Petr Holický; Jan Pelant
Radon spaces which are not σ-fragmentable

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 36 (1995), No. 2, 19--25

Persistent URL: http://dml.cz/dmlcz/702021

Radon Spaces Which Are Not σ-Fragmentable

P. HOLICKÝ and J. PELANT

Praha*)

Received 15. March 1995

We give examples of Radon Banach spaces which are not σ-fragmented by any metric, or by the norm, respectively.

At the 22nd Winter School on Abstract Analysis in Poděbrady 1994, John Jayne had a series of lectures on σ-fragmented Banach spaces, a class of spaces introduced by J. Jayne, I. Namioka and C. A. Rogers (see e.g. [JNR]). J. Jayne asked whether, for Banach spaces endowed with the weak topology, σ-fragmented and Radon is the same provided that the space does not contain a relatively discrete subset of real-valued measurable cardinality. We answer here the question in the negative by considering spaces of continuous functions on tree spaces. They are Radon with the above restriction on relatively discrete subsets due to results of Tortrat ([E, Proposition 3.5]), Gardner and Pfeffer ([GP]), and Dow, Junnila and Pelant ([DJP]). We describe some tree spaces for which the weak topology of the corresponding Banach space of continuous functions turns out to be not σ-fragmented by any metric, or at least, by the supremum norm.

We need to recall a series of notions and results, so we do it successively in the following text, denoting the known results as Propositions not regarding their strength.

1. Radon spaces

Definition. We say that the topological space (X, τ) is a *(w-)Radon space* if (there is no relatively discrete subset of real-valued measurable cardinality, and if) every finite Borel measure μ on X is a *Radon measure*, i.e. given a Borel set $B \subset X$ there are compact sets $C_n \subset B$, $n \in \mathbb{N}$, such that $\mu(\bigcup_{n \in \mathbb{N}} C_n) = \mu(B)$.

*) Department of Math. Anal., Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic
Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

We say that the finite Borel measure μ on X is *τ-additive* if for every family $\mathscr{H}$ of open subsets of X, which is directed upwards with respect to $\subset$, we have $\mu(\bigcup\mathscr{H}) = \sup\{\mu(H) \mid H \in \mathscr{H}\}$.

Unless stated otherwise, all Banach spaces will be considered with their weak topology. We recall a sufficient condition for a Banach space to be a w-Radon space. First we recall a covering property of a topological space. The standard terminology of spaces with this property is "hereditarily weakly θ-refinable". We are going to use a perhaps more descriptive name suggested by H. Junnila in [DJP].

Definition. The topological space X is called hereditarily σ-relatively metacompact if every open family in X has a point-finite relatively open refinement.

Let us recall that it is equivalent with the property that each open family has a σ-relatively discrete refinement.

Now, we can recall the following assertion following from Theorems 10.2 and 11.6 of [GP].

Proposition 1.

(a) If X is hereditarily σ-relatively metacompact, then every finite Borel measure on X is τ-additive.

(b) Let every τ-additive finite Borel measure be Radon on Y. Let X be measurable with respect to the completion of any Radon measure on Y. Then every τ-additive measure is Radon on X.

A deep theorem of Tortrat ([E, Proposition 3.5]) says that a Banach space is measurable with respect to the completion of any Radon measure on (X^{**}, *weak**). Due to preceding Proposition 1(b) and to the fact that any τ-additive finite Borel measure on the σ-compact space (X^{**}, *weak**) is Radon, we get

Proposition 2. *Let X be a Banach space and τ_w its weak topology. Then any finite Borel and τ-additive measure on X is Radon.*

Using Proposition 1(a), we get from Proposition 2 immediately

Proposition 3. *If a Banach space X is hereditarily σ-relatively metacompact, then it is w-Radon.*

2. σ-fragmentability of topological spaces

The notion of σ-fragmentability was introduced by Jayne, Namioka and Rogers (see e.g. [JNR]) in the context of Banach spaces for which the σ-fragmentability of the weak or weak* topologies by the norm were studied mainly. In a more

Both authors were partially supported by the grant GAČR 201/94/0069, the first author was partially supported by GAUK 362, the second author was partially supported by the grant of Acad. Sci. of Czech Rep. No. 119 401.

general setting, also the σ-fragmentability of any topology by a lower semi-continuous metric was investigated.

Definition. The topological space (X, τ) is *σ-fragmented by the metric ϱ* if, for every $\varepsilon > 0$, there are $X_n \subset X$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} X_n = X$ and each X_n is *ε-fragmented by ϱ*, i.e. for every nonempty set $S \subset X_n$ there is a τ-relatively open and nonempty subset of (S, τ) with the ϱ-diameter less than ε.

Using this basic definition, we introduce still some related notions for our convenience.

Definition. We say that (X, τ) is *σ-fragmentable* is there is a metric ϱ such that (X, τ) is σ-fragmented by ϱ.

We say that the subset *Y of a Banach space X is σ-fragmented* if the space (Y, τ), Y endowed with the weak topology, is σ-fragmented by the norm-metric.

Significant examples of σ-fragmented Banach spaces are duals of Asplund spaces, for which even the w^*-topology is σ-fragmented by the norm.

The property to be σ-fragmentable is rather weak for small spaces as the following result and the remark behind it say.

We may derive immediately from [R, Theorem 1.9], where fragmentable spaces are characterized, an analogous characterization of σ-fragmentable spaces.

Proposition 4. *The topological space X is σ-fragmentable if and only if there is a sequence of partitions $\mathscr{C}_n$ of X such that*

(i) *each $\mathscr{C}_n$ is σ-scattered, i.e. $\mathscr{C}_n = \bigcup_{m \in \mathbb{N}} \mathscr{C}_m^{(n)}$, where every nonempty subfamily $\mathscr{D}$ of $\mathscr{C}_m^{(n)}$ contains a nonempty set $D \in \mathscr{D}$ which is open in $\bigcup \mathscr{D}$, and*

(ii) *$\bigcup \mathscr{C}_n$ separates points of X, i.e. for every $x, y \in X$, $x \neq y$, there is a $C \in \mathscr{C}_n$ for some $n \in \mathbb{N}$ with $x \in C$, $y \notin C$.*

We omit the proof which is even easier when one assumes that $\mathscr{C}_n \succ \mathscr{C}_{n+1}$ for each $n \in \mathbb{N}$.

So in particular, if the cardinality of X is at most $\mathfrak{c} = 2^{\aleph_0}$, then the space X is σ-fragmentable because we can consider any one-to-one map f of $[0, \infty)$ onto X and take $\mathscr{C}_n = \{X_k^{(n)} \mid k \in \mathbb{N}\}$ with $X_k^{(n)} = f([(k-1)/n, k/n))$. We may also transfer the metric of $[0, \infty)$ to X by defining $\varrho(f(x), f(y)) = |x - y|$ and realize that X is σ-fragmented by ϱ directly.

On the other hand we give an auxiliary necessary condition for the space X to be σ-fragmentable.

Lemma 1. *Let (X, τ) be a topological space and $\mathscr{B}$ some pseudobasis of τ, i.e. $\mathscr{B}$ is a family of nonempty open subsets of X such that for every nonempty open U in X there is a $B \in \mathscr{B}$ such that $B \subset U$. Let us suppose that the cardinality of $\bigcap_{n \in \mathbb{N}} B_n$ is greater than $\mathfrak{c}$ whenever B_n, $n \in \mathbb{N}$, is a sequence of elements of $\mathscr{B}$ with $B_{n+1} \subset B_n$ for $n \in \mathbb{N}$.*

Then (X, τ) *is not* σ*-fragmentable.*

Proof. Let us suppose that (X, τ) is σ-fragmentable. Due to Proposition 4, there is a sequence of families $\mathscr{C}_n = \bigcup_{m \in \mathbb{N}} \mathscr{C}_m^{(n)}$, $n \in \mathbb{N}$, (of partitions) of X which fulfils (i) and (ii). Let $\{\mathscr{D}_k \mid k \in \mathbb{N}\} = \{\mathscr{C}_m^{(n)} \mid m,\ n \in \mathbb{N}\}$. Each $\mathscr{D}_k$ can be well ordered by choosing, successively, the elements $D_k(\alpha)$ of $\mathscr{D}_k$ such that

$$\mathscr{D}_k = \{D_k(\alpha) \mid \alpha \in [0, \varkappa_k')\} \neq \emptyset$$

for some ordinal $\varkappa_k$ and such that $D_k(\alpha)$ is relatively open in $\bigcup\{D_k(\beta) \mid \beta \in [\alpha, \varkappa_k)\}$. Thus there are open sets $U_k(\alpha)$ with

$$D_k(\alpha) \subset U_k(\alpha) \setminus \bigcup\{U_k(\beta) \mid \beta < \alpha\}.$$

Now, we choose a decreasing sequence B_k, $k = 0, 1, \ldots$, of elements of $\mathscr{B}$ inductively. For $k = 0$ choose any $B_0 \in \mathscr{B}$. If $B_0, \ldots, B_k$ are already chosen, we find a $B_{k+1} \in \mathscr{B}$ such that $B_{k+1} \subset B_k \cap U_{k+1}(\alpha_{k+1})$ where α_{k+1} is the smallest ordinal α such that $D_{k+1}(\alpha) \cap B_k \neq \emptyset$. If there is no such ordinal, put $B_{k+1} = B_k$.

The intersection $X_0 = \bigcap_{k=0}^{\infty} B_k$ has cardinality greater than $\mathfrak{c}$ by our hypothesis. On the other hand, since $\bigcup_{k \in \mathbb{N}} \mathscr{D}_k$ separates points of X_0 and $X_0 \cap D \neq \emptyset$ for at most countably many elements D of $\bigcup_{k \in \mathbb{N}} \mathscr{D}_k$, the cardinality of X_0 is at most $\mathfrak{c}$ which is a contradiction.

3. Tree spaces

Now, we introduce the class of Banach spaces of continuous functions on tree spaces which turned out to be very useful to get counterexamples. R. Haydon used them, for example, to get Asplund spaces with no Gateaux smooth norm ([H]).

Definition. By a *tree space* T here we understand any partially ordered set which has, for simplicity, the only root r_T (the smallest element) and which, endowed with the *interval topology* generated by the intervals of the form $(s, t]$ for $s, t \in T$, is Hausdorff, i.e. if $s, t \in T$ are of the same height which is a limit ordinal, then $[r_T, s) = [r_T, t)$ implies $s = t$.

Let us notice that the tree topology is locally compact and we may consider its Alexandrov (one-point) compactification αT and the Banach space $C(\alpha T)$ of real continuous functions on αT endowed with the supremum norm. Notice that αT is scattered and so the weak topology of $C(\alpha T)$ and the topology of pointwise convergence τ_p coincide on bounded subsets of $C(\alpha T)$. Thus any subset of $C(\alpha T)$ is σ-fragmented with respect to the weak topology τ_w if and only if it is σ-fragmented with respect to the topology τ_p.

The following result is from [DJP, Corollary 1.7].

Proposition 5. *For every tree space T, the space $C(\alpha T)$ endowed with the topology τ_p is hereditarily σ-relatively metacompact. The same holds for the weak topology τ_w.*

As an immediate corollary of Proposition 5 and Proposition 3 we get

Proposition 6. *For any tree space T, the space $(C(\alpha T), \tau_w)$ is w-Radon.*

Using Lemma 1, we get our first example.

Example 1. *There is a tree space T such that the space $(C(\alpha T), \tau_w)$ is w-Radon and not σ-fragmentable.*

Proof. By Proposition 6, $(C(\alpha T), \tau_w)$ is w-Radon for any tree space T. So it is enough to describe some tree space T for which $(C(\alpha T), \tau_w)$ is not σ-fragmentable.

Let T be a tree space such that the cardinality of $T_t = \{s \in T \mid s \geq t\}$ is greater than $\mathfrak{c}$ for every $t \in T$, such that every $t \in T$ has infinitely many (immediate) successors and such that any sequence $t_1 \leq t_2 \leq \dots$ has an upper bound in T. There are many such tree spaces, as an example consider the tree spaces such that every maximal chain is of ordinal type ω_1 and every element has more than $\mathfrak{c}$ (immediate) successors. Another possibility occurs by considering the tree spaces such that every maximal chain of T_t for $t \in T$ has cardinality of uncountable cofinality and greater or equal to $\mathfrak{c}$ and such that every its element has infinitely many successors. Clearly, it is enough when a tree contains a subtree of such a type.

We shall show that there is a pseudobasis (in the fact a basis) $\mathscr{B}$ for the topology $\tau_p = \tau_w$ on $Y = \{\chi_{[r_T, t]} \mid t \in T\} \subset X = C(\alpha T)$ which fulfils the conditions of Lemma 1.

Let $\mathscr{B}$ be the family of subsets of Y of the form

$$B(t_0; t_1, \dots, t_p) = \{\chi_{[r_T, t]} \mid t \geq t_0, t \not\geq t_1, \dots, t \not\geq t_p\}$$

for $t_0, t_1, \dots, t_p \in T$. Obviously, $\mathscr{B}$ forms a basis for $\tau_p = \tau_w$ on Y. Let $B_k \in \mathscr{B}$ be such that $B_{k+1} \subset B_k$ for $k \in \mathbb{N}$. So there are $t_0^{(k)}, t_1^{(k)}, \dots, t_{p_k}^{(k)} \in T$ such that $B_k = B(t_0^{(k)}, t_1^{(k)}, \dots, t_{p_k}^{(k)})$. We find $t_1 \geq t_0^{(1)}$ such that $T_{t_1} \cap T_{t_i^{(1)}} = \emptyset$ for $i = 1, \dots, p_1$. Similarly, we find t_2 in T_{t_1} because every element of T has infinitely many successors, and by induction, we get a sequence of elements of T such that $t_1 \leq t_2 \leq \dots$ and such that $\{\chi_{[r_T, t]} \mid t \in T_{t_k}\} \subset B_k$. Find $t_\infty \geq t_k$ for every $k \in \mathbb{N}$. Now, the cardinality of T_{t_∞} is greater than $\mathfrak{c}$ by our assumptions on T and $\{\chi_{[r_T, t]} \mid t \in T_{t_\infty}\} \subset \bigcap_{k \in \mathbb{N}} B_k$.

In the next example, we use Lemma 2 which contains a part of a criterion due to R. Haydon (private communication) for $C(\alpha T)$ to be σ-fragmented by the supremum norm. In the fact, R. Haydon proves using his criterion that $C(\alpha T)$ is σ-fragmented if and only if it has an equivalent Kadec norm. We use only the necessity of his conditions for $C(\alpha T)$ to be σ-fragmented, and for the reader's convenience we indicate the proof of it.

Lemma 2. *Let $C(\alpha T)$ is σ-fragmented. Then the following hold*

(a) $Y = \{\chi_{[r_T, t]} \mid t \in T\}$ is σ-scattered with respect to the "reverse" topology τ_r generated by the sets $\{s \in T \mid s \geq t\}$ for $t \in T$.

(b) Y is σ-relatively discrete with respect to τ_r.

(c) There is a real function $\varrho : T \to \mathbb{R}$ such that $\varrho(s) \geq \varrho(t)$ for $s \leq t$, and such that for every $t \in T$ there is an $\varepsilon > 0$ with $\{s \in t^+ \mid \varrho(s) - \varrho(t) < \varepsilon\}$ finite.

Proof. The condition (a) follows immediately from the fact that the distance of every two distinct elements of Y is one and $Y = \bigcup_{n \in \mathbb{N}} Y_n$ where every Y_n is $\frac{1}{2}$-fragmented.

The condition (b) can be derived as follows. Let $T^{(\alpha)}$ denote the α^{th} derivative of the topological space T. Let $\alpha(t)$ denote the smallest ordinal such that $t \notin T^{(\alpha)}$ for every $t \in T$. Now, for every $t \in T$, we find a finite subset F_t of the set t^+ of its successors such that, for every $t' \in I_t = t^+ \setminus F_t$ and every $s \in T_{t'}$, we have $\alpha(s) < \alpha(t)$. Let us denote $I = \bigcup_{t \in T} I_t$ and realize that, for every $t \in T$, the set $I \cap [r_T, t]$ is finite because α on it gives a decreasing sequence of ordinals. So the sets $I_n = \{s \in I \mid s \in I_t, |[r_T, t] \cap I| = n - 1\}$ cover I. The subsets $T^{(k)} = \{t \in T \mid |[r_T, t] \cap I| = k - 1\}$ satisfy

$$|\min(\{s \in T \mid s > t\} \cap T^{(k)})| < \aleph_0$$

for each $t \in T^{(k)}$ and $k \in \mathbb{N}$, so they form the cover of T by relatively discrete sets.

To prove (c), suppose that we have the σ-discrete decomposition $T = \bigcup_{k \in \mathbb{N}} T^{(k)}$. We shall define the function ϱ by induction over the height of $t \in T$. Put $\varrho(r_T) = 0$. Suppose that $t \in T^{(k)}$ and $\varrho(t)$ is already defined. Then there are only finitely many elements s of t^+ such that $T_s \cap T^{(k)} \neq \emptyset$. Denote F_t the set of such s. Let us put $\varrho(s) = \varrho(t)$ for them. Let $s \in t^+ \setminus F_t$. Then we put $\varrho(s) = \varrho(t) + 1/2^k$. For $s \in T$ being a supremum of $[r_T, s)$ we put $\varrho(s) = \sup\{\varrho(t) \mid t \in [r_T, s)\}$.

We conclude by a little more subtle example of a non-σ-fragmented space $C(\alpha T)$.

Example 2. *There is a tree space T such that the space $C(\alpha T)$ endowed with the weak topology τ_w, or, equivalently, with the topology τ_p, is not σ-fragmented (by the supremum norm), and such that it is still σ-fragmentable (by some metric).*

Proof. We consider the tree (sometimes denoted by $\sigma\mathbb{Q}$) of bounded well ordered subsets of the rationals ordered by the relation $s \leq t$ if s is an initial segment to t ($r_T = \emptyset$). Our proof follows an idea of the standard one (see [T, Corollary 9.9]).

Since the cardinality of $Y = \{\chi_{[\emptyset, t]} \mid t \in T\}$ is $\mathfrak{c}$, we get by the Stone-Weierstrass theorem that $C(\alpha T)$ has also the cardinality $\mathfrak{c}$. By the remark behind Proposition 4, the space $C(\alpha T)$ is σ-fragmentable.

Let us suppose that there is a real function ϱ described in Lemma 2. We may notice that it can be supposed bounded, e.g. let $\varrho(T) \subset [0, 1]$ as in the proof of Lemma 2. Now, we proceed by the induction over countable ordinals as follows. For $F_0 = \emptyset \in T$ consider the value $\varrho(\emptyset) = 0$. We choose singleton F_1 in negative irrationals for which $\varrho(F_1) > \varrho(F_0)$. By induction, we construct increasing sequence of elements F_α of T such that $r < \varrho(F_\alpha)$ for all $r \in F_\alpha$ and all countable ordinals α.

Let F_β, $\beta < \alpha$, were constructed. If $\alpha = \beta + 1$, we may find an $\varepsilon > 0$ such that, for a finite set S_β of successors F_β, we have $\varrho(t) > \varrho(F_\beta) + \varepsilon$ for $t \notin S_\beta$. So we find a rational number r_α in $(\varrho(F_\beta), \varrho(F_\beta) + \varepsilon)$ which is not contained in any element of S_β. We put $F_\alpha = F_\beta \cup \{r_\alpha\}$ and we get that $\varrho(F_\alpha) > \varrho(F_\beta)$.

If α is a limit ordinal, we put $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$, $\varrho(F_\alpha) \geq \sup \{\varrho(F_\beta \mid \beta < \alpha\}$, and from the preceding construction we get that $r < \varrho(F_\alpha)$ for $r \in F_\alpha$.

Notice that since we assume that $\varrho(t) \leq 1$ (in the fact $\varrho(t) < 1$) for every $t \in T$, we may always find successors for F_α, $\alpha < \omega_1$. But in this way we get an increasing sequence $\varrho(F_\alpha)$, $\alpha < \omega_1$, of reals which is impossible. So our assumption that $C(\alpha T)$ is σ-fragmented by the supremum norm is absurd.

Remark. Presented construction of trees whose function spaces are not σ-fragmented are quite straightforward. Examples of similar nature were used by Haydon to present a function space on a tree space without the Namioka property, hence it cannot be σ-fragmented by results of Jayne, Namioka and Rogers.

References

[DJP] Dow, A., Junnila, H. and Pelant, J. *Weak covering properties of weak topologies*, in preparation.

[E] Edgar, G. A. *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), 663–677.

[GP] Gardner, R. J. and Pfeffer, W. E. *Borel Measures*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, New York, Oxford, 1984, pp. 961–1043.

[H] Haydon, R. *A counterexample to several questions about scattered compact spaces*, J. London Math. Soc. **22** (1990), 261–268.

[JNR] Jayne, J., Namioka, I. and Rogers, C. A. *Topological properties of Banach spaces*, Proc. London Math. Soc. **66** (1993), 651–672.

[R] Ribarska, N. K. *Internal characterization of fragmentable spaces*, Mathematika **34** (1987), 243–257.

[T] Todorčević, S. *Trees and Linearly Ordered Sets*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, New York, Oxford, 1984, pp. 235–293.