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Smoothness in $\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))$

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Let $\mathbf{B}(\mathscr{L}(\mathsf{C}(X), C(Y)))$ be the unit ball of the space of operators acting from the space of continuous functions $\mathsf{C}(X)$ into $\mathsf{C}(Y)$ (X, Y – compact metric spaces). The purpose of this paper is to give a characterization of the smooth points of $\mathbf{B}(\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y)))$.

Let $\mathbf{E}$, $\mathbf{F}$ be Banach spaces. The unit ball of $\mathbf{E}$ is denoted by $\mathbf{B}(\mathbf{E})$ and its smooth points by smooth $\mathbf{B}(\mathbf{E})$. Recall that $x \in \text{smooth } \mathbf{B}(\mathbf{E})$ if there exists a unique continuous linear functional $\xi \in E^*$ such that $\xi(x) = \|\xi\| = 1$ (note that such a $\xi \in \text{ext } \mathbf{B}(E^*)$). We point out that there is a connection between smoothness and the differentiability of the norm (see e.g. [2]). We denote the linear space of all (compact) bounded linear operators from $\mathbf{E}$ into $\mathbf{F}$ by $(\mathscr{K}(\mathbf{E}, \mathbf{F}))$ $\mathscr{L}(\mathbf{E}, \mathbf{F})$.

The investigations of the smooth points in the spaces of operators were started by Holub [7] considering compact operators on Hilbert space. This was extended by Heinrich [6] to the compact operators acting on arbitrary Banach spaces and by Kittaneh and Younis [8] to the space of bounded operators on Hilbert space. We also have a description of smooth points in $\mathscr{L}(l^p, l^r)$, $p, r \in [1, \infty)$ ([3, 4]).

The aim of this paper is to present a description of smooth points of the unit ball of $\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))$.

Note that if an operator $T: \mathbf{E} \to \mathbf{F}$ is a smooth point of the unit ball then T attains its norm on at most one vector (up to constant multiple) and moreover if $\|T\| = \|T^*\mathbf{u}\| = \|\mathbf{u}\| = 1$ for some $\mathbf{u} \in \mathbf{F}^*$ then $T^*\mathbf{u} \in \text{smooth } \mathbf{B}(E^*)$.

Let X, Y be compact Hausdorff spaces. By $\mathsf{C}(X)$ we denote the Banach space of scalar valued continuous functions on X equipped with the supremum norm. Note that smooth $\mathbf{B}(C(X)) = \{f \in \mathsf{C}(X)$: there exist $x_0 \in X$ such that $1 = |f(x_0)| > |f(x)|$ for all $x \neq x_0\}$ (cf. Banach classical monograph [1], p. 168). This was extended by Sundaresan [9] to the space of vector valued continuous functions $\mathsf{C}(X, \mathbf{E})$. Moreover if $\text{card } X > \aleph_0$ then smooth $\mathbf{B}(C(X)^*) = \emptyset$, and if $\text{card } X \leq \aleph_0$ then smooth $\mathbf{B}(C(X)^*) = \{\mu \in \mathsf{C}(X)^*: \|\mu\| = 1 \text{ and supp } \mu = X\}$.

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Remark. Consider a closed subspace $E_1 = \{(u_n) : \lim_{n\to\infty} u_{2n-1} \text{ exists}, u_{2n} = 0, n \in \mathbb{N}\}$ of l^∞. We can find a functional $\bar{\eta}_1$ on E_1 such that $\bar{\eta}_1((u_n)) = \lim_{n\to\infty} u_{2n-1}$ with $\|\bar{\eta}_1\| = 1$. And we can extend $\bar{\eta}_1$ into η_1 acting on the whole l^∞ with $\|\eta_1\| = 1$. Analogously we can find $\eta_2 \in (l^\infty)^*$ such that $\|\eta_2\| = 1$ and $\eta_1((u_n)) = \lim_{n\to\infty} u_{2n}$ for all $(u_n) \in l^\infty$ having the limit $\lim_{n\to\infty} u_{2n}$. We may also build η_i with the above properties choosing cluster points $v_1, v_2 \in \beta\mathbb{N} \setminus \mathbb{N}$ of the sets $D_1 = \{2n - 1 : n \in \mathbb{N}\}$ and $D_2 = \{2n : n \in \mathbb{N}\}$, respectively, and putting $\eta_i = \delta_{v_i}$, $i = 1, 2$ (δ_{x_0} denotes the point mass measure at x_0). In this construction we use well known identification $l^\infty = \mathrm{C}(\mathbb{N})$ with $\mathrm{C}(\beta\mathbb{N})$, and $(l^\infty)^*$ with $C(\beta\mathbb{N})^* = \mathscr{M}(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Čech–Stone compactification of the positive integers $\mathbb{N}$.

Lemma. *Let X and Y be compact metric spaces and let* $\operatorname{card} X \geq \aleph_0$, *and let $T \in \mathbf{B}(\mathscr{L}(\mathrm{C}(X), \mathrm{C}(Y)))$ be such that there exists a sequence $\{y_n\}$ of distinct points of Y with $\|T^*\delta_{y_n}\| \to 1$. Then $T \notin$ smooth* $\mathbf{B}(\mathscr{L}(\mathrm{C}(X), \mathrm{C}(Y)))$.

Proof. Let T satisfies the conditions from the Lemma. We may and do assume that the sequence $\{y_n\}_{n=1}^\infty$ is converging to $y_0 \in Y$, and we can choose open neighborhood U_n of y_n such that $\overline{U_{k_1}} \cap \overline{U_{k_2}} = \emptyset$ if $k_1 \neq k_2$. Now we fix $h_n \in C(Y)$ such that $h_n(y_n) = 1$, $h_n(U_n^c) = 0$, $0 \leq h_n \leq 1$ (eventually we consider a subsequence). Now we choose a converging sequence $\{x_k\}$ $(\lim_{k\to\infty} x_k = x_0)$ of distinct points of X. Then we choose its subsequence $\{x_n\}$ such that $|T^*\delta_{y_n}|(X \setminus \{x_{2n-1}, x_{2n}\}) \to 1$. Let A_n^+, A_n^- be the Hahn decomposition of X to positive and negative part with respect to the measure $T^*\delta_{y_n}$. Put $B_n^+ = A_n^+ \setminus \{x_{2n-1}, x_{2n}\}$, $B_n^- = A_n^- \setminus \{x_{2n-1}, x_{2n}\}$. For any $R \in \mathscr{L}(\mathrm{C}(X), \mathrm{C}(Y))$ we define a sequence (u_n^R) by

$$u_n^R = (R^*\delta_{y_n})(B_n^+) - (R^*\delta_{y_n})(B_n^-) + (-1)^n [(R^*\delta_{y_n})(\{x_{2n-1}\}) - (R^*\delta_{y_n})(\{x_{2n}\})].$$

Obviously $(u_n^R) \in l^\infty$ and $\|(u_n^R)\|_\infty \leq \|R\|$ and $\|(u_n^T)\|_\infty = \|T\| = 1$. Now we define functionals ξ_i $(i = 1, 2)$ on $\mathscr{L}(\mathrm{C}(X), \mathrm{C}(Y))$ by $\xi_i(R) = \eta_i((u_n^R))$, where η_i is constructed as in Remark. We have $\|\xi_i\| = 1 = \xi_i(T)$, e.i. ξ_i supports $\mathbf{B}(\mathscr{L}(\mathrm{C}(X), \mathrm{C}(Y)))$ at T. To finish the proof we need to show that $\xi_1 \neq \xi_2$. To get it we define S: $\mathrm{C}(X) \to \mathrm{C}(Y)$ by

$$(Sg)(y) = \sum_{n=1}^\infty h_n(y)[g(x_{2n-1}) - g(x_{2n})], \qquad g \in \mathrm{C}(X)$$

or equivalently

$$S^*\delta_y = \sum_{n=1}^\infty h_n(y)(\delta_{x_{2n-1}} - \delta_{x_{2n}}).$$

Obviously S is linear. If $g \in \mathrm{C}(X)$ then $a_n = g(x_{2n-1}) - g(x_{2n}) \to 0$. Because h_n have norm equal to one and disjoint supports the series $\sum_n a_n h_n$ is uniformly convergent.

Thus $Sg = \sum h_n(y)\,[g(x_{2n-1}) - g(x_{2n})] = \sum a_n h_n \in \mathsf{C}(X)$ and $\|Sg\| \le 2\|g\|$, i.e. $S \in \mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))$. We have $u_n^S = (-1)^n \cdot 2$. Therefore $\xi_1(S) = \eta_1((u_n^S)) = -2 \neq 2 = \eta_2((u_n^S)) = \xi_2(S)$. □

Theorem. *Let X and Y be compact metric spaces.*
(a) If card $X > \aleph_0$ *then smooth* $\mathbf{B}(\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))) = \emptyset$.
(b) If card $X \le \aleph_0$ *then smooth* $\mathbf{B}(\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))) = \{T \in \mathbf{B}(\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y)))$: *there exists an isolated point* y_0 *of Y such that* $T^*\delta_{y_0} \in$ *smooth* $\mathbf{B}(C(X)^*)$ *and* $\sup_{y \neq y_0} \|T^*\delta_y\| < 1\}$.

Proof. Let $T \in \mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))$. Suppose that card $X \ge \aleph_0$. In view of the Lemma the operator T could only be a smooth point if $\|T^*\delta_{y_0}\| = 1$ for y_0 isolated point of Y, and if $T^*\delta_{y_0} \in$ smooth $\mathbf{B}(C(X)^*)$. Because smooth $\mathbf{B}(C(X)^*) = \emptyset$ if card $X > \aleph_0$, we get *(a)*.

To finish *(b)* we need to show that any operator from the right side set actually is a smooth point. Obviously a functional ξ_0 defoned by $\xi_0(R) = \mathfrak{n}_0(R^*\delta_{y_0})$, where $\mathfrak{n}_0 \in C(X)^{**}$ supports $\mathbf{B}(C(X)^*)$ at $T^*\delta_{y_0}$. Suppose that ξ with $\|\xi\| = \xi(T) = 1$ supports $\mathbf{B}(\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y)))$ at T, too. We denote $1_{\{y_0\}} \otimes \mu_0 \in (\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))$ defined by

$$(1_{\{y_0\}} \otimes \mu_0)(g)(y) = \begin{cases} \int g\, d\mu_0 & \text{if} \quad y = y_0 \\ 0 & \text{if} \quad y \neq y_0 \end{cases},$$

$g \in \mathsf{C}(X)$. Each $R \in (\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y))$ has a representation $R = R_0 + R_1$, where $R_0 = 1_{\{y_0\}}R = 1_{\{y_0\}} \otimes R^*\delta_{y_0}$, $R_1 = 1_{Y \setminus \{y_0\}}R$. Obviously $\|R_1\| < \infty$. We have $\xi(R_1) = 0$ since $1 \pm \varepsilon\xi(R_1) = \xi(T \pm \varepsilon R_1) \le \|\xi\|\,\|T \pm \varepsilon R_1\| \le 1$ for sufficiently small $\varepsilon > 0$. Hence $\xi(R) = \xi(R_0) = \xi(1_{\{y_0\}} \otimes R^*\delta_{y_0})$. Now we consider a functional

$$\mu \in \mathsf{C}(X)^* \to \xi(1_{\{y_0\}} \otimes \mu) \in \mathbb{R}\,.$$

It has norm equal to one and supports $\mathbf{B}(C(X)^*)$ at $T^*\delta_{y_0} \in$ smooth $\mathbf{B}(C(X)^*)$. Hence $\xi(R) = \xi(\delta_{y_0} \otimes R^*\delta_{y_0}) = \mathfrak{n}_0(R^*1_{\{y_0\}})$ which shows the uniqueness of the supporting functional and smoothness at T. □

Note that if card $X < \aleph_0$ ($\mathsf{C}(X)$ is finite dimensional) we have

$$\mathscr{L}(\mathsf{C}(X), \mathsf{C}(Y)) = \mathscr{K}(\mathsf{C}(X), \mathsf{C}(Y)) = \mathsf{C}(Y, \mathsf{C}(X)^*) = \mathsf{C}(Y, l_n^1)\,,$$

and in this case we get the above characterization by Heinrich's result [6] for compact operators or by Sunderesan's results [9] for $\mathsf{C}(Y, l_n^1)$. □

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