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Frolík's Theorem for Basically Disconnected Spaces

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It is proved that every embedding of the Stone space of a $\varkappa$-complete Boolean algebra has a fixed-point set which is a $P_\varkappa$-set.

0. Introduction

The intention of this paper is to obtain the famous theorem of Frolík:
"The fixed-pint set of a self embedding of a compact extremally disconnected space is clopen"
as a kind of a "limit theorem" on self-embeddings of Stone spaces of $\varkappa$-complete Boolean algebras.
We would also like to obtain the generalization of the Frolík theorem, due to Abramovich, Arenson and Kitover as a limit construction.
Unfortunately, the type of maps we need between these spaces to succeed in the second goal are rather odd. (Fortunately, there is a large nice class of maps with this property, the open maps.) We will show that the fixed-point set of an self--embedding of a Stone space of a $<\varkappa$-complete Boolean algebra has a $P_{<\varkappa}$-set as fixed-point set.
One of the conclusions of this paper is that an autohomeomorphism φ of a basically disconected space behaves very good. Not only is the fixed-point set F a P-set (and therefore basically disconnected), we will also show that each fixed point has a base of clopen sets which are both φ and φ^{-1}-invariant. This generalizes another result of Frolík on extremally disconnected spaces to the class of basically disconnected spaces.
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1. Preliminaries

All spaces under consideration are assumed to be zero-dimensional. The words maps and functions stand for continuous maps and continuous functions. For a compact space X, $\mathscr{B}(X)$ denotes the algebra of clopen sets. An algebra $\mathscr{B}$ is called $<\varkappa$-complete if any family of cardinality less than $\varkappa$ has a supremum.

1.1. Definition. A space X is called $<\varkappa$-basically disconnected (we say: $\varkappa$-b.d.) if $\mathscr{B}(X)$ is $<\varkappa$-complete.
When the notion $\varkappa$-b.d. appears, it is assumed that $\varkappa > \omega_0$. The standard notion basically disconnected coincides with the notion $<\omega_1$-b.d.

1.2. Lemma. *If X is $<\varkappa$-b.d. then the union of any family $\mathscr{A}$ of clopen subsets of cardinality less than $\varkappa$ has a clopen closure and $\cup\mathscr{A}$ is C^*-embedded in the closure.*

Let X be a space and φ be a selfmap of X. A subset $A \subset X$ is called φ-invariant (resp. φ^{-1}-invariant) if $\varphi(A) \subseteq A$ (resp. $\varphi^{-1}(A) \subseteq A$).
For any subset $A \subseteq X$, put $A^\#$ to be the smallest closed φ^{-1}-invariant subset that contains A. $A^\#$ can be obtained in the following way.
Put $A(0) = \operatorname{cl} A$, $A(\alpha+1) = A(\alpha) \cup \varphi^{-1}(A(\alpha))$ and $A(\beta) = \operatorname{cl}[\cup\{A(\alpha): \alpha < \beta\}]$ if β is a limit. At the moment that $A(\alpha+1) = A(\alpha)$ it is clear that $A^\# = A(\alpha)$.
Note that the set $A^\#$ depends on the given selfmap φ of X.
The sharp-operator was defined in [A, A, K]. For more information see also [V].
In the sequel $F^\#$ always denotes the smallest φ^{-1}-invariant set that contains the fixed-point set of φ.

1.3. Lemma. *([A, A, K], [V]) Let φ be a selfmap of a compact space X.*
1. If the map φ is open, then $A^\# = A(\omega_0)$.
2. If A is φ-invariant, then $A^\#$ is both φ and φ^{-1}-invariant.
3. If U is open, $\varphi(U) \subseteq U$ and $U \cap F = \emptyset$, then $U \cap F^\# = \emptyset$.

1.4. Definitions. Let φ be a selfmap of a space X.
1. The selfmap φ of X is called a $\# < \varkappa$ map if for every clopen subset $C \subseteq X$ we have $C^\# = C(\alpha)$, for some ordinal α with $\alpha < \varkappa$.
2. A closed set A is called a $P_{<\varkappa}$-set if there exists an open set V with $A \subseteq V \subseteq \cap U_\alpha$, whenever $A \subseteq \cap\{U_\alpha: \alpha < \gamma\}$ ($\gamma < \varkappa$, U_α is open).
$P_{<\omega_1}$-sets will be refered to as P-sets.
3. If $A \subseteq X$, then $\{C^1, C^2, C^3\}$ is called a 3-partition of A, if the sets C^i are clopen in X, pairwise disjoint, $A = C^1 \cup C^2 \cup C^3$ and $\varphi(C^i) \cap C^i = \emptyset$.

Remark. 1. Note that closed $P_{<\varkappa}$-sets of $<\varkappa$-basically disconnected spaces are again $<\varkappa$-basically disconnected. (In Boolean algebra language this is clear. The dual statement is that the quotient algebra of a $<\varkappa$-complete algebra by a $<\varkappa$-ideal is again a $<\varkappa$-complete algebra.)

2. Note that what is called #-finite in [V] coincides with the notion $\# < \omega_0$.

We collect in one theorem all the result from [V] which remain true for $\# < \varkappa$ maps on $<\varkappa$-b.d. spaces. The proofs are omitted, but using lemma 1.2 and 1.3 the proofs of these statement can be copied from [V].

1.5. Theorem. *[V] Assume X is $<\varkappa$-b.d. and $\varphi: X \to X$ is a $\# < \varkappa$ selfmap. Then:*

1. *If C is clopen, then $C^\#$ is clopen.*
2. *If C is clopen and $C \cap F = \emptyset$, then there exists $f: C^\# \to \{0, 1, 2\}$ such that $f(x) \neq f(\varphi(x))$, for all $x \in C^\#$ with $\varphi(x) \in C^\#$.*
   In particular we have $C^\# \cap F = \emptyset$.
3. *A fixed-point x of φ is a strong attractor.*
   (This means that $\exists U_x \exists$ clopen V_x with $\varphi(V_x) \subseteqq V_x \subseteqq U_x$).
4. *F is a retract of $F^\#$ and a retraction $r: F^\# \to F$ exists with $r = r \circ \varphi$.*
5. *If C is clopen, $C \cap F = \emptyset$, $\varphi(C) \subseteqq C$ and $\{C^1, C^2, C^3\}$ is a 3-partition of C, then there exists a 3-partition $\{S^1, S^2, S^3\}$ of $C^\#$ with $S^i \cap C = C^i$.*
6. *If C is clopen and $\varphi(C) \subseteqq C$ and $C \cap F = \emptyset$ then $C^\# \cap F^\# = \emptyset$.*

1.6. Theorem. *Assume X is $<\varkappa$-b.d. and φ is $\# < \varkappa$ selfmap. Let H be a clopen subset of X with $\varphi(H) \subseteqq H$ and $\varphi^{-1}(H) \subseteqq H$. If C is a clopen subset of X with $C \cap (F^\# \cup H) = \emptyset$ then there exists a clopen subset G of X with $C \subseteqq G$, $G \cap (F^\# \cup H) = \emptyset$, $\varphi(G) \subseteqq G$ and $\varphi^{-1}(G) \subseteqq G$.*

Proof. One way of proving this is to follow [[V], 7.3 and 6.2].
But I found an easy proof, which I present here.
Note first that $A \cap H = \emptyset$ implies $A^\# \cap H = \emptyset$.
Put $C_0 = C$ and if C_n is defined let C_{n+1} be a clopen set with

$$C_n \cup \varphi(C_n) \subseteqq C_{n+1} \subseteqq X - (F^\# \cup H).$$

If $U = \cup C_n$, then U is a cozeroset, $\varphi(U) \subseteqq U$ and $U \cap (F^\# \cup H) = \emptyset$ and $H \cap \mathrm{cl}\, U = \emptyset$.
The following theorem 1.7 implies that $\beta(\varphi | U)$ has no fixed-points.
From U being C^*-embedded in X, it follows that $F \cap \mathrm{cl}\, U = \emptyset$.
Since $\mathrm{cl}\, U$ is clopen and $\varphi(\mathrm{cl}\, U) \subseteqq \mathrm{cl}\, U$ it follows from 1.5.6. that $[\mathrm{cl}\, U]^\# \cap F^\# = \emptyset$. Clearly, we can take $G = [\mathrm{cl}\, U]^\#$.

In the previous theorem I already used the following theorem, due to A. Krawczyk and J. Stepràns. Unfortunately, I was not aware of these results when I was preparing the manuscript of [V].

1.7. Proposition. *[K, S] 1. Let X be a σ-compact 0-dimensional Hausdorff space and let φ be a selfmap of X. Then φ has a fixed-point if and only if $\beta\varphi$ has a fixed point.*

2. *Any selfmap φ without fixed points of a compact 0-dimensional space X has a 3-partition.*

Statement 1.7.2 was also proved in [B, K], but only in the case that the map is injective. A close look at their proof gives the following:

1.8. Lemma. *[B, K] Let X be a zerodimensional compact space and let φ be a selfembedding of X. If $A_i \subseteqq X$ are pairwise disjoint closed sets (i = 1, 2, 3) with $\varphi(A_i) \cap A_i = \emptyset$, then there exists a 3-partition C_i (i = 1, 2, 3) of X with $A_i \subseteqq C_i$.*

2. The theorem of Frolik as a limittheorem.

The following theorem is the basis of the goal to see Frolík's theorem as a limit.

2.1. Theorem. *Assume X is a $<\varkappa$-basically disconnected space and φ is a $\# < \varkappa$ selfmap of X.*
Then: $F^\#$ is a $P_{<\varkappa}$-subset of X.
In particular, both F and $F^\#$ are $<\varkappa$-basically disconnected spaces.

Proof. Assume $F^\# \subseteqq \cap\{U_\alpha: \alpha < \gamma\}$, where γ is an ordinal of cardinality less than $\varkappa$ and the sets U_α are clopen.
By induction we construct, using 1.2. clopen sets D_α ($\alpha < \gamma$) with:
$X - U_\alpha \subseteqq D_\alpha$, $D_\alpha \cap F^\# = \emptyset$, $\varphi(D_\alpha) \subseteqq D_\alpha$, $\varphi^{-1}(D_\alpha) \subseteqq D_\alpha$ and $\alpha < \beta < \gamma$ implies $D_\alpha \subseteqq D_\beta$.
together with a 3-partition C^1_α, C^2_α and C^3_α of D_α
such that $C^i_\beta \cap D_\alpha = C^i_\alpha$ $(\alpha < \beta)$.
Choose D_0 to be some clopen subset of X which is φ and φ^{-1}-invariant with $X - U_0 \subseteqq D_0$ and with $D_0 \cap F^\# = \emptyset$. This is possible by 1.6.
And 1.7.2 implies that there exists a 3-partition C^1_0, C^2_0, C^3_0 of D_0 as required.
Assume for $\alpha < \beta$, the D_α with partitions are defined.
case 1. β is a limit.
Then the set $D = \cup\{D_\alpha: \alpha < \beta\}$ is a union of $<\varkappa$ many clopen sets, so D is C^* embedded in X and cl D is clopen in X.
Moreover, $C^i = \cup\{C^i_\alpha: \alpha < \beta\}$ for $i \in \{1, 2, 3\}$ are pairwise disjoint open subsets of D with $\varphi(C^i) \cap C^i = \emptyset$.

It follows that φ has no fixed-point sets on D, and the existence of the 3-partition on D implies that $\beta\varphi$ has no fixed points. But $\beta D = \text{cl } D$. Since $\varphi(\text{cl } D) \subseteqq D$, cl D clopen and $\text{cl } D \cap F = \emptyset$, we see from 1.5.6 that $[\text{cl } D]^\# \cap F^\# = \emptyset$. (By the way, this is why the proof does not work for F-spaces, not even when the map is open. The set cl D need not be open, so 1.5.6. cannot be applied) By 1.5.6 the 3-partition $\{\text{cl } C^i\}$ of cl D can be extended to a 3-partition S^1, S^2, S^3 of $D^\#$.

Note that the clopen set D is φ-invariant, and so is cl D. But then $[\text{cl } D]^\#$ is both φ and φ^{-1}-invariant.

According to 1.6 there exists a clopen set G with $[\mathrm{cl}\, D]^{\#} \cup (X - U_\alpha) \subseteqq G$, $G \cap F^{\#} = \emptyset$ and also G is both φ and φ^{-1} invariant.
Also, the set G-$[\mathrm{cl}\, D]^{\#}$ is both φ and φ^{-1}-invariant.
But then G-$[\mathrm{cl}\, D]^{\#}$ has a 3-partition, say T^1, T^2, T^3.
Finally, we put $D_\beta = G$ and $C^i = S^i \cup T^i$, for $i \in \{1, 2, 3\}$.
case 2. $\beta = \lambda + 1$. This is essentially the last part of case 1. Read D_γ instead of $D^{\#}$.
We conclude that the sequence $\{D_\alpha: \alpha < \gamma\}$ is found.

Next, we proceed as above. The set $D = \cup\{D_\alpha: \alpha < \gamma\}$ is C^*-embedded and has a clopen closure and has a 3-partition. As above, we see that $[\mathrm{cl}\, D] \cap F^{\#} = \emptyset$ and clearly $F^{\#} \subseteqq X - \mathrm{cl}\, D \subseteqq \cap U_\alpha$.
We conclude with the observation that the $P_{<\varkappa}$-set $F^{\#}$ in a $<\varkappa$-b.d. space is necessarily $<\varkappa$-b.d.
Next 1.5.4 implies that F, being a retract of $F^{\#}$, is $<\varkappa$-b.d. too.

We obtain the theorem of Frolík as a limitcase. (Note that if X is extremally disconnected, then the algebra $\mathscr{B}(X)$ is $<\varkappa$-complete for all $\varkappa$).
Also the generalization from [A, A, K] shows up as a limit.

2.2. Corollary. *[A, A, K]. If X is extremally disconnected and φ is a selfmap then $F^{\#}$ is clopen.*
In particular, if φ is an embedding then F is clopen.

Proof. Put $\varkappa = \mathrm{card}\, X$. Then X is $<\varkappa^+$-basically disconnected and each map φ is $\# < \varkappa$.
Moreover, a closed $P_{<\varkappa^+}$-set is necessarily clopen.
we conclude from 2.1 that $F^{\#}$ is clopen.

The following "Theorem of Frólik" for $<\varkappa$-basically disconnected spaces appears.

2.3. Corollary. *Let X be a $<\varkappa$-basically disconnected space.*
1. If γ is an embedding which is $\# < \varkappa$, then F is a $P_{<\varkappa}$-set.
2. If φ is an autohomeomorphism of X (or just any open embedding), then F is a $P_{<\varkappa}$-set.

Proof. The first part follows from 2.1 and the second statement follows from 1.3.1 and the first part.

It would have been nice if all embeddings of a $<\omega_1$-basically disconnected space are $\# < \omega_1$. However, this is not the case.
To see this, we first show the following lemma.

2.4. Lemma. *Let X be a $<\varkappa$-basically disconnected space and let φ be a continuous selfmap with the property that $F = F^{\#}$.*
If the map φ is $\# < \varkappa$, then:
each point $x \in F$ has the property that there exists a clopen local base of sets which are both φ- and φ^{-1}-invariant.

Proof. Choose $x_0 \in F$ and let U be a clopen set with $x_0 \in U$.
Put $A = \{x: x \in U$ and $\varphi(x) \notin U\} = U \cap \varphi^{-1}(X - U)$ and put $B = \varphi^{-1}(U) - U$.
Then A and B are clopen and $(A \cup B) \cap F = \emptyset$.
But $F = F^{\#}$, so according to 1.5.6., there exists a clopen set G with:
$A \cup B \subseteqq G, \quad \varphi(G) \subseteqq G, \quad \varphi^{-1}(G) \subseteqq G$ and $G \cap F^{(\#)} = \emptyset$.

Put $V = U - G$. Then V is clopen and $x_0 \in V \subseteqq U$.
We check that V is both φ and φ^{-1} invariant.
Choose $x \in V = U - G$.

1. Clearly: $x \notin G$, so $\varphi(x) \notin G$. But then $\varphi(x) \notin A$, so $\varphi(x) \in U$. It follows that $\varphi(x) \in U - G = V$. We see that V is φ-invariant.
2. Consider $\varphi^{-1}(x)$. Since $x \notin G$ and G is φ-invariant we see that $\varphi^{-1}(x) \cap G = \emptyset$.
   Assume $\varphi^{-1}(x)$ is not a subset of U, i.e. $x = \varphi(b)$ for some $b \in B$.
   Hence V is φ^{-1}-invariant.

This shows that the neighborhood V is as required.

In [Wal], 6.3.5. an example is constructed of an embedding φ of a $< \omega_1$-basically disconnected space with one fixed-point that does not have a local base of φ-invariant neighborhoods. It follows from 2.4. that this particular embedding cannot be $\# < \omega_1$.
Fortunately we still can show that the fixed-point set is a P-set. This follows from the following theorem.

2.5. Theorem. *Let X be a basically disconnected space and let φ be a selfmap. Then $F^{\#}$ is a P-set.*

Proof. Let $F^{\#} \subseteqq \cap U_n$, with U_n clopen. Consider the F_σ-set $\cup(X - U_n)$.
We know that $F^{\#}$ is both φ- and φ^{-1}-invariant.
Fix $n \in \mathbb{N}$. Define by induction clopen sets C_k with

$$\varphi^k(X - \cap U_n) \subseteqq C_k, \quad C_k \cap F^{\#} = \emptyset, \quad C_k \cup \varphi(C_k) \subseteqq C_{k+1} \subseteqq X - F^{\#}.$$

Put $O_n = \cup\{C_k: k \geqq 0\}$. Then O_n is a cozeroset with $\varphi(O_n) \subseteqq O_n$.
Next, put $D = \cup\{O_n: n \geqq 0\}$.

Then D is an open σ-compact subset of X disjoint from $F^{\#}$. Moreover, $\varphi(D) \subseteqq D$. So the restriction of the map φ to D has no fixed-point and we can conclude that $\beta(\varphi|D)$ has no fixed points. But D is C^*-embedded, and we see that $\varphi|\text{cl}\, D$ has no fixed points.

But $\text{cl}\, D$ is clopen and $\text{cl}\, D \cap F = \emptyset$ and $\varphi(\text{cl}\, D) \subseteqq \text{cl}\, D$
By 1.3.3: $\text{cl}\, D \cap F^{\#} = \emptyset$. We see that $X - \text{cl}\, D$ is an open set with $F^{\#} \subseteqq$ $\subseteqq X - \text{cl}\, D \subseteqq \cap U_n$.
The conclusion follows.

I did not succeed in answering the following:

2.6. Question. "If X is $<\varkappa$-b.d. and φ is an arbitrary selfmap, does this imply that $F^{\#}$ is a $P_{<\varkappa}$-set?"
Fortunately the question can be answered for embeddings.

2.7. Theorem. *Let X be a $<\varkappa$-basically disconnected space.*
If φ is an embedding of X then the fixed-point set F is a $P_{<\varkappa}$-set.
In particular, F itself is $<\varkappa$-basically disconnected.

Proof. We use transfinite induction.
Note that 2.5 implies that the statement is true for $\varkappa = \omega_1$.

Assume $\gamma > \omega_1$ is a cardinal such that for all $\delta < \gamma$ the theorem is correct.
Let Y be a $<\gamma$-basically disconnected space and let φ be an self-embedding.
Note that Y is $<\delta$-basically disconnected, for all $\delta < \gamma$, so F is a $F_{<\delta}$ set.
This already proves the result in the case that γ is a limit cardinal.
Next, assume that γ is a successor, say $\gamma = \lambda^+$.

Note: If $U \subseteqq Y$ is a union of δ clopen sets with $\delta < \lambda$ such that $(+)$
U is both φ and φ^{-1}-invariant and $U \cap F = \emptyset$, then $\operatorname{cl} U \cap F = \emptyset$.

Indeed, cl U is (being clopen, see 1.2) $<\lambda^+$-basically disconnected, and the fixed-point set F' of $\varphi|\operatorname{cl} U$ must be a P_{δ^+}-set in $\operatorname{cl} U - U$. So $F' = \emptyset$.
Let $\{A_\alpha: \alpha < \lambda\}$ be a collection of closed sets with $A_\alpha \cap F = \emptyset$.
First we extend the A_α to suitable clopen sets C_α with:

$$A_\alpha \subseteqq C_\alpha \subseteqq X - F$$

C_α is both φ- and φ^{-1}-invariant.

$$C_\alpha \subseteqq C_\beta \text{ if } \alpha \leqq \beta < \lambda .$$

This can be done by induction using 1.6 and $(+)$.
Indeed, if for $\alpha < \beta$ $(\beta < \lambda)$ the C_α are defined, then $(+)$ implies that $\operatorname{cl}(\cup C_\alpha)$ is clopen and disjoint from F.
Next use 1.6 to find a φ and φ^{-1}-invariant clopen set $G = C_\beta$ with $\operatorname{cl}(\cup C_\alpha) \cup \cup A_\beta \subseteqq G \subseteqq X - F$.

Next, we want to show that if $U = \cup\{C_\alpha: \alpha < \lambda\}$ has the property that $\operatorname{cl} U \cap F = \emptyset$.
Note that by 1.2., U is C^*-embedded in Y, so it suffices to find a 3-partition U^i $(i = 1, 2, 3)$ of U into pairwise-disjoint open sets with $\varphi(U^i) \cap U^i = \emptyset$.
As follows: use 1.7 and 1.8 to obtain increasing 3-partitions of C_α.

Indeed, find a 3-partition D_0^i $(i = 1, 2, 3)$ of C_0.
If D_α^i $(i = 1, 2, 3)$ is a 3-partition of C_α, use 1.8 to extend this to a 3-partition $D_{\alpha+1}^i$ $(1 = 1, 2, 3)$ of $C_{\alpha+1}$.
For limit ordinals β, put $E_\beta^i = \operatorname{cl}[\cup D_\alpha^i: \alpha < \beta]$ $(i = 1, 2, 3)$. Then E_β^i is clopen in Y, and $\varphi(E_\beta^i) \cap E_\beta^i = \emptyset$
Next use 1.8 to extend the E_β^i to a 3-partition D_β^i $(i = 1, 2, 3)$ of C_β.
Finally, put

$$U^i = \cup\{D_\alpha^i: \alpha < \lambda\}$$

Then the U^i are pairwise disjoint open sets with $\varphi(U^i) \cap U^i = \emptyset$, and they cover U.

The reason that the method in 2.7 does not work for arbitrary maps is that for such maps no lemma similar to 1.8 is available. If however the map is $\# < \varkappa$, then 1.6 is used as a substitute for 1.8.

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