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Cyclic Approximation of Ergodic Step Cocycles Over Irrational Rotations

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Let $x \to x + \alpha$ be an irrational rotation of the circle group. We construct a step cocycle $\varphi(x) = \gamma 1_{[0,\beta)}(x)$ such that associated Anzai skew product T_φ admits a cyclic approximation with speed controlled by α, and is a weakly mixing extension. In particular, given any value $d(T) \geqq 3/2$ for the Katok–Stepin exponent of cyclic approximation, we find T_φ as above such that $d(T_\varphi)$ is off by at most 1/2. Moreover, for almost every rotation, T_φ is rigid and rank-1.

1 Introduction

Let T be an automorphism of a Lebesgue probability space (X, μ). The invariant $d(T)$ introduced by Katok and Stepin [5] informs us of the speed of cyclic approximation which T admits. In [3] (see also [2]) it was observed that for irrational rotations all values $2 \leqq d(T) \leqq \infty$ occur. Therefore, by result in [3] and [4], for every $2 \leqq d \leqq \infty$ there exist an irrational number α and a measurable function φ: $\mathbf{T} \to \mathbf{T}$ such that the associated Anzai skew product T_φ is a weakly mixing extension of the α-rotation and satisfies $d(T_\varphi) = d$. In fact, for a fixed α the set of such φ's is residual for the topology of convergence in measure. On the other hand, it has not been clear how to produce the function φ in a more constructive way and within a limited class of functions such as, e.g., the step functions. In the present note we are able to find, for every $2 \leqq d \leqq \infty$, an irrational number α and a step function φ such that $d - 1 \leqq d(T) \leqq d$ (Corollary 1). A result of Gabriel, Lemańczyk, and Liardet [1] allows φ to be a weakly mixing cocycle. Moreover, for almost every α we obtain a step function φ such that the extension T_φ is weakly mixing, rigid, and rank-1 (Corollary 2).

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2 Definitions and notation

Denote by ε the decomposition of X into singletons and let $0 < f(n) \to 0$. According to [5], the automorphism T admits *cyclic approximation by periodic transformations (cyclic a.p.t.) with speed* $f(n)$ if there exist a sequence of partitions

$$\xi_n = \{C_0, \ldots, C_{h_n - 1}\} \to \varepsilon$$

and automorphisms T_n such that T_n cyclically permutes ξ_n and

$$\sum_{j=0}^{h_n - 1} \mu(TC_j \Delta T_n C_j) < f(h_n).$$

As in [5], we let

$$d(T) = \sup\{r > 0 \colon T \text{ admits cyclic a.p.t. with speed } 1/n^r\}.$$

In the sequel we consider transformation of the 2-torus $\mathbf{T}^2$. It will be convenient to identify the circle group $\mathbf{T}$ with the interval $[0, 1)$, with addition modulo 1. For every $\alpha \in \mathbf{T}$ and a measurable function $\varphi \colon \mathbf{T} \to \mathbf{T}$ (a *cocycle*), we define the (Anzai) *skew product*

$$T_\varphi(x, y) = (x + \alpha, y + \varphi(x))$$

over the α-rotation. The cocycle φ is said to be *weakly mixing*, in which case T_φ is referred to as a *weakly mixing extension*, if T_φ is ergodic and its only eigenvalues are the numbers $\exp(2\pi i n\alpha)$, $n \in \mathbf{Z}$.

We say that α admits a *diophantine approximation with speed* $f(n)$ if there exists a sequence of integers $q_n \to \infty$ such that for some integers p_n we have

$$|\alpha - p_n/q_n| < f(q_n).$$

It is well known that α always admits $f(n) = 1/n^2$ (see e.g. [6]). We denote by $\|x\|$ the norm in $\mathbf{T}$, i.e. the distance from x to the nearest integer. The above condition now reads $\|q_n\alpha\| < q_n f(q_n)$.

3 Construction of step cocycles

We are going to define a family of step cocycles depending on three parameters $\alpha, \beta, \gamma \in \mathbf{T}$. More precisely, for every irrational rotation α we define a step cocycle $\varphi(x) = \gamma 1_{[0,\beta)}(x)$ which satisfies, up to a certain error, a preassigned speed of cyclic approximation.

Lemma 1. *Let* $C > 1$, $0 < c < C - 1$, *and* $1 \leqq j_n \leqq n$. *Then for every sufficiently large* n *there exists a prime number* Q_n *such that*

$$c \log n < Q_n \leqq C \log n$$

and Q_n *does not divide* j_n.

Proof. Choose $1 < C' < C - c$. By Prime Number Theorem the number of primes in the interval $(c \log n, C \log n]$, equal to $\pi(C \log n) - \pi(c \log n)$, exceeds

$$C' \log n / \log \log n$$

for all sufficiently large n. It follows that their product Π exceeds

$$(c \log n)^{C' \log n / \log \log n} .$$

This implies

$$\log \Pi > (\log \log n + \log c)\, C' \log n / \log \log n > C'' \log n$$

for all sufficiently large n, provided $C'' < C'$. We may choose $c' > C'' > 1$, whence $\log \Pi > \log n \geqq \log j_n$. Consequently, $j_n < \Pi$ so at least one prime Q_n in $(c \log n, C \log n]$ does not divide j_n.

Theorem 1. *Let $f(x) > 0$, $g(x) > 0$ decrease to 0 as $x \to \infty$ and let $C > 1$. Let α be an irrational number such that $\|q_n \alpha\| < g(q_n)$ for some sequence $q_n \to \infty$.. Then there exists a residual set $\mathrm{B}(\alpha) \subset \mathbf{T}$ and, for each $\beta \in \mathrm{B}(\alpha)$, a residual set $\Gamma(\alpha, \beta)$ such that for every $\gamma \in \Gamma(\alpha, \beta)$ the Anzai skew product T_φ defined by the cocycle*

$$\varphi(x) = \gamma 1_{[0, \beta)}(x)$$

admits cyclic a.p.t. with speed

$$2g(n/C \log n) + f(n) .$$

Proof. We can find two positive monotone functions $f_1(x)$, $f_2(x)$ such that $f_1(x) < 1/x$ and

$$2f_1(x/C \log x) + 2f_2(x/C \log x) \leqq f(x) .$$

Denote by V_q the union of the open intervals

$$(j/q - f_1(q), j/q) ,$$

where $j = 1, 2, \ldots, q$. The set $\bigcup_{n=N}^{\infty} V_{q_n}$ is open and dense, so the intersection

$$\mathrm{B}(\alpha) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} V_{q_n}$$

is residual. Now fix $\beta \in \mathrm{B}(\alpha)$. There exists a subsequence q_{n_k} such that $\beta \in V_{q_{n_k}}$, whence

$$j_{n_k}/q_{n_k} - f_1(q_{n_k}) < \beta < j_{n_k}/q_{n_k} ,$$

where $1 \leqq j_{n_k} \leqq q_{n_k}$, $k = 1, 2, \ldots$

Let $c > 0$ be as in Lemma 1. Cosequently, there exist prime numbers Q_{n_k} such that

$$c \log q_{n_k} < Q_{n_k} < C \log q_{n_k}$$

and j_{n_k} is not a multiple of Q_{n_k} (for k sufficiently large). Note that the sequence Q_{n_k} depends on α and β. We denote by W_Q the union of the open intervals

$$(P/Q - f_2(\exp(Q/c)), P/Q + f_2(\exp(Q/c))),$$

where $P = 1, 2, \ldots, Q - 1$. Observe as above that the set

$$\Gamma(\alpha, \beta) = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} W_{Q_{n_k}}$$

is residual. Now for every $\gamma \in \Gamma(\alpha, \beta)$ there exists a subsequence n_{k_l} such that

$$|\gamma - P_{n_{k_l}}/Q_{n_{k_l}}| < f_2(\exp(Q_{n_{k_l}}/c),$$

where $1 \leqq P_{n_{k_l}} < Q_{n_{k_l}}$, and $Q_{n_{k_l}}, j_{n_{k_l}}, P_{n_{k_l}}$ are relatively prime for $l = 1, 2, \ldots$

We are now in a position to construct a cyclic approximation of the skew product T_φ, where $\varphi = \gamma 1_{[0,\beta)}$. To simplify the notation we abbreviate the subscripts n_{k_l} and write n. Let $\alpha_n = p_n/q_n$, where $|q_n\alpha - p_n| < g(q_n)$. Since $g(x)$ is monotone, we may assume without loss of generality that p_n, q_n are relatively prime. This implies that

$$\{0, \alpha_n, \ldots, (q_n - 1)\,\alpha_n\} = \{0, 1/q_n, \ldots, (q_n - 1)/q_n\}.$$

To define the approximating partition $\xi_n = \{C_0, \ldots, C_{h_n-1}\}$ and the cyclic automorphism T_n we first let

$$C_0 = [0, 1/q_n) \times [0, 1/Q_n)$$

and define T_n on C_0 by the formula

$$T_n(x, y) = (x + \alpha_n, y + \varphi(0)).$$

Next let $C_1 = T_n C_0$ and, on C_1, define

$$T_n(x, y) = (x + \alpha_n, y + \varphi(\alpha_n)).$$

We let $C_2 = T_n C_1$ and continue is the same manner up to C_{q_n-2}, on which T_n is defined by

$$T_n(x, y) = (x + \alpha_n, y + \varphi((q_n - 2)\,\alpha_n)),$$

and $C_{q_n-1} = T_n C_{q_n-2}$. To define T_n on C_{q_n-1} we use the same α_n-translation along the x-axis but slightly alter the vertical shift. Note that

$$C_{q_n-1} = [(q_n - 1)\,\alpha_n, (q_n - 1)\,\alpha_n + 1/q_n) \times [z, z + 1/Q_n),$$

where $z = \varphi(0) + \varphi(\alpha_n) + \ldots + \varphi((q_n - 1)\alpha_n)$. If the value $\varphi((\omega_v - 1)\alpha_n)$ were used to define the vertical shift of C_{q_n-1} we would obtain the rectangle

$$[0, 1/q_n) \times [y_1, y_1 + 1/Q_n),$$

where

$$y_1 = z + \varphi((q_n - 1)\alpha_n) = \varphi(0) + \varphi(\alpha_n) + \ldots + \varphi((q_n - 1)\alpha_n)$$
$$= \varphi(0) + \varphi(1/q_n) + \ldots + \varphi((q_n - 1)/q_n) = j_n\gamma$$

(the last equality follows from the definition of $\varphi(x)$). Instead, we define

$$C_{q_n} = [0, 1/q_n) \times [y_2, y_2 + 1/Q_n),$$

where $y_2 = j_nP_n/Q_n \pmod 1$. The transformation T_n is defined on C_{q_n-1} accordingly in order to ensure $T_nC_{q_n-1} = C_{q_n}$. Observe that

$$|y_1 - y_2| = j_n|\gamma - P_n/Q_n| < Q_nf_2(\exp(Q_n/c)).$$

The construction continues in the same manner (mod q_n) until we reach $C_{Q_nq_n-1}$. The definition of T_n is completed on $C_{Q_nq_n-1}$ so that T^{h_n} becomes the identity transformation, where $h_n = Q_nq_n$. Since j_nP_n, Q_n are relatively prime, it is clear that the sets C_j are pairwise disjoint and T_n permutes cyclically the partition

$$\xi_n = \{C_0, \ldots, C_{h_n-1}\}.$$

Since the diameters of the rectangles C_j tend to zero, we have $\xi_n \to \varepsilon$. It remains to estimate the approximation error

$$E = \sum_{j=0}^{h_n-1} \mu(T_\varphi C_j \,\Delta\, T_nC_j).$$

Note that E decomposes into three parts:

1. The error E_α caused by the approximation of α by α_n consists of $2q_n$ vertical stripes of width $|\alpha - \alpha_n| < g(q_n)/q_n$ each. Therefore

$$E_\alpha < 2g(q_n) = 2g(h_n/Q_n) \leqq 2g(h_n/C\log h_n).$$

2. The error E_β caused by the jump of the function φ at β occurs as a vertical split of those rectangles C_j which cross the vertical line $x = \beta$. The right part of each split rectangle produces the error so we have

$$E_\beta \leqq 2|\beta - j_n/q_n| < 2f_1(q_n) \leqq 2f_1(h_n/C\log h_n).$$

3. The error E_γ caused by the approximation of y_1 by y_2 occurs for each rectangle in the first column $[0, 1/q'_n) \times [0, 1)$ so

$$E_\gamma \leqq 2|y_1 - y_2|\, Q_n/q_n < 2f_2(\exp(Q_n/c))\, Q_n^2/q_n$$
$$\leqq 2f_2(\exp(Q_n/c)) \leqq 2f_2(q_n) \leqq 2f_2(h_n/C\log h_n)$$

for n large enough.

By the choice of f_1 and f_2 we obtain $E_\beta + E_\gamma < f(h_n)$. Consequently, $E < 2g(h_n/C\log h_n) + f(h_n)$, which ends the proof of the theorem.

4 Corollaries

Our next aim is to improve the construction of φ in order to obtain a weakly mixing extension. To this end we apply a result of Gabriel, Lemańczyk, and Liardet ([1], Cor. 1.6), which gives a criterion for a step cocycle to be weakly mixing. We say, as in [1], that β is *α-separated* if

$$\limsup_{n\to\infty} \min_{0\le k\le q_n'} q_n'\|\beta - k\alpha\| > 0,$$

where q_n' is the sequence of denominators of α. The result of [1] asserts that if $\beta \notin \mathbf{Z}\alpha$, $\pm\beta$ are α-separated, and $\gamma \neq 0$, then T_φ is a weakly mixing extension of the α-rotation. It is also observed in [1] that if α has bounded partial quotients then β is α-separated whenever $\beta \notin \mathbf{Z}\alpha$. In the general case we have the following simple lemma whose proof is left to the reader.

Lemma 2. *Let α be an irrational number. Then the set* $\mathrm{B}'(\alpha)$ *of all numbers β such that $\pm\beta$ are α-separated is residual.*

Now by taking $\beta \in \mathrm{B}(\alpha) \cap \mathrm{B}'(\alpha)\setminus\mathbf{Z}\alpha$ in Theorem 1, we obtain immediately.

Theorem 2. *Let f, g, C, α be as in Theorem 1. Then there exist numbers β, γ such that the cocycle $\varphi = \gamma 1_{[0,\beta)}$ is weakly mixing and admits cyclic a.p.t. with speed* $2g(n/C\log n) + f(n)$.

It was shown in [2] (see also [3]) that the speed of cyclic approximation of an authomorphism is never better than the speed of (simultaneous) diophantine approximation of its eigenvalues. Now let $2 \leqq d \leqq \infty$. Using continued fractions, it is easy to construct a number α admitting diophantine approximation with speed $1/n^r$ for all $r < d$, but not for $r > d$. The following corollary is now a consequence of Theorem 2.

Corollary 1. *For every $2 \leqq d \leqq \infty$ there exist a rotation $\alpha \in \mathbf{T}$ and a step cocycle φ as above such that T_φ is a weakly mixing extension and $d - 1 \leqq d(T_f) \leqq d$.*

It is known (see [6]) that almost every α (with respect to Lebesgue measure) admits diophantine approximation with speed

$$o(1/n^2 \log n \log\log n).$$

Corollary 2. *For a.e. α in* $\mathbf{T}$ *there exists a step cocycle φ as above such that T_φ is a weakly mixing extension and admits a cyclic a.p.t. with speed* $o(1/n\log\log n)$. *In particular, T_φ is rigid and rank-1.*

Proof. Choose $f(x) = g(x) = o(1/x \log x \log \log x)$ in Theorem 2 to obtain the first part of the assertion. To get the second part, we recall that an authomorphism which admits cyclic approximation with speed $o(1/n)$ is necessarily rigid (see [5]) and rank-1 (see e.g. [4]).

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