B_{s^\frown n} \subset B_s$$

(2) $$\forall_{s \in \omega^n \cap T} \forall_{L \subset Y} diam(L) < \varepsilon_s \Rightarrow \underset{k}{\exists}(L \times B_{s^\frown k}) \cap F_n = \emptyset$$

For every n we define $\delta_n = \min\{\varepsilon_s: |s| = n \,\&\, s \in T\}$

We know that $\delta_n > 0$. Now let $X \subset Y$ be a strong measure zero. From the definition of the property of strong measure zero we know that there exists a sequence of balls $\{L_n\}_{n \in \omega}$ contained in Y such that $X \subset \bigcap_m \bigcup_{n>m} L_n$ and $diam(L_n) < \delta_n$. Now we construct a function $f: \omega \to \omega$ such that $(L_n \times B_{f|n+1}) \cap F_n = \emptyset$. Let $x \in \bigcap_n B_{f|n}$. From this we obtain that $(\bigcap_n \bigcup_{m>n} L_m \times \{x\}) \cap \bigcup_n F_n) = \emptyset$.

Theorem 2. *Let Y be a separable metric space, Z Hausdorff, second-countable dense in itself space, $X \subset Y$. Then if*

(*) $$\forall_{D \subset Y \times Z,\, closed\, set} \left(\underset{x \in Y}{\forall} D_x \text{ is meagre} \Rightarrow \bigcup_{x \in X} D_x \neq Z \right)$$

then X is strong measure zero.

Proof. Let us assume that we have a sequence $(\varepsilon_n)_{n \in \omega}$ of positive real numbers. We will construct a cover of X $(K_n)_{n \in \omega}$ with open balls such that $diam(K_n) < \varepsilon_n$. Let $(U_n)_{n \in \omega}$ be a countable base of Z, and for any $n < \omega$ let $(U_{n,m})_{m \in \omega}$ be open disjoint sets in U_n and let $(B_{n,m})_{m \in \omega}$ be a cover of Y with open balls of diameter less than ε_n. Let us fix $n < \omega$ and put $W_n = \bigcup_m B_{n,m} \times U_{n,m}$. Put also $D_n = Y \times Z \setminus W_n$ and $D = \bigcap_n D_n$.

Clearly D is a closed set in $Y \times Z$. Next we show that $\forall_{x \in Y} D_x$ is meagre. For that if U_n is any base set in Z and $x \in Y$ than let $m < \omega$ be such that $x \in B_{n,m}$. We have $U_{n,m} \subset (\bigcup_m B_{n,m} \times U_{n,m})_x = (W_n)_x = Z \setminus (D_n)_x \subset Z \setminus (\bigcap_m D_n)_x = Z \setminus D$, so D is meager. From the (*) we obtain $\bigcup_{x \in X} (D)_x \neq Z$ so let $z \in Z \setminus \bigcup_{x \in X} (D)_x$. Let $n < \omega$ be fixed. We have that $\bigcup_{x \in X} (D)_x = \bigcup_{x \in X} \bigcap_n (D_n)_x = \bigcup_{x \in X} \bigcap_n (Y \times Z \setminus W_n)_x = Z \setminus \bigcap_{x \in X} \bigcup_n (W_n)_x$ so (+) $z \in \bigcap_{x \in X} \bigcup_n (W_n)_x$.

Now we define sets $(K_n)_{n \in \omega}$: For any $n \in \omega$ let $m \in \omega$ be such, that $z \in U_{n,m}$, if there exists any. In this case we put $K_n = B_{n,m}$. If there does not exist any $m \in \omega$ such that $z \in U_{n,m}$ we will take as K_n any open ball in Y with diameter less than ε_n. We must check that $X \subset \bigcup_n K_n$. So let $x \in X$. From the statement (+) we know, that there exists $n \in \omega$ such that $z \in (W_n)_x$, that means $z \in \bigcup_m (B_{n,m} \times U_{n,m})_x$, so we must have for some $m \in \omega$: $z \in U_{n,m}$ and $x \in B_{n,m}$, and this implies that $x \in K_n$ from the definition of K_n. This ends the proof.

References

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