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A Note on Carathéodory Type Function

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In this note we discuss some problems related to the product measurability of functions which are measurable in one and continuous in the second variable.

Let $\mathscr{A}$ be a family of subsets of a set T. By $\sigma(\mathscr{A})$ we denote the σ-field generated by $\mathscr{A}$, and by $\mathscr{A}|_S$, where $S \subset T$, the family $\{A \cap S: A \in \mathscr{A}\}$. The family of all subsets of T is denoted by $\mathscr{P}(T)$. If $\mathscr{B}$ is a family of subsets of X, then $\mathscr{A} \times \mathscr{B}$ stands for the family of rectangles, i.e. $\mathscr{A} \times \mathscr{B} = \{A \times B: A \in \mathscr{A}, B \in \mathscr{B}\}$. If $(T, \mathscr{A})$ and $(X, \mathscr{B})$ are measurable spaces, then $\mathscr{A} \otimes \mathscr{B}$ denotes the product σ-field on $T \times X$; $\mathscr{A} \otimes \mathscr{B} = \sigma(\mathscr{A} \times \mathscr{B})$. A topological space X is always considered with the Borel σ-field $\mathscr{B}(X)$. The letter Y always stands for a topological space.

We say that $f: T \times X \to Y$ is a Carathéodory type function if it is measurable in t and continuous in x. In general, such a function need not be $\mathscr{A} \otimes \mathscr{B}(X)$-measurable. But if we assume that X has a countable base and Y is perfectly normal (e.g., metrizable), then each Carathéodory type function is $\mathscr{A} \otimes \mathscr{B}(X)$-measurable.

Suppose $(T, \mathscr{A}, \mu)$ is a measure space. Denote by $\mathscr{A}_\mu$ the completion of $\mathscr{A}$, i.e. $\mathscr{A}_\mu = \sigma(\mathscr{A} \cup \mathscr{A}_0)$, where $\mathscr{A}_0 = \{D: D \subset E \in \mathscr{A} \text{ and } \mu(E) = 0\}$. Remind some elementary facts from the measure theory:

(1) $\mathscr{A}_\mu = \{C \setminus D: C \in \mathscr{A}, D \in \mathscr{A}_0\}$.

(2) If Y has a countable base and $g: T \to Y$ is $\mathscr{A}_\mu$-measurable, then there is a set $S \in \mathscr{A}$ such that $\mu(T \setminus S) = 0$ and $g|_S$ is $\mathscr{A}$-measurable.

Professor W. Zygmunt (Lublin) has posed the following question: Let $f: T \times X \to \mathbb{R}$ be $\mathscr{A} \otimes \mathscr{B}(X)$-measurable and upper semicontinuous in x. Does there exist $S \in \mathscr{A}$ such that $\mu(T \setminus S) = 0$ and $f|_{S \times X}$ is $\mathscr{A} \otimes \mathscr{B}(X)$-measurable?

Note that f from this problem is a Carathéodory type function. Indeed, if we endow the real line with the topology generated by intervals $(-\infty, r)$, $r \in \mathbb{R}$, then the continuity with respect to this topology is just the upper semicontinuity.

The following theorem gives the positive answer to the above question.

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Theorem 1. *Let $(T, \mathscr{A}, \mu)$ be a measure space, $(X, \mathscr{B})$ an arbitrary measurable space, and $f\colon T \times X \to Y$. If Y has a countable base and f is $\mathscr{A}_\mu \otimes \mathscr{B}$-measurable, then there is a set $S \in \mathscr{A}$ such that $\mu(T \setminus S) = 0$ and $f|_{S\times X}$ is $\mathscr{A} \otimes \mathscr{B}$-measurable.*

The proof is based on two known facts:

(3) If $\mathscr{T}$ is a family of sets and $A \in \sigma(\mathscr{T})$, then there is a countable subfamily $\mathscr{T}_0 \subset \mathscr{T}$ such that $A \in \sigma(\mathscr{T}_0)$.

(4) Let S be a subset of T. Then $\sigma(\mathscr{T}|_S) \subset \sigma(\mathscr{T})$ whenever $S \in \sigma(\mathscr{T})$.

Proof of Theorem 1. Let $\{V_n : n \in \mathbb{N}\}$ be a base in Y. By (3), there is a countable family $\mathscr{R}_n \subset \mathscr{A}_\mu \times \mathscr{B}$ such that $f^{-1}(V_n) \in \sigma(\mathscr{R}_n)$, $n \in \mathbb{N}$. Enumerate the family $\mathscr{R} = \bigcup\{\mathscr{R}_n : n \in \mathbb{N}\}$ in a single sequence; $\mathscr{R} = \{A_n \times B_n : n \in \mathbb{N}\}$. We have $f^{-1}(V) \in \sigma(\mathscr{R})$ for each open $V \subset Y$. Since $A_n \in \mathscr{A}_\mu$, there exist $C_n, E_n \in \mathscr{A}$ with $\mu(E_n) = 0$ and $D_n \subset E_n$ such that $A_n = C_n \setminus D_n$. Put $E = \bigcup\{E_n : n \in \mathbb{N}\}$ and $S = T \setminus E$. Of course, $S \in \mathscr{A}$ and $\mu(T \setminus S) = 0$. Moreover,

$$(A_n \times B_n) \cap (S \times X) = [(C_n \setminus D_n) \cap S] \times B_n =$$
$$= (C_n \times B_n) \cap (S \times X) \in \mathscr{A} \times \mathscr{B}|_{S\times X},$$

which means that $\mathscr{R}|_{S\times X} \subset \mathscr{A} \times \mathscr{B}|_{S\times X}$. Hence, by (4), for each open $V \subset Y$ we have

$$(f|_{S\times X})^{-1}(V) \in \sigma(\mathscr{R})|_{S\times X} = \sigma(\mathscr{R}|_{S\times X}) \subset \sigma(\mathscr{A} \times \mathscr{B}|_{S\times X}) =$$
$$= \mathscr{A} \otimes \mathscr{B}|_{S\times X} \subset \mathscr{A} \otimes \mathscr{B}.$$

It completes the proof.

Remarks. 1. If X and Y are topological spaces with countable basis, Y is metrizable, and $f\colon T \times X \to Y$ is $\mathscr{A}_\mu$-measurable in t and continuous in x, then the conclusion of Theorem 1 holds with $\mathscr{B} = \mathscr{B}(X)$.

2. In the same way one can prove that if $D \subset T \times X$ and $f\colon D \to Y$ is $\mathscr{A}_\mu \otimes \mathscr{B}|_D$-measurable, then $f|_{(S\times X)\cap D}$ is $\mathscr{A} \otimes \mathscr{B}|_{(S\times X)\cap D}$-measurable for some $S \in \mathscr{A}$ with $\mu(T \setminus S) = 0$.

Throughout the rest of this note T is a topological space and $\mathscr{A} = \mathscr{B}(T)$. We say that a measure μ on $\mathscr{A}$ is regular (Radon) if for each $A \in \mathscr{A}$ and each $\varepsilon > 0$ there is a closed (compact) subset $F \subset A$ such that $\mu(A \setminus F) < \varepsilon$. It is well known that if T is metrizable (Polish), then each finite measure on $\mathscr{B}(T)$ is regular (Radon).

In the case of regular μ, the proposition (2) is a consequence of the Lusin theorem.

Lusin Theorem. *Suppose μ is regular (Radon and T is Hausdorff), Y has a countable base and $g\colon T \to Y$. If g is $\mathscr{A}_\mu$-measurable then for each $\varepsilon > 0$ there is closed (compact) $S \subset T$ such that $\mu(T \setminus S) < \varepsilon$ and $g|_S$ is continuous.*

The Lusin theorem was generalized for Carathéodory type functions. The first result of this type was obtained by Scorza Dragoni [2]. He proved that if $f: [a, b] \times [c, d] \to \mathbb{R}$ is Lebesgue measurable in the first and continuous in the second variable, then for each $\varepsilon > 0$ there is a compact set $K \subset [a, b]$ such that the Lebesgue measure of its complement is less then ε and $f|_{K \times [c,d]}$ is continuous. There are many generalizations of this result (also for set-valued functions). The following theorem is an example (for the proof see e.g. [1]).

Theorem 2. *Suppose $(T, \mathscr{A}, \mu)$ is a σ-finite measure space with μ regular (Radon and T is Hausdorff), X is Polish, Y has a countable base, and $f: T \times X \to Y$ is continuous in x. If f is $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurable then for each $\varepsilon > 0$ there is closed (compact) $S \subset T$ such that $\mu(T \setminus S) < \varepsilon$ and $f|_{S \times X}$ is continuous.*

Remark. If Y is metrizable it suffices to assume that f is $\mathscr{A}_\mu$-measurable in t instead of the product measurability.

Note that the converse implication in the Lusin theorem also holds. The same is true for Scorza Dragoni type theorems, i.e. the continuity of restricted functions implies the $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurability (see e.g. [4] for the case when f is real-valued and semicontinuous in x; [3] and [5] for f set-valued). Since known proofs are rather complicated, we propose a straightforward proof of the reversed implication to that one from Theorem 2.

Theorem 3. *Let T, X and Y be topological spaces, $\mathscr{A} = \mathscr{B}(T)$, μ a measure on $\mathscr{A}$, and $f: T \times X \to Y$. Assume X has a countable base and f is continuous in x. If for each $\varepsilon > 0$ there is $S \in \mathscr{A}_\mu$ such that $\mu(T \setminus S) < \varepsilon$ and $f|_{S \times X}$ is continuous, then f is $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurable.*

It is immediate that a function f satisfying assumptions of this theorem is $\mathscr{A}_\mu$-measurable in t. Hence, if Y is metrizable then f is $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurable. In the general case the proof of Theorem 3 is based on the following lemma.

Lemma. *Let Z be a set, X a topological space with a countable base and $A \subset Z \times X$. If A has open vertical sections $A_z = \{x \in X: (z, x) \in A\}$, $z \in Z$, then $A \in \mathscr{P}(Z) \otimes \mathscr{B}(X)$.*

Proof. Let $\{U_n: n \in \mathbb{N}\}$ be a base in X. Then $A = \bigcup\{M_n \times U_n: n \in \mathbb{N}\}$, where $M_n = \{z \in Z: U_n \subset A_z\}$.

Proof of Theorem 3. For each $n \in \mathbb{N}$ there is $S_n \in \mathbb{N}$ there is $S_n \in \mathscr{A}_\mu$ such that $\mu(T \setminus S_n) < \frac{1}{n}$ and $f|_{S_n \times X}$ is continuous. Since X has a countable base, $\mathscr{B}(S_n \times X) = \mathscr{B}(S_n) \otimes \mathscr{B}(X) \subset \mathscr{A}_\mu \otimes \mathscr{B}(X)$. Thus $f|_{S_n \times X}$ is $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurable. Now put $S = \bigcup\{S_n: n \in \mathbb{N}\}$. Of course, $S \in \mathscr{A}_\mu$, $\mu(T \setminus S) = 0$ and $f|_{S \times X}$ is $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurable. Let $Z = T \setminus S$. In order to complete the proof it suffices to show that $f|_{Z \times X}$ is also $\mathscr{A}_\mu \otimes \mathscr{B}(X)$-measurable. Note that $\mathscr{A}_\mu|_Z = \mathscr{P}(Z) \subset \mathscr{A}_\mu$, because Z is a null set. If $V \subset Y$ is open, then $(F|_{Z \times X})^{-1}(V) = f^{-1}(V) \cap$

$\cap$ $(Z \times X)$ has open vertical sections, since f is continuous in x. By our Lemma, this preimage belongs to $\mathscr{P}(Z) \otimes \mathscr{B}(X) \subset \mathscr{A}_\mu \otimes \mathscr{B}(X)$, which completes the proof.

Remark. The same proof holds for f being a set-valued function lower (upper) semicontinuous in x.

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