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On Carathéodory Type Multifunctions

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We study multifunctions defined on the product of metric spaces, which are product Borel-measurable and semicontinuous in one variable.

Let T, X and Y be topological spaces. $\mathscr{P}(Y)$ stands for the family of all nonempty subsets of Y, and $\mathscr{B}(T)$ for the Borel σ-field on T. By a multifunction from T to Y we mean any map $\phi : T \to \mathscr{P}(Y)$. We say that ϕ is lower (upper) semicontinuous if for each open $G \subset Y$ the set $\phi^{-}(G) = \{t : \phi(t) \cap G \neq 0\}$ (respectively, $\phi^{+}(G) = \ t : \phi(t) \subset G\})$ is open in T. A multifunction is called continuous if it is lower and upper semicontinuous. We say that ϕ is Borel-measurable if for each open $F \subset Y$, $\phi^{-}(G) \in \mathscr{B}(T)$. Note, that Himmelberg [4] calls such ϕ weakly measurable.

In this paper we study multifunctions from $T \times X$ to Y which are $\mathscr{B}(T \times X)$-measurable and semicontinuous in the second variable. We propose a method which allows to obtain some results on such multifunctions from known theorems on semicontinuous multifunctions. As an application of this method, we prove the existence of Carathéodory selectors, some approximation results, and a "sandwich theorem" for multifunctions.

Our approach is based on two known results:

Theorem 1. *(see e.g. [3], II-18). Let T and X be Polish spaces, $\{G_n : n \in \mathbb{N}\}$ a base in X, and A a Borel subset of $T \times X$ with open vertical section $A_t = \{x : (t, x) \in A\}$, $t \in T$. Then A has a representation*

$$A = \cup \{B_n \times G_n : n \in \mathbb{N}\},$$

where $B_n \in \mathscr{B}(T)$.

The next theorem is a consequence of the result of Miller ([8], Th. 2.3).

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Theorem 2. *Let T with the topology $\mathcal{G}$ be a separable and metrizable space. Then for any sequence (B_n) of Borel sets in T there is a separable and metrizable topology $\mathcal{G}'$ on T such that $\mathcal{G} \subset \mathcal{G}'$, each $B_n \in \mathcal{G}'$, and the σ-fields generated by $\mathcal{G}$ and $\mathcal{G}'$ are the same.*

Let ϕ be a multifunction from $T \times X$ to Y. A function f: $T \times X \to Y$ is called a Carathéodory selector of ϕ if it is $\mathcal{B}(T \times X)$-measurable, continuous in x, and $f(t, x) \in \phi(t, x)$ for all $(t, x) \in T \times X$. The next theorem follows from a result of Michael ([7], Lemma 5.2).

Theorem 3. *Suppose T and X are Polish spaces, Y is a separable Banach space, and $\phi : T \times X \to \mathcal{P}(Y)$ is $\mathcal{B}(T \times X)$-measurable, lower semicontinuous in x, and closed convex-valued. Then there exists a sequence (f_n) of Carathéodory selectors of ϕ such that $\phi(t, x) = \mathrm{cl}\{f_n(t, x) : n \in \mathbb{N}\}, (t, x) \in T \times X$.*

Proof. Let $\mathcal{V}$ be a countable base of Y. For each $V \in \mathcal{V}$ the set $\phi^{-}(V)$ is Borel in $T \times X$ and has open vertical sections. Because of Theorem 1,

$$\phi^{-}(V) = \cup \{B_n(V) \times G_n : n \in \mathbb{N}\},$$

where $B_n(V) \in \mathcal{B}(T)$ and G_π are open in X. Now we apply Theorem 2 to the countable family of Borel sets $\{B_n(V) : V \in \mathcal{V}, n \in \mathbb{N}\}$. Denote by T' the set T with the new topology $\mathcal{G}'$. Then all rectangles $B_n(V) \times G_\pi$ are open in $T' \times X$. It implies that ϕ treated as a multifunction on $T' \times X$ is lower semicontinuous. By the result of Michael ([7], Lemma 5.2), there is a sequence $f_n : T' \times X \to Y$ of continuous selectors of ϕ such that $\phi(t, x) = \mathrm{cl}\ \{f_n(t, x) : n \in \mathbb{N}\}$ for all $(t, x) \in T' \times X$. Since $\mathcal{B}(T' \times X) = \mathcal{B}(T \times X)$ and the topology of X does not change, each f_π is $\mathcal{B}(T \times X)$-measurable and continuous in x. It completes the proof.

Remark. This result seems to be new. In known theorems on the existence of Carathéodory selectors one assumes that T is a measurable space with a complete σ-field (cf. [9]).

It is known that a semicontinuous multifunction can be approximated by a monotone sequence of continuous ones. Results of this type were given e.g. by Hukuhara [5], Aseev [1] and De Blasi [2].

In the case of a lower semicontinuous multifunction such an approximation can be easily obtained. In fact, let $\psi : X \to \mathcal{P}(Y)$, where X is metrizable and Y a separable Banach space, be lower semicontinuous and closed convex-valued. Then there exists a sequence (f_n) of continuous selectors of ψ such that for each $x \in X$, $\psi(x) = \mathrm{cl}\ \{f_n(x) : n \in \mathbb{N}\}$ ([7], Lemma 5.2). Now define $\psi_n : X \to \mathcal{P}(Y)$ as $\psi_n(x) = \mathrm{conv}\ \{f_1(x), \ldots, f_n(x)\}, n \in \mathbb{N}$. It is not difficult to see that (ψ_n) is an increasing sequence of continuous and compact convex-valued multifunctions, such that $\psi(x) = \mathrm{cl} \cup \{\psi_n(x) : n \in \mathbb{N}\}$, $x \in X$. Moreover, if ψ is compact-valued then the sequence (ψ_n) converges pointwise to ψ in the Hausdorff distance.

By the above observation we obtain the following:

Theorem 4. *Let T, X, Y and ϕ be such as in Theorem 3. Then there exists a sequence $\phi_n : T \times X \to \mathscr{P}(Y)$ of $\mathscr{B}(T \times X)$-measurable multifunctions which are continuous in x, compact convex-valued, and for each $(t, x) \in T \times X$ the following conditions hold:*

(i) $\phi_n(t, x) \subset \phi_{n+1}(t, x) \subset \phi(t, x),\ n \in \mathbb{N}$.

(ii) $\phi(t, x) = \mathrm{cl} \cup \{\phi_n(t, x) : n \in \mathbb{N}\}$.

Proof. In the same way as in the proof of Theorem 3, we can extend the topology of T such that ϕ treated as a multifunction on $T' \times X$ is lower semicontinuous. Thus there is a sequence of multifunctions $\phi_n : T' \times X \to \mathscr{P}(Y)$ which are continuous and compact convex-valued, and the conditions (i) — (ii) are satisfied. Since ϕ_π are $\mathscr{B}(T \times X)$-measurable, it completes the proof.

The next theorem follows from a result of Aseev ([1], Th. 2).

Theorem 5. *Suppose T and X are Polish spaces, and $\phi : T \times X \to \mathscr{P}(\mathbb{R}^m)$ is $\mathscr{B}(T \times X)$-measurable, upper semicontinuous in x and compact convex valued. Then there exists a sequence $\phi_n : T \times X \to \mathscr{P}(\mathbb{R}^m)$ of $\mathscr{B}(T \times X)$-measurable multifunctions, which are continuous in x, compact convex-valued, and for each $(t, x) \in T \times X$ the following conditions are satisfied:*

(i) $\phi_n(t, x) \supset \phi_{n+1}(t, x) \supset \phi(t, x),\ n \in \mathbb{N}$.

(ii) $\phi(t, x) = \cap \{\phi_n(t, x) : n \in \mathbb{N}\}$.

Proof. Let $\mathscr{V}$ be a countable base of $\mathbb{R}^m$. Since ϕ is compact-valued, for each open $G \subseteq \mathbb{R}^m$

$$\phi^+(G) = \cup \{\phi^+(V_1 \cup \ldots \cup V_k) : V_i \in \mathscr{V} \text{ and } V_i \subset G \text{ for } i = 1, \ldots, k;\quad k \in \mathbb{N}\}.$$

Moreover, the measurability of ϕ is equivalent to the following condition: $\phi^+(G) \in \mathscr{B}(T \times X)$ for each open $G \subset \mathbb{R}^m$ (see e.g. [4], Th. 3.1).

We may assume that the base $\mathscr{V}$ is closed under finite unions. For each $V \in \mathscr{V}$ the set $\phi^+(V)$ is Borel in $T \times X$ and has open vertical sections. Thus it has a representation

$$\phi^+(V) = \cup \{B_n(V) \times G_n : n \in \mathbb{N}\},$$

where $B_n(V) \in \mathscr{B}(T)$ and G_π are open in X. Now we apply Theorem 2 to the family $\{B_n(V) : V \in \mathscr{V},\ n \in \mathbb{N}\}$. Denote by T' the set T with the extended topology $\mathscr{G}'$. If we treat ϕ as a multifunction on $T' \times X$, it is upper semicontinuous. Hence, there is a sequence of multifunctions $\phi_n : T \times X \to \mathscr{P}(\mathbb{R}^m)$ which are continuous and compact convex-valued, and the conditions (i) — (ii) hold (see [1], Th. 2). This completes the proof.

Remark. Approximations of Carathéodory type multifunctions were studied by Tsalyuk [10] and Zygmunt [11], [12], but our Theorems 4 and 5 seem to be new.

The last part of this paper is devoted to a sort of sandwich theorem for multifunctions. Aseev ([1], Th. 4) proved that if $\phi, \psi : T \to \mathscr{P}(\mathbb{R}^n)$ are compact convex-valued, and such that ϕ is upper semicontinuous, ψ is lower semicontinuous and $\phi(t\} \subset \psi(t\}$ for all $t \in T$, then there exists a continuous and compact con vex-valued multifunction $\chi : T \to \mathscr{P}(\mathbb{R}^n)$ satisfying $\phi(t) \subset \chi(t) \subset \psi(t)$ $t \in T$. Related result was obtained by De Błasi ([2], Th. 5.1). Recently the first author has generalized the result the result of Aseev for multifunctions with values in a Banach space [6]. As a consequence, we have the following theorem:

Theorem 6. *Let T and X be Polish spaces, Y a separable Banach space, and $\varphi, \psi : T \times X \to \mathscr{P}(Y)$ two $\mathscr{B}(T \times X)$-measurable multifunctions with compact convex values. If ϕ is upper semicontinuous in x, ψ is lower semicontinuous ın x and $\phi(t, x) \subset \psi(t, x)$ for $(t, x) \in T \times X$, then there is a $\mathscr{B}(T \times X)$ measurable multifunction $\chi : T \times X \to \mathscr{P}(Y)$ which is continuous in x, compact convex-valued, and such that $\phi \subset \chi \subset \psi$ pointwise.*

Proof. Combining the proofs of Theorems 3 and 5, we can extend the topology of T such that ϕ and ψ treated as multifunctions on $T' \times X$ are, respectively, upper semicontinuous and lower semicontinuous. By the result of the first author [6], there exists a continuous multifunction $\chi : T' \times X \to \mathscr{P}(Y)$ with compact convex values, such that $\varphi \subset \chi \subset \psi$ pointwise. It completes the proof.

Remark. Theorem 1 and, consequently, Theorem 3 – 6 hold, when T and X are analytic subsets of Polish spaces. Further, our results can be generalized for the case when all multifunctions are defined on an analytic subset D of $T \times X$. It can be proved by the same method.

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