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Amoeba Relation and Galois-Tukey Connections

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In this paper we continue the research concerning generalized Galois-Tukey connections between explicit relations on classical objects of real analysis. We introduce a new (internal) structure related with the amoeba forcing (in contrast with external structures related with generic objects) and we show it is equivalent to the inclusion restricted to measure zero sets. Finally we complete results adding some extremal relations in the Galois-Tukey lattice of binary relations.

This paper is a continuation of the research we begun in [V1] and continued in [V2] and [V3] where we refer the reader concerning the motivation to.

Definitions and the theory

For the paper to be selfcontained we present following notation. Assume R is a binary relation. The complement (or negation) of R is denoted by $\neg R = \{(x, y) : x \in \mathrm{dom}(R) \ \&\ y \in \mathrm{rng}(R) \ \&\ (x, y) \notin R\}$. The inverse $R^{-1} = \{(y, x) : (x, y) \in R\}$.

A set $B \subseteq \mathrm{dom}(R)$ is said to be R-unbounded if $(\forall x \in \mathrm{rmg}(R))(\exists x \in B)((x, y) \notin R)$. A set $B \subseteq \mathrm{rng}(R)$ is said to be R-dominating if $(\forall x \in \mathrm{dom}(R))(\exists y \in D)((x, y) \in R)$. The corresponding "unboundedness"-like "$\mathfrak{d}$ominatedness"-like (abbreviated $\mathfrak{b}$- and $\mathfrak{d}$-like) cardinal characteristics are the following:

$$\mathfrak{b}(R) = \min\{|B| : B \subseteq \mathrm{dom}(R) \ \&\ B \text{ is an } R\text{-unbounded set}\}$$

and

$$\mathfrak{d}(R) = \min\{|D| : D \subseteq \mathrm{rng}(R) \ \&\ D \text{ is an } R\text{-dominating set}\}$$

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In order to avoid undefined or trivial cases we consider in the future only relations R such that rng $(\neg R)$ = rng (R) and dom $(\neg R)$ (when necessary we restrict R to some $X \times Y$). Note that in this case $\mathfrak{b}(R) \geqq 2$ and $\mathfrak{d}(R) \geqq 2$.

A restriction of R to $X \times Y$ is the relation $R \cap (X \times Y) = \{(x, y) \in R : x \in X \ \& \ y \in Y\}$. This restriction will be sometimes abbreviated as $R_{X \times Y}$. We now list definitions of some binary relations and set-theoretic representation of objects of real analysis.

$\mathcal{O}_{<\frac{1}{2}}$ is the set of all open subsets of $[0, 1]$ with measure less than $\frac{1}{2}$;

$$\mathbb{S} = \{f \in {}^{\omega}([\omega]^{<\omega}) : \forall_n^{\infty} |f(n)| < n^2\};$$

$\mathbb{L}$ is the ideal on reals of measure zero sets and $\mathbb{K}$ of sets of first category;

$[\omega]^{\omega}$ is the set of infinite subsets of natural numbers and ${}^{\omega}\omega$ is the set of all functions from natural to natural numbers;

ℓ^1 are absolutely summable series of reals and ℓ^{∞} is the set of all bounded positive sequences of reals, $h_0 \subseteqq \ell^{\infty}$ are those having 0 as an accumulation point;

$$L = \{(a, X) : a \in \ell^{\infty} \ \& \ X \in [\omega]^{\omega} \ \& \ \lim_{n \in X} a(n) \text{ exists}\};$$

$$C = \{(a, X) : a \in \ell^{\infty} \ \& \ X \in [\omega]^{\omega} \ \& \ \sum_{n \in X} |a(n)| < +\infty\};$$

$$D_{{}^{\omega}\omega \times ({}^{\omega \times \omega}\omega)} = \{(f, g) : f \in {}^{\omega}\omega \text{ and } g \in {}^{\omega \times \omega}\omega \text{ and } \exists n \forall k f(k) = g(n, k)\}$$

(see [B]).

We also consider the equality restricted to pairs of reals, $(=_{(\mathbb{R})^2})$ (in the context of Galois-Tukey connection mentioned first in [T]).

An ordered pair of functions (E, F) is called a (generalized) Galois-Tukey connection (also abbreviated as a GT-connection) from R to S if the following holds:

(a) $E : \text{dom}(R) \to \text{dom}(S)$

(b) $F : \text{rng}(S) \to \text{rng}(R)$

(c) $(\forall x \in \text{dom}(R))(\forall v \in \text{rng}(S))(E(x), v) \in S$ implies $(x, F(v)) \in R$.

The fact that there is a GT-connection from R to S is often rephrased as "R is simpler than S" (using a motivation of J. W. Tukey coming from convergence structures). The relation "to be simpler than" forms (after some necessary factorization) a partial order on the class of all binary relations, which we denote by $R \leqq_{GT} S$.

Assume $X_0 \subseteqq \text{dom}(R)$ and $Y_0 \subseteqq \text{rng}(R)$ and consider the relation $R_0 = R \cap (X_0 \times Y_0)$. We say that R_0 is a substantial part of R if R and R_0 are equivalent in the sense of generalized GT-connection in a special way, that is, that the F-mapping of $R \leqq_{GT} R_0$ and the E-mapping of $R_0 \leqq_{GT} R$ are identities. So for instance $\subseteqq$ restricted to G_{δ} sets of measure zero is a substantial part of $\subseteqq_{(\mathbb{L})^2}$. We will not complicate the topic and in the sequel we always assume we are working with some definable substantial part of the relation in question.

We studied this structure from the point of view of algebraic theory of categories in [V2] and from the point of view of lattice theory in [V3]. Here we continue the research of [V1], which is oriented mainly to establish connections between explicitly defined relations and their influence on forcing, moreover we add a consideration about the lattice theoretical point of view.

Consider two binary relations R and S and define

$$R \oplus S = \{((a, \varepsilon), (z, v)) : aRz \ \& \ \varepsilon = 0 \quad \text{or} \quad aSv \ \& \ \varepsilon = 1\}$$

and similarly

$$R \otimes S = \{((w, u), (a, \varepsilon)) : wRa \ \& \ \varepsilon = 0 \quad \text{or} \quad uSa \ \& \ \varepsilon = 1\}$$

The relation $R \otimes S$ is the infimum in the partial order $\leqq_{GT}$ and the relation $R \oplus S$ is the supremum in the partial order $\leqq_{GT}$ of R and S. So $\leqq_{GT}$ forms a lattice (or equivalently a category with products and coproducts, see [V2] and [V3]).

We repeat following observations from [V1] on preservation of forcing properties to recall what is known on influence of GT-connections to forcing. Consider two models $M \subseteqq N$ of set theory and a definable binary relation R. We assume moreover that all relations are absolute i.e. for x, $y \in M$, xR^My iff $x^NR^Ny^N$. (Note that all explicit relatons in our paper are absolute.) Then $x \in \operatorname{dom}(R^N)$ is said to be an R^M-unbounded object if for all $y \in \operatorname{rng}(R^M)$ it is not the case that xR^Ny^N. An $y \in \operatorname{rng}(R^N)$ is said to be an R^M-dominating object if for all $x \in \operatorname{dom}(R^M)$ we have x^NR^Ny. (Our convention about domains and ranges in the begining of the paper avoids trivial cases.)

Sometimes properties of generic extensions are investigated concerning the question whether $\operatorname{dom}(R^M)^N$ is R-unbounded in N and whether $\operatorname{rng}(R^M)^N$ is R-dominating in N. Observe that

(a) $\{x^N : x \in \operatorname{dom}(R^M)\} = \operatorname{dom}(R^M)^N$ is R-unbounded set in N iff there are no R^M-dominating objects in N

and

(b) $\{x^N : x \in \operatorname{rng}(R^M)\} = \operatorname{rng}(R^M)^N$ is R-dominating set in N iff there are no R^M-unbounded objects in N.

A definable mapping $f : X \to Y$ is absolute according to models $M \subseteqq N$ if $(\forall x \in X^M)(f^N(x^N) = (f^M(x))^N$. That is: if f as interpreted in N acts on objects with codes in M, it gives the same result as when interpreted in M.

Assume $M \subseteqq N$ and R is simpler than S witnessed by (E, F) and moreover E is absolute according to M and N. Then if there is an S^M-dominating object then there is a R^M-dominating object. So the existence of dominating objects is preserved in the opposite direction as absolute E-mappings are going. Of course equivalently, under the same assumptions if $\operatorname{dom}(R^M)^N$ is an R-unbounded set in N then $\operatorname{dom}(S^M)^N$ is an S-unbounded set in N.

Analogously consider $M \subseteq N$ and R is simpler than S witnessed by (E, F) and moreover F is absolute according to $M \subseteq N$. Then if there is an R^M-unbounded object then there is an S^M-unbounded object; and similarly $\operatorname{rng}(S^M)^N$ being S-dominating set implies $\operatorname{rng}(R^M)^N$ is R-dominating.

Observe moreover that an R^M-unbounded object is an $(\neg R^{-1})^M$-dominating object and R^M-dominating object is an $(\neg R^{-1})^M$-unbounded object. Using this preservation properties we can derive lot of informations about generic extensions (having some basic informations).

Recal that in a simple Cohen extension $N_1 = M[c]$ there is an $(\in \cap([0,1] \times \mathbb{K}))^M$-unbounded object (the very Cohen real) and that there are no $(\in \cap([0, 1] \times \mathbb{K}))^M$-dominating objects. Analogously for $N_2 = M[r]$ a random extension there is an $(\in \cup([0, 1] \times \mathbb{L}))^M$-unbounded object (the very random real) and no $(\in \cap([0, 1] \times \times \mathbb{K}))^M$-dominating object ([V1]). We extend this dependencies between forcing and GT-connections in the sequel by an example of another type.

Some new relations and connections

The motivation for the structure $\subseteq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}$ came first from the paper [M] of A. W. Miller, where he noticed the constructive character of the proof (namely (2) of Theorem 1). Further the structure $\subseteq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}$ is motivated by amoeba forcing-it is rather more internal like structure (i.e. connected (at least in one coordinate) with forcing conditions)-on contrast with other external like structures (connected with generic objects). Though the following theorem follows from results below, we present it for historical reasons and to give a direct constructive proof.

The numbering of indexes of mappings in connections is a continuation of the numbering from [V1].

Theorem 1. *(J. Chichoń, F. Galvin (see [M]))*

(1) *The relation $\leqq^*_{({}^{\omega}\omega)^2}$ is simpler than $\subseteq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}$*

(2) *or equivalently, there are mappings $E_{14} : {}^{\omega}\omega \to \mathbb{L}$ and $F_{14} : \mathcal{O}_{<\frac{1}{2}} \to {}^{\omega}\omega$ such that if $E_{14}(g) \subseteq G$ then $g \leqq {}^*F_{14}(G)$*

(3) *and consequently* $\mathfrak{b}(\subseteq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}) \leqq \mathfrak{b}$ and $\mathfrak{d}(\subseteq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}) \geqq \mathfrak{d}$.

Proof. As in [M], namely put

$$E_{14}(g) = \{x \in {}^{\omega}2 : \exists^{\infty}_n x \restriction [g(n), g(n+1)) \text{ is identical } 0\}.$$

For an open $G \in \mathcal{O}_{<\frac{1}{2}}$ define first $G_n = \cup\{[s] : s \in {}^n2 \text{ and } [s] \subseteq G\}$ and fix a sequence of positive reals ε_n such that $\sum_{n=0}^{\infty} 2^{n^2}\varepsilon_n < \frac{1}{2}$. Then put

$$F_{14}(G)(n) = \min\{k : \mu(G \setminus G_n) < \varepsilon_n\}$$

Theorem 2.

(1) *The relation* $\subseteqq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}$ *is simpler than* $\subseteqq_{(\mathbb{L})^2}$

(2) *or equivalently, there are mappings* $E_{15} : \mathbb{L} \to \mathbb{L}$ *and* $F_{15} : \mathbb{L} \to \mathcal{O}_{<\frac{1}{2}}$ *such that if* $E_{15}(X) \subseteqq Y$ *then* $X \subseteqq F_{15}(Y)$

(3) *(implicitly in [M]) and consequently* $\operatorname{add}(\mathbb{L}) \leqq \mathfrak{b}(\subseteqq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}})$ *and* $\operatorname{cof}(\mathbb{L}) \geqq \mathfrak{d}(\subseteqq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}})$.

Proof. Is easy, E_{15} is identity and $F_{15}(Y)$ is any $G \in \mathcal{O}_{<\frac{1}{2}}$ with $Y \subseteqq G$.

The situation is simplified by following theorem, depending on and quoting an idea of J. Cichoń with his permision ([C]).

Theorem 3.

(1) *The relation* $\subseteqq_{(\mathbb{L})^2}$ *is simpler than* $\subseteqq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}$

(2) *or equivalently, there are mappings* $E_{16} : \mathbb{L} \to \mathbb{L}$ *and* $F_{16} : \mathcal{O}_{<\frac{1}{2}} \to \mathbb{L}$ *such that if* $E_{16}(A) \subseteqq G$ *then* $A \subseteqq F_{16}(G)$

(2) *([C], J. Chichoń's answer to our question) and consequently* $\mathfrak{b}(\subseteqq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}) \leqq \operatorname{add}(\mathbb{L})$ *and* $\mathfrak{d}(\subseteqq_{\mathbb{L} \times \mathcal{O}_{<\frac{1}{2}}}) \geqq \operatorname{cof}(\mathbb{L})$.

Proof. Put

$$E_{16}(A) = A + \mathcal{Q} = \{x + q : x \in A \ \&\ q \in \mathcal{Q}\}.$$

For an open $G \in \mathcal{O}_{<\frac{1}{2}}$ define

$$F_{16}(G) = \cap\{G + q : q \in \mathcal{Q}\}$$

The structure $\boldsymbol{D}_{\omega_\omega \times (\omega \times \omega_\omega)}$ comes from [B] and represents the relation with $\mathfrak{b}(R) = \aleph_1$ and $\mathfrak{d}(R) = 2^{\aleph_0}$. The relation $(=_{(R)^2})$ and its role in our context was motivated by the paper of S. Todorčevič ([T]). In [V3] it is proved that whenever $2 \leqq \mathfrak{b}(R), \mathfrak{d}(R) \leqq 2^{\aleph_0}$ then $(\neq_{\mathfrak{R}}) \leqq_{GT} R \leqq_{GT} (=_{\mathfrak{R}})$. So this sublattice has the least and greatest element. It is natural to ask whether $\boldsymbol{D}_{\omega_\omega \times (\omega \times \omega_\omega)}$ does play a similar role for relations with $\aleph_1 \leqq \mathfrak{b}(R), \mathfrak{d}(R) \leqq 2^{\aleph_0}$. In our case for relations from [V1] and all those restricted to fields of size at most $\mathfrak{c}$ it is so.

Theorem 4. *Recall we restricted to relations with* $\operatorname{dom}(\neg R) = \operatorname{dom}(R)$ *and* $\operatorname{rng}(\neg R) = \operatorname{rng}(R)$. *In this case* $\mathfrak{b}(R) \leqq |\operatorname{rng}(R)|$ *and* $\mathfrak{d}(R) \leqq |\operatorname{dom}(R)|$. *Moreover having a relation* R *with* $\mathfrak{b}(R) \geqq \aleph_1$ *and* $|\operatorname{dom}(R)| \leqq \mathfrak{c}$ *it follows* $R \leqq_{GT} \boldsymbol{D}_{\omega_\omega \times (\omega \times \omega_\omega)}$ *(and for dual conditions dually for* $\neg R^{-1}$ *and* $\neg\boldsymbol{D}^{-1}_{\omega_\omega \times (\omega \times \omega)}$*).*

Proof. Just take the E mapping any injection of $\operatorname{dom}(R)$ into ${}^{\omega}\omega$. Then as any uncountable subset of ${}^{\omega}\omega$ is $\boldsymbol{D}_{\omega_{\omega} \times (\omega \times \omega_{\omega})}$-unbounded, E maps R-unbounded sets onto $\boldsymbol{D}_{\omega_{\omega} \times (\omega \times \omega_{\omega})}$-unbounded ones.

We give following corollaries, first because in the top of diagrams in [V1] there were two relations, namely $L_{\ell^{\infty} \times [\omega]^{\omega}}$ and $\leqq^{*}_{(\ell^{1})^{2}}$ and second to give constructive proofs.

Corollary 5.

(1) *The relation* $L_{\ell^{\infty} \times [\omega]^{\omega}}$ *is simpler than* $\boldsymbol{D}_{\omega_{\omega} \times (\omega \times \omega_{\omega})}$

(2) *or equivalently, there are mappings* $E_{17} : \ell^{\infty} \to {}^{\omega}\mathcal{Q}$ *and* $F_{17} : {}^{\omega \times \omega}\mathcal{Q} \to [\omega]^{\omega}$ *such that if* $\exists n \forall k E_{17}(a)(k) = g(n, k)$ *then* $\lim\limits_{n \in F_{17}(g)} a(n)$ *exist.*

Proof. E_{17} is an approximation of a sequence $a \in \ell^{\infty}$ by rationals and for an $g \in {}^{\omega \times \omega}\mathcal{Q}$ (seen as countably many sequences) there is an $F_{17}(g) \in [\omega]^{\omega}$ such that all $g(n, -)$'s converge on it (this is just the fact that $\mathfrak{s}_{\sigma} \geqq \aleph_1$)).

Corollary 6.

(1) *The relation* $\leqq^{*}_{(\ell^{1})^{2}}$ *is simpler than* $\mathbf{D}_{\omega_{\omega} \times (\omega \times \omega_{\omega})}$

(2) *or equivalently, there are mappings* $E_{18} : \ell^{1} \to {}^{\omega}\mathcal{Q}$ *and* $F_{18} : {}^{\omega \times \omega}\mathcal{Q} \to \ell^{1}$ *such that if* $\exists n \forall k E_{18}(a)(k) = g(n, k)$ *then* $a \leqq {}^{*}F_{18}(g)$

Proof. Take $E_{18}(a) \in {}^{\omega}\mathcal{Q} \cap \ell^{1}$ to be some upper bound of a and $F_{18}(g)$ is an upper bound for all those $g(n, -)$'s which are in ℓ^{1} (this is possible because $\mathfrak{b}(\leqq^{*}_{(\ell^{1})^{2}}) = \operatorname{add}(\mathbb{L}) \geqq \aleph_1$).

In [V1] we considered relations $\precsim^{*}_{(\ell^{1}_{+})^{2}}$, $\supseteqq^{*}_{([\omega]^{\omega})^{2}}$ and $\geqq^{*}_{(\ell^{\infty} \vee 1)^{2}}$ which are inverse of ordering generating complete Boolean algebras (which are at least consistently isomorphic each other) and the ordering $C_{h_0 \times [\omega]^{\omega}}$. We show they are all GT-equivalent to $(=_{\mathscr{R}})$.

Lemma 7. *If* $P, \leqq$ *is a partial ordering with a pairwise* $\leqq$*-disjoint family of size at least* $\mathfrak{c}$ *then* $=_{(\mathscr{R})^{2}}$ *is simpler than* $\geqq$ *(i.e. simpler than* $\leqq^{-1}$*).*

Proof. Let $A \in [P]^{\geqq \mathfrak{c}}$ is the antichain. Let $E : Real \to A$ is an injection. For $x \in P$, if there is an $a \in Real$ such that $E(a) \geqq x$, then (this a is unique) define $F(x) = a$, else F is defined arbitralily.

Lemma 8. $=_{(\mathscr{R})^{2}}$ *is simpler than* $C_{h_0 \times [\omega]^{\omega}}$.

Proof. Similarly as before, take an $\mathscr{A} \subseteqq [\omega]^{\omega}$ a maximal disjoint family of size $\mathfrak{c}$. For $A \in \mathscr{A}$ define $a_A(n) = \dfrac{1}{2}$ if $n \notin A$ and $a(n) = \dfrac{1}{|A \cap n| + 1}$ if $n \in A$. Let $E : Real \to \{a_A : A \in \mathscr{A}\}$ is an injection and for $B \in [\omega]^{\omega}$ is $(a_A, B) \in C_{h_0 \times [\omega]^{\omega}}$ iff $B \subseteqq^{*} A$. So put $F(B) = E^{-1}(a_A)$ if $B \subseteqq^{*} A$, if there is such A, or arbitrarily if not.

The constructions of following four results are essentialy due to T. Bartoszynski ([Ba]) and J. Raisonier and J. Stern ([SR]), (in our language) they are observed in

[T] and some of them implicitly in [V1] (we add them just for the sake of completeness and to give constructive proofs).

Theorem 9. *([Ba], [SR], [T])*

(1) *The relation* $\subseteqq^* \cap ({}^\omega\omega \times \mathbb{S})$ *is simpler than* $\subseteqq_{(\mathbb{L})^2}$

(2) *or equivalently, there are mappings* $E_{19} : {}^\omega\omega \to \mathbb{L}$ *and* $F_{19} : \mathbb{L} \to \mathbb{S}$ *such that if* $E_{19}(f) \subseteqq X$ *then* $f \subseteqq^* F_{19}(X)$

Proof. Put

$$E_{19}(X) = \bigcap_{l=0}^{\infty} \bigcup_{i=l}^{\infty} G(i, f(i)) .$$

$$F_{19}(Y)(i) = \bigcup_{n=0}^{\infty} \{j : i \geqq h_Y(n) \quad \& \quad G(i,j) \cap U_n^{K^Y} = \emptyset\}$$

where $h_Y(n) = \min\left\{k : \forall i \geqq k \,|\{j : U_n^{K_Y} \cap G(i,j) = \emptyset\}| \leqq \frac{(i+1)^2}{2^{n+1}}\right\}$

For the following theorem take a base of topology U_n with $\sum_{n=0}^{\infty} \mu(U_n) < \infty$ and we work with increasing functions from ${}^\omega\omega$ and increasing pipes from $\mathbb{S}$ (i.e. with $\min(S(i))$ tending to infinity, as a substantial part of relation).

Theorem 10. *([Ba], [SR], [T])*

(1) *The relation* $\subseteqq_{(\mathbb{R})^2}$ *is simpler than* $\subseteqq^*_{({}^\omega\omega \times \mathbb{S})}$

(2) *or equivalently, there are mappings* $E_{20} : \mathbb{L} \to {}^\omega\omega$ *and* $F_{20} : \mathbb{S} \to \mathbb{L}$ *such that if* $E_{20}(X) \subseteqq^* S$ *then* $X \subseteqq F_{20}(S)$

Proof. Take the required functions such that following holds

$$X \subseteqq \bigcap_{l=0}^{\infty} \bigcup_{i=l}^{\infty} U_{E_{20}(X)(i)}$$

and

$$F_{20}(S) = \bigcap_{l=0}^{\infty} \bigcup_{i=l}^{\infty} \{U_j : j \in S(i)\}$$

Theorem 11. *([Ba], [SR])*

(1) *The relation* $\subseteqq_{(\mathbb{K})^2}$ *is simpler than* $\subseteqq^*_{({}^\omega\omega \times \mathbb{S})}$

(2) *or equivalently, there are mappings* $E_{21} : \mathbb{K} \to {}^\omega\omega$ *and* $F_{21} : \mathbb{S} \to \mathbb{K}$ *such that if* $E_{21}(X) \subseteqq^* S$ *then* $X \subseteqq F_{21}(S)$

Proof. Put

$$E_{21}(X)(i\} = \min\{j : H_i^X \cap V(i,j) = \emptyset\}$$

and

$$F_{21}(S) = [0, 1]\setminus\bigcap_{l=0}^{\infty} \bigcup_{i=l}^{\infty} \bigcap \{V(i,j) : j \in S(i)\}.$$

in the notation of [V1].

Theorem 12. *([Ba])*

(1) *The relation* $\subseteq^*_{({}^\omega\omega \times \mathbb{S})}$ *is simpler than* $\subseteq^*_{(\ell^1)^2}$

(2) *or equivalently, there are mappings* $E_{22} : {}^\omega\omega \to \ell^1$ *and* $F_{22} : \ell^1 \to \mathbb{S}$ *such that if* $E_{22}(f) \leqq^* a$ *then* $f \supseteq_* F_{22}(a)$

Proof. Put $E_{22}(f)(n) = \max\{k^{-2} : n = f(k)\}$ if $n \in \operatorname{rng}(f)$ and 0 else. And $F_{22}(a)(k) = \{n : a(n) \geqq k^{-2}\}$.

There are many possible questions on the structure and interrelations between external and internal relations respective to some forcing notion. We will not formulate any of them, we would like just to stress this type of problems by recalling a question of T. Jech ([J]) he posed after our lecture at Oberwolfach 93, namely: given a forcing notion, describe a relation which is the canonical internal and external one for this forcing.

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