Acta Universitatis Carolinae. Mathematica et Physica

F. Oertel

Operators with extension property and the principle of local reflexivity

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 37 (1996), No. 2, 55--63

Persistent URL: http://dml.cz/dmlcz/702035

Operators with Extension Property and the Principle of Local Reflexivity

F. OERTEL

Zurich*)

Received 15. March 1996

Given an arbitrary p-Banach ideal $(\mathscr{A}, \mathbf{A})$ $(0 < p \leq 1)$, we ask geometrical properties of $(\mathscr{A}, \mathbf{A})$ which are sufficient (and necessary) to allow a transfer of the principle of local reflexivity to $(\mathscr{A}, \mathbf{A})$.

1. Introduction

Given Banach spaces E, F and a maximal Banach ideal $(\mathscr{A}, \mathbf{A})$, we are interested in reasonable sufficient conditions on E, F and $(\mathscr{A}, \mathbf{A})$ such that $(\mathscr{A}, \mathbf{A})$ is accessible. In general it is a nontrivial subject to prove accessibility of maximal Banach ideals since non-accessibility only appears on Banach spaces without the metric approximation property, and in 1991, Pisier made use of such a Banach space (the Pisier space P) to construct a non-accessible maximal Banach ideal (cf. [2], 31.6). On the other hand, accessible Banach ideals allow a suggestive calculus which leads to further results concerning the local structure of operator ideals (e.g., a transfer of the principle of local reflexivity from the operator norm to suitable ideal norms $\mathbf{A}$ (cf. [2], [8], [9], [10] and [11]).

This paper is mainly devoted to the investigation of a class of (maximal) Banach ideals which do allow the transfer of the operator norm estimation in the principle of local reflexivity to the norm of the given operator ideal.

We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard and can be found in the detailed monographs [2] and [12]. However, if $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$ are given quasi-Banach ideals, we will use the shorter notation $(\mathscr{A}^d, \mathbf{A}^d)$ for the dual ideal (instead of $(\mathscr{A}^{\text{dual}}, \mathbf{A}^{\text{dual}})$) and the abbreviation $\mathscr{A} \stackrel{1}{=} \mathscr{B}$ for the isometric identity $(\mathscr{A}, \mathbf{A}) = (\mathscr{B}, \mathbf{B})$. The inclusion $(\mathscr{A}, \mathbf{A}) \subseteq (\mathscr{B}, \mathbf{B})$ is often shortened by $\mathscr{A} \stackrel{1}{\subseteq} \mathscr{B}$, and if $T: E \to F$ is an operator, we indicate that it is a metric injection by writing $T: E \stackrel{1}{\hookrightarrow} F$. Each section of this paper includes the more special terminology which is not so common to specialists in the geometry of Banach spaces.

*) Swiss Reinsurance Company, Dpt. PM-PH, Mythenquai 50/60, CH - 8022 Zurich, Switzerland

Let E, F be arbitrary Banach spaces and $T \in \mathscr{L}(E, F)$. Given a p-Banach ideal $(\mathscr{A}, \mathbf{A})$ $(0 < p \leq 1)$ and a q-Banach ideal $(\mathscr{B}, \mathbf{B})$ $(0 < q \leq 1)$, we can construct further ideals:

- T belongs to $\mathscr{A} \circ \mathscr{B}^{-1}(E, F)$ if $TS \in \mathscr{A}(G, F)$ for all operators $S \in \mathscr{B}(G, E)$ and for all Banach spaces G. Let $\mathbf{A} \circ \mathbf{B}^{-1}(T) := \sup\{\mathbf{A}(TS): S \in \mathscr{B}(G, E), \mathbf{B}(S) = 1\}$. Then $(\mathscr{A} \circ \mathscr{B}^{-1}, \mathbf{A} \circ \mathbf{B}^{-1})$ defines a p-Banach ideal, called the *right-$\mathscr{B}$-quotient of $\mathscr{A}$*.
- T belongs to $\mathscr{B}^{-1} \circ \mathscr{A}(E, F)$ if $ST \in \mathscr{A}(E, G)$ for all operators $S \in \mathscr{B}(F, G)$ and for all Banach spaces G. Let $\mathbf{B}^{-1} \circ \mathbf{A}(T) := \sup\{\mathbf{A}(ST): S \in \mathscr{B}(F, G), \mathbf{B}(S) = 1\}$. Then $(\mathscr{B}^{-1} \circ \mathscr{A}, \mathbf{B}^{-1} \circ \mathbf{A})$ defines a p-Banach ideal, called the *right-$\mathscr{B}$-quotient of $\mathscr{A}$*.
- T belongs to $\mathscr{A} \circ \mathscr{B}(E, F)$ if there exists a Banach space G and operators $R \in \mathscr{B}(E, G)$ and $S \in \mathscr{A}(G, F)$ such that $T = SR$. If we set $\mathbf{A} \circ \mathbf{B}(T) := \inf\{\mathbf{A}(S) \cdot \mathbf{B}(R)\}$, the infimum being taken over all possible factorizations of T, then $(\mathscr{A} \circ \mathscr{B}, \mathbf{A} \circ \mathbf{B})$ is a r-Banach ideal, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Note, that in general $(\mathscr{B}^{-1} \circ \mathscr{A}, \mathbf{B}^{-1} \circ \mathbf{A}) \neq (\mathscr{A} \circ \mathscr{B}^{-1}, \mathbf{A} \circ \mathbf{B}^{-1})$. Using Pisier's counterexample of a non-accessible maximal Banach ideal, it follows that there even exists a Banach ideal $(\mathscr{A}, \mathbf{A})$ such that $(\mathscr{A}^{-1} \circ \mathscr{I}, \mathbf{A}^{-1} \circ \mathbf{I}) \neq (\mathscr{I} \circ \mathscr{A}^{-1}, \mathbf{I} \circ \mathbf{A}^{-1})$ (cf. [11]), where $(\mathscr{I}, \mathbf{I})$ denotes the class of all integral operators.

Using the class of all approximable operators $(\bar{\mathscr{F}}, \|\cdot\|)$. important special cases of the previous constructions are given by:

- the *minimal kernel* of $(\mathscr{A}, \mathbf{A})$:

$$(\mathscr{A}^{\min}, \mathbf{A}^{\min}) := (\bar{\mathscr{F}} \circ \mathscr{A} \circ \bar{\mathscr{F}}, \|\cdot\| \circ \mathbf{A} \circ \|\cdot\|)$$

- the *maximal hull* of $(\mathscr{A}, \mathbf{A})$:

$$(\mathscr{A}^{\max}, \mathbf{A}^{\max}) := (\bar{\mathscr{F}}^{-1} \circ \mathscr{A} \circ \bar{\mathscr{F}}^{-1}, \|\cdot\|^{-1} \circ \mathbf{A} \circ \|\cdot\|^{-1})$$

$(\mathscr{A}, \mathbf{A})$ is said to be *minimal* if $(\mathscr{A}, \mathbf{A}) = (\mathscr{A}^{\min}, \mathbf{A}^{\min})$. If $(\mathscr{A}, \mathbf{A}) = (\mathscr{A}^{\max}, \mathbf{A}^{\max})$, we say that $(\mathscr{A}, \mathbf{A})$ is *maximal*. Obviously $(\mathscr{A}^{\min}, \mathbf{A}^{\min})$ is the largest minimal operator ideal which is contained in $(\mathscr{A}, \mathbf{A})$, and $(\mathscr{A}^{\max}, \mathbf{A}^{\max})$ is the smallest maximal operator ideal which contains $(\mathscr{A}, \mathbf{A})$. Although the product of Banach operator ideals need not to be normed again, it can be shown that if $(\mathscr{A}, \mathbf{A})$ is a Banach ideal then $(\mathscr{A}^{\min}, \mathbf{A}^{\min})$ is also a *Banach* ideal (cf. [1]).

Concerning a deeper investigation of local properties of operator ideals, two further important *Banach* ideals play a key role. As before, let $(\mathscr{A}, \mathbf{A})$ be a given p-Banach ideal $(0 < p \leq 1)$, E, F be arbitrary Banach spaces and $T \in \mathscr{L}(E, F.)$ (The notion of the conjugate of an operator ideal was introduced by Gordon, Lewis, Retherford (cf. [4], [6]).)

- $T \in \mathscr{A}^*(E, F)$, if there exists a constant $c \geq 0$ such that for all Banach spaces E_0, F_0 and for all operators $b \in \mathscr{F}(E_0, E)$, $S \in \mathscr{A}(F_0, E_0)$, $A \in \mathscr{F}(F, F_0)$

$$|\mathrm{tr}(TBSA)| \leq c \cdot \|B\| \cdot \mathbf{A}(S) \cdot \|A\| .$$

Setting

$$\mathbf{A}^*(T) := \inf(c),$$

where the infimum is taken over all such constants c, we obtain a Banach ideal $(\mathscr{A}^*, \mathbf{A}^*)$, the *adjoint* of $(\mathscr{A}, \mathbf{A})$.

- $T \in \mathscr{A}^\Delta(E, F)$, if there exists a constant $c \geq 0$ such that for all *finite* operators $L \in \mathscr{F}(E, F)$

$$|\mathrm{tr}(TL)| \leq c \cdot \mathbf{A}(L) .$$

Setting

$$\mathbf{A}^\Delta(T) := \inf(c),$$

where the infimum is taken over all such constants c, we obtain a Banach ideal $(\mathscr{A}^\Delta, \mathbf{A}^\Delta)$, the *conjugate* of $(\mathscr{A}, \mathbf{A})$.

3. On tensor norms and associated Banach ideals

First we recall the basic notions of Grothendieck's metric theory of tensor products (cf., e.g., [2], [3], [5], [7]), which will be used throughout this paper.

A *tensor norm* α is a mapping which assigns to each pair (E, F) of Banach spaces a norm $\alpha(\cdot; E, F)$ on the algebraic tensor product $E \otimes F$ (shorthand: $E \otimes_\alpha F$ and $E \tilde{\otimes}_\alpha F$ for the completion) such that

(1) $\varepsilon \leq \alpha \leq \pi$

(2) α satisfies the metric mapping property: If $S \in \mathscr{L}(E, G)$ and $T \in \mathscr{L}(F, H)$, then $\|S \otimes T : E \otimes_\alpha F \to G \otimes_\alpha H\| \leq \|S\| \, \|T\|$.

Well-known examples are the injective tensor norm ε, which is the smallest one, and the projective tensor norm π, which is the largest one. For other important examples we refer to [2], [3] or [7]. Each tensor norm α can be extended in two natural ways. For this, denote for given Banach spaces E and F

$$\mathrm{FIN}(E) := \{M \subseteq E \mid M \in \mathrm{FIN}\} \quad \text{and} \quad \mathrm{COFIN}(E) := \{L \subseteq E \mid E/L \in \mathrm{FIN}\},$$

where FIN stands for the class of all finite-dimensional Banach spaces. Let $z \in E \otimes F$. Then the *finite hull* $\vec{\alpha}$ of α is given by

$$\vec{\alpha}(z; E, F) := \inf\{\alpha(z; M, N\} \mid M \in \mathrm{FIN}(E), N \in \mathrm{FIN}(F), z \in M \otimes N\}$$

and the *cofinite hull* $\overleftarrow{\alpha}$ of α is given by

$$\overleftarrow{\alpha}(z; E, F) := \sup(\alpha(Q_K^E \otimes Q_L^F(z); E/K, F/L) \mid K \in \mathrm{COFIN}(E), L \in \mathrm{COFIN}(F)\}.$$

α is called *finitely generated* if $\alpha = \vec{\alpha}$, *cofinitely generated* if $\alpha = \overleftarrow{\alpha}$ (it is always true that $\overleftarrow{\alpha} \leq \alpha \leq \vec{\alpha}$). α is called *right-accessible* if $\overleftarrow{\alpha}(z; M, F) = \vec{\alpha}(z; M, F)$ for all $(M, F) \in \mathrm{FIN} \times \mathrm{BAN}$, *left-accessible* if $\overleftarrow{\alpha}(z; E, N) = \vec{\alpha}(z; E, N)$ for all

$(E,N) \in \mathrm{BAN} \times \mathrm{FIN}$, and *accessible* if it is right- and left-accessible. α is called *totally accessible* if $\overleftarrow{\alpha} = \overrightarrow{\alpha}$.

The injective norm ε is totally accessible, the projective norm π is accessible — but not totally accessible, and Pisier's counterexample implies the existence of a (finitely generated) tensor norm which is neither left- or right-accessible (see [2], 31.6).

There exists a powerful one-to-one correspondence between finitely generated tensor norms and maximal Banach ideals which links thinking in terms of operators with "tensorial" thinking and which allows to transfer notions in the "tensor-language" to the "operator-language" and conversely. We refer the reader to [2] and [9] for detailed informations concerning this subject. Let E, F be Banach spaces and $z = \sum_{i=1}^{n} a_i \otimes y_i$ be an Element in $E' \otimes F$. Then $T_z(x) := \sum_{i=1}^{n} < x, a_i > y_i$ defines a finite operator $T_z \in \mathscr{F}(E, F)$ which is independent of the representation of z in $E' \otimes F$. Let α be a finitely generated tensor norm and $(\mathscr{A}, \mathbf{A})$ be a maximal Banach ideal. α and $(\mathscr{A}, \mathbf{A})$ are said to be *associated*, notation:

$$(\mathscr{A}, \mathbf{A}) \sim \alpha \ (\text{shorthand: } \mathscr{A} \sim \alpha, \text{ resp. } \alpha \sim A)$$

if for all $M, N \in \mathrm{FIN}$

$$\mathscr{A}(M, N) = M' \otimes_\alpha N$$

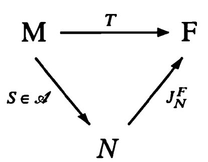
holds isometrically: $\mathbf{A}(T_z) = \alpha(z; M', N)$.

Important examples of Banach ideals are given by $(\mathscr{I}, \mathbf{I}) \sim \pi$ (integral operators), $(\mathscr{L}_2, \mathbf{L}_2) \sim w_2$ (operators which factor through a Hilbert space), $(\mathscr{D}_2, \mathbf{D}_2) \overset{1}{=} (\mathscr{L}_2^*, \mathbf{L}_2^*) \sim w_2^*$ (2-dominated operators), $(\mathscr{P}_p, \mathbf{P}_p) \sim g_p \backslash = g_q^*$ (absolutely p-summing operators), $1 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$, $(\mathscr{L}_\infty, \mathbf{L}_\infty) \overset{1}{=} (\mathscr{P}_1^*, \mathbf{P}_1^*) \sim w_\infty$ and $(\mathscr{L}_1, \mathbf{L}_1) \overset{1}{=} (\mathscr{P}_1^{*d}, \mathbf{P}_1^{*d}) \sim w_1$.

4. Accessible conjugate operator ideals

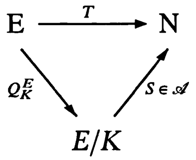
Let α be an arbitrary finitely generated tensor norm and $(\mathscr{A}, \mathbf{A}) \sim \alpha$ the associated maximal Banach ideal. Accessibility conditions of tensor norms can be transfered to the operator ideal language in the following sense:

$(\mathscr{A}, \mathbf{A})$ is called *right-accessible*, if for all $(M, F) \in \mathrm{FIN} \times \mathrm{BAN}$, operators $T \in \mathscr{L}(M, F)$ and $\varepsilon > 0$ there are $N \in \mathrm{FIN}(F)$ and $S \in \mathscr{L}(M, N)$ such that the following diagram commutes:



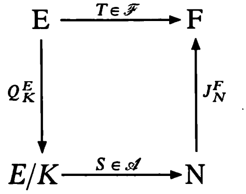
and such that $\mathbf{A}(S) \leq (1 + \varepsilon) \cdot \mathbf{A}(T)$.

$(\mathscr{A}, \mathbf{A})$ is called *left-accessible*, if for all $(E, N) \in \mathrm{BAN} \times \mathrm{FIN}$, operators $T \in \mathscr{L}(E, N)$ and $\varepsilon > 0$ there are $K \in \mathrm{COFIN}(E)$ and $S \in \mathscr{L}(E/K, N)$ such that the following diagram commutes



and such that $\mathbf{A}(S) \leq (1 + \varepsilon) \cdot \mathbf{A}(T)$.

$(\mathscr{A}, \mathbf{A})$ is *totally accessible*, if for every finite rank operator $T \in \mathscr{F}(E, F)$ between arbitrary Banach spaces E, F and $\varepsilon > 0$ there are $(K, N) \in \mathrm{COFIN}(E) \times \mathrm{FIN}(F)$ and $S \in \mathscr{L}(E/K, N)$ such that the following diagram commutes



and such that $\mathbf{A}(S) \leq (1 + \varepsilon) \cdot \mathbf{A}(T)$.

The problem whether each *maximal* Banach ideal is accessible, was negatively answered by G. Pisier at Oberwolfach in 1991 (cf. [2], 31.6):

Theorem 4.1. (G. Pisier, 1991) *There exists a maximal Banach ideal, which is not accessible.*

Let $(\mathscr{A}, \mathbf{A})$ be a given *Banach* ideal. Looking at the following "increasing sequence",

$$(\mathscr{A}^{\min}, \mathbf{A}^{\min}) \subseteq (\mathscr{A}^{*\triangle}, \mathbf{A}^{*\triangle}) \subseteq (\mathscr{A}^{\max}, \mathbf{A}^{\max})$$

then it follows that:

- $(\mathscr{A}^{\min}, \mathbf{A}^{\min})$ is accessible and in general not totally accessible.
- $(\mathscr{A}^{*\triangle}, \mathbf{A}^{*\triangle})$ is *right*-accessible. In particular $(\mathscr{A}^{*\triangle dd}, \mathbf{A}^{*\triangle dd})$ is accessible (cf. [9], [10]).
- $(\mathscr{A}^{**}, \mathbf{A}^{**}) = (\mathscr{A}^{\max}, \mathbf{A}^{\max})$ in general is not accessible.

Hence, the "larger" the ideal, the "fewer" accessible is it.

OPEN PROBLEM: *Is* $(\mathscr{A}^{*\triangle}, \mathbf{A}^{*\triangle})$ *always left-accissible?*

Note, that if $(\mathscr{A}^{*}, \mathbf{A}^{*})$ is right-accessible, then $(\mathscr{A}^{**}, \mathbf{A}^{**})$ is left-accessible. In particular $(\mathscr{A}^{*\triangle}, \mathbf{A}^{*\triangle})$ is left-accessible. Hence, if there exists a non-left-accessible conjugate of a maximal Banach ideal, then this ideal is not right-accessible. To solve this difficult problem, we look for conditions which are *equivalent* to the left-accessibility of $(\mathscr{A}^{*\triangle}, \mathbf{A}^{*\triangle})$. To this end remember the

Theorem 4.2. (Principle of local reflexivity) *Let M and F be Banach spaces, M finite-dimensional and $T \in \mathscr{L}(M, F'')$. Then for every $\varepsilon > 0$ and $N \in \mathrm{FIN}(F')$ there exists an operator $S \in \mathscr{L}(M, F)$ such that*
(1) $\|S\| \leq (1 + \varepsilon) \cdot \|T\|$
(2) $\langle Sx, b \rangle = \langle b, Tx \rangle$ *for all* $(x, b) \in M \times N$
(3) $j_F Sx = Tx$ *for all* $x \in M \cap T^{-1}(R(j_F))$

A transfer of the principle of local reflexivity (from the classical operator norm) to arbitrary p-norms of operator ideals, which is directly related to the left-accessibility of conjugate operator ideals, is given in the following sense (cf. [9] and [10]):

Definition 4.1. Let M and F be Banach spaces, M finite-dimensional, $N \in \mathrm{FIN}(F')$ and $T \in \mathscr{L}(M, F'')$. Let $(\mathscr{A}, \mathrm{A})$ be an arbitrary p-Banach ideal $(0 < p \leq 1)$ and $\varepsilon > 0$. We say that the *principle of $\mathscr{A}$-local reflexivity* (short: *$\mathscr{A}$ − LRP*) holds, if there exists an operator $S \in \mathscr{L}(M, F)$ such that
(1) $\mathrm{A}(S) \leq (1 + \varepsilon) \cdot \mathrm{A}^{**}(T)$
(2) $\langle Sx, b \rangle = \langle b, Tx \rangle$ for all $(x, b) \in M \times N$
(3) $j_F Sx = Tx$ for all $x \in M \cap T^{-1}(R(j_F))$

Using an analogous proof as in [12], ch. 28, it can be shown that for any p-Banach ideal $(\mathscr{A}, \mathrm{A})$ $(0 < p \leq 1)$, the $\mathscr{A}$-LRP is already satisfied if only the condition (1) and (2) of the previous definition are assumed (cf. [10]). In which sense does this reflect accessibility conditions? The answer (cf. [10]) is given by the gollowing.

Theorem 4.3. *Let $(\mathscr{A}, \mathrm{A})$ be an arbitrary p-Banach ideal $(0 < p \leq 1)$. Then the following statements are equivalent:*
(1) $(\mathscr{A}^{\Delta}, \mathrm{A}^{\Delta})$ *is left-accessible*
(2) $\mathscr{A}^{**}(M, F'') \cong \mathscr{A}(M, F)''$ *for all* $(M, F) \in \mathrm{FIN} \times \mathrm{BAN}$
(3) *The $\mathscr{A}$-LRP holds.*

To obtain operator ideals which satisfy the transfer of the norm estimation in the $\mathscr{L}$-LRP to their ideal norm, we need further geometrical properties of such operators (for an interesting connection of the $\mathscr{A}$-LRP for injective Banach ideals $\mathscr{A}$ with Grothendieck's inequality we refer the reader to [11]). First let us note the following.

Theorem 4.4. *Let $(\mathscr{A}, \mathrm{A})$ be an arbitrary Banach ideal. If the $\mathscr{A}^*$-LRP holds, then $\mathscr{A}^{\mathrm{inj}} \circ \mathscr{F}$ is totally accessible.*

Proof. Let E, F be arbitrary Banach spaces and $L \in \mathscr{F}(E, F)$ an arbitrary finite operator. We put $\mathscr{B} := \mathscr{A}^{*\Delta}$. Without further assumptions on the Banach spaces (such as approximation properties), the inclusion $(\mathscr{A}^{\mathrm{inj}})^{\min}(E, F) \overset{1}{\subseteq} (\mathscr{A}^{\min})^{\mathrm{inj}}(E, F)$ is always true (cf. [2], 25.11). Since $\mathscr{A}^{\min} \overset{1}{\subseteq} \mathscr{B}$, it follows therefore that $(\mathscr{A}^{\mathrm{inj}})^{\min} \overset{1}{\subseteq} (\mathscr{A}^{\min})^{\mathrm{inj}} \overset{1}{\subseteq} \mathscr{B}^{\mathrm{inj}}$. By theorem 4.3 and the assumption, $\mathscr{B}$ is left-accessible, which implies that $\mathscr{B}^{\mathrm{inj}}$ is totally accessible. Hence, given $\varepsilon > 0$, there exist

Banach spaces $K \in \text{COFIN}(E)$, $N \in \text{FIN}(F)$ and an operator $A \in \mathscr{L}(E/K, N)$, such that $L = J_N^F A Q_K^E$ and

$$(\mathbf{A}^{\text{inj}})^{\min}(L) \overset{1}{=} (\mathbf{B}^{\text{inj}})^{\min}(L) \leq \mathbf{B}^{\text{inj}}(A) < (1+\varepsilon) \cdot \mathbf{B}^{\text{inj}}(L) \leq (1+\varepsilon) \cdot (\mathbf{A}^{\text{inj}})^{\min}(L).$$

Therefore, for *all finite operators* we have obtained the following identities:

$$(\mathbf{A}^{\text{inj}})^{\min}(L) = \mathbf{B}^{\text{inj}}(L) = (\mathbf{A}^{\min})^{\text{inj}}(L).$$

Since $(\mathscr{A}^{\text{inj}})^{\min} \overset{1}{=} \mathscr{A}^{\text{inj}} \circ \overline{\mathscr{F}}$ (cf. [2], 25.2), the proof is finished. □

This theorem leads to the conjecture that $\mathscr{A}^{\text{inj}}$ even is totally accessible, if we "only" assume that the $\mathscr{A}^*$-LRP holds; this conjecture remains still open. However, an additional extension property leads to further interesting aspects concerning relations between accessibility conditions and the local reflexivity principle for operator ideals.

Definition 4.2. Let $(\mathscr{A}, \mathbf{A})$ be a p-Banach ideal $(0 < p \leq 1)$. We say that the *$\mathscr{A}$-extension property* (short: *$\mathscr{A}$-EP*) holds, if for every $\varepsilon > 0$, for every metric injection $J : E \overset{1}{\hookrightarrow} G$ and $T \in \mathscr{A}(E, F)$ there exists a $\tilde{T} \in \mathscr{A}(G, F)$ such that $T = \tilde{T}J$ and $\mathbf{A}(\tilde{T}) \leq (1+\varepsilon) \cdot \mathbf{A}(T)$.

One example of a maximal and injective Banach ideal for which this extension property holds, is given by the class of all absolutely 2-summing operators: $(\mathscr{P}_2, \mathbf{P}_2) = (\mathscr{P}_2^*, \mathbf{P}_2^*)$ satisfies the $\mathscr{P}_2 \overset{1}{=} \mathscr{P}_2^*$-EP. This follows immediately by a well known factorization theorem for absolutely 2-summing operators cf. ([12], 17.3.7) and the metric extension property of spaces of type $C(K)$, where K is a compact space.

Further examples are given by certain minimal Banach ideals: Let $\mathscr{A} \sim \alpha$ be associated. Then $\mathscr{A}^{\text{inj}*} \overset{1}{=} \backslash\mathscr{A}^* \sim \backslash\alpha^*$. By the representation theorem for minimal operator ideals the canonical map $E' \tilde{\otimes}_{\backslash\alpha^*} F \to (\backslash\mathscr{A}^*)^{\min}(E, F)$ is a metric surjection for *all* Banach spaces E and F (cf. [2], 22.2). Since $\backslash\alpha^*$ is left-projective, it follows therefore that the $(\backslash\mathscr{A}^*)^{\min}$-EP always holds.

Note that the $\mathscr{L}$-EP is false, since there is no Hahn-Banach theorem for operators.

To prepare the proof of the following theorem, observe that for a given p-Banach ideal $(\mathscr{A}, \mathbf{A})$ we have the inclusion $\mathscr{A} \overset{1}{\subseteq} \mathscr{A}^{\Delta\Delta}$ on the class of all *finite* operators and the (global) inclusion $\mathscr{A}^{\Delta} \overset{1}{\subseteq} \mathscr{A}^*$ which implies that $\mathscr{A}^{*\Delta} \overset{1}{\subseteq} \mathscr{A}^{\Delta\Delta}$. These inclusions follow directly by the definition of conjugate and adjoint operator ideals. Since there exists a maximal Banach ideal which is not right-accessible and since $\mathscr{A}^{*\Delta}$ always is right-accessible, it follows that in general $\mathscr{A}^{*\Delta} \overset{1}{\neq} \mathscr{A}^{\Delta\Delta}$.

Theorem 4.5. *Let $(\mathscr{A}, \mathbf{A})$ be an arbitrary p-Banach ideal $(0 < p \leq 1)$.*

(1) If the $\mathscr{A}$-EP holds, then $\mathscr{A}$ is left-accessible and $\mathscr{A} \overset{1}{\subseteq} \mathscr{A}^{\Delta\Delta}$. In particular $\mathscr{A}^{\text{inj}}$ is totally accessible.

(2) If the $\mathscr{A}^*$-EP and the $\mathscr{A}^{\Delta}$-LRP both are given, then $\mathscr{A}$ is accessible and $\mathscr{A}^{\Delta\Delta} \overset{1}{=} \mathscr{A}^{*\Delta}$.

Proof. To prove (1), let $T \in \mathscr{F}(E, N)$ an arbitrary (finite) operator, considered as an element of $\mathscr{A}(E, N)$, where $(E, N) \in \mathrm{BAN} \times \mathrm{FIN}$. Let $J_E : E \overset{1}{\hookrightarrow} E^\infty$ the canonical injection from the Banach space E into the Banach space $E^\infty = C(B_{E'})$. Given $\varepsilon > 0$, the assumption of the $\mathscr{A}$-EP implies the existence of an operator $\tilde{T} \in \mathscr{A}(E^\infty, N)$, such that $T = \tilde{T} J_E$ and $\mathbf{A}(\tilde{T}) \leq (1 + \varepsilon) \cdot \mathbf{A}(T)$. Since the dual of E^∞ has the metric approximation property, it follows that there exists a *finite* operator $A \in \mathscr{F}(E^\infty, E^\infty)$, such that $\tilde{T} = \tilde{T} A$ and $\|A\| < 1 + \varepsilon$. Hence we obtain that $T = Id_N \tilde{T} A J_E \in \mathscr{A}^{\min}(E, N)$ and

$$\mathbf{A}^{\min}(T) = \mathbf{A}^{\min}(Id_N \tilde{T} A J_E) \leq \mathbf{A}(\tilde{T}) \cdot \|A J_E\| \leq (1 + \varepsilon)^2 \cdot \mathbf{A}(T) \leq (1 + \varepsilon)^2 \cdot \mathbf{A}^{\min}(T).$$

Since minimal p-Banach ideals are always accessible, the last estimation shows that $\mathscr{A}$ is left-accessible. A straightforward calculation shows that $\mathscr{A}(E^\infty, F) \overset{1}{\subseteq} \mathscr{A}^{\triangle\triangle}(E^\infty, F)$ for all Banach spaces E and F. Hence, the $\mathscr{A}$-EP implies that $\mathscr{A} \overset{1}{\subseteq} \mathscr{A}^{\triangle\triangle}$.

To prove statement (2), let E, F be arbitrary Banach spaces, $\varepsilon > 0$ and set $\mathscr{B} := \mathscr{A}^{*\triangle}$. First we show that

$$(*) \qquad \mathbf{B}(T'') \leq \mathbf{A}(T) \quad \text{for all} \quad T \in \mathscr{F}(E, F).$$

To this end let $T \in \mathscr{F}(E, F)$ and $L \in \mathscr{F}(F'', E'')$ be arbitrary finite operators. By the assumed $\mathscr{A}^*$-EP there exists an operator $\tilde{L} \in \mathscr{A}^*((F'')^\infty, E'')$ such that $L = \tilde{L} J_{F''}$ and $\mathbf{A}^*(\tilde{L}) \leq (1 + \varepsilon) \cdot \mathbf{A}^*(L)$. As in the proof of (1), we find an operator $A \in \mathscr{F}((F'')^\infty, (F'')^\infty)$, such that $T''\tilde{L} = T''\tilde{L}A$ and $\|A\| \leq 1 + \varepsilon$. Canonical factorization of A leads to a finite dimensional Banach space N, operators $U \in \mathscr{L}(N, (F'')^\infty)$ and $V \in \mathscr{L}((F'')^\infty, N)$ with $A = UV$, $\|V\| \leq 1$ and $\|U\| \leq \|A\| \leq 1 + \varepsilon$. Since $\tilde{L}U \in \mathscr{F}(N, E'')$, the assumed validity of the $\mathscr{A}^\triangle$-LRP implies the existence of an operator $S \in \mathscr{L}(N, E)$ such that $T''\tilde{L}U = T''j_E S = j_F T S$ and $\mathbf{A}^\triangle(S) \leq (1 + \varepsilon) \cdot \mathbf{A}^*(\tilde{L}U)$ (cf. [10], lemma 1.1). Hence $T''L = T''\tilde{L}J_{F''} = j_F T S V J_{F''}$, which implies that:

$$|\mathrm{tr}(T''L)| = |\mathrm{tr}(j_F T S V J_{F''})| = |\mathrm{tr}(S V J_{F''} j_F T)|.$$

Since T is a finite operator, we therefore obtain the following estimation:

$$|\mathrm{tr}(T''L)| \leq \mathbf{A}^\triangle(SV) \cdot \mathbf{A}(T) \leq (1 + \varepsilon) \cdot \mathbf{A}^*(\tilde{L}U) \cdot \mathbf{A}(T) \leq (1 + \varepsilon) \cdot \mathbf{A}^*(L) \cdot \mathbf{A}(T),$$

which implies $(*)$. Obviously $\mathscr{B}^{dd} \overset{1}{\subseteq} \mathscr{B}$, and therefore it follows that $\mathscr{A} \overset{1}{\subseteq} \mathscr{B}^{dd} \overset{1}{\subseteq} \mathscr{B}$ on the class of all *finite* operators. Conjugation of this inclusion implies $\mathscr{B}^\triangle \overset{1}{\subseteq} \mathscr{A}^\triangle$, and a further conjugation leads to $\mathscr{A}^{\triangle\triangle} \overset{1}{\subseteq} \mathscr{B}^{\triangle\triangle} \overset{1}{=} \mathscr{B} \overset{1}{\subseteq} \mathscr{A}^{\triangle\triangle}$. Hence $\mathscr{A}^{\triangle\triangle} \overset{1}{=} \mathscr{A}^{*\triangle}$. Obviously, the $\mathscr{A}^\triangle$-LRP implies the $\mathscr{A}^*$-LRP. Hence, $\mathscr{B} \overset{1}{=} \mathscr{A}^{\triangle\triangle}$ is accessible, which immediately leads to the accessibility of $\mathscr{A}$. □

Corollary 4.1. *Let $(\mathscr{A}, \mathbf{A})$ be a maximal Banach ideal with the $\mathscr{A}$-EP. Then the $\mathscr{A}^*$-LRP holds.*

Recall the following (cf. [9], Satz 3.10).

Theorem 4.6. *Let* $(\mathscr{A}, \mathbf{A})$ *be an arbitrary left-accessible Banach ideal. Then the following statements are equivalent:*

(i) the $\mathscr{A}^{\Delta}$*-LRP holds*

(ii) the $\mathscr{A}^{*}$*-LRP holds.*

Combining the last two theorems, another interesting fact follows.

Theorem 4.7. *Let* $(\mathscr{A}, \mathbf{A})$ *be an arbitrary maximal and left-accessible Banach ideal, such that the* $\mathscr{A}^{*}$*-EP holds. Then* $\mathscr{A}$ *is accessible and* $\mathscr{A}^{\Delta\Delta} \stackrel{1}{=} \mathscr{A}^{*\Delta}$.

The previous considerations naturally lead to the following (still unsolved) problem: Is it possible to drop the assumption "left-accessible" in the theorems 4.6 and 4.7?

References

[1] BRAUNSS A., *Multi-ideals with special properties*, preprint, Potsdam, Pädagogische Hochschule "Karl-Liebknecht", 1987.

[2] DEFANT A. AND FLORET K., *Tensor norms and operator ideals*, North-Holland Amsterdam, London, New York, Tokio 1993.

[3] GILBERT J. E. AND LEIH T., *Factorization, tensor products and bilinear forms in Banach space theory*, Notes in Banach spaces, pp. 182–305, Univ. of Texas Press, Austin, 1980.

[4] GORDON Y., LEWIS D. R. AND RETHERFORD J. R., *Banach ideals of operators with applications*, J. Funct. Analysis 14 (1973), 85–129.

[5] GROTHENDIECK A., *Résumé de la thérie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo 8 (1956), 1–79.

[6] JARCHOW H. AND OTT R., *On trace ideals*, Math. Nachr. 108 (1982), 23–37.

[7] LOTZ H. P., *Grothendieck ideals of operators in Banach spaces*, Lecture notes, Univ. Illinois, Urbana, 1973.

[8] LINDENSTRAUSS J. AND ROSENTHAL H. P., *The* $\mathscr{L}_p$*-spaces*, Israel J. Math. 7 (1969), 325–349.

[9] OERTEL F., *Konjugierte Operatorenideale und das* $\mathscr{A}$*-lokale Reflexivitätsprinzip*, Dissertation, Kaiserslautern 1990.

[10] OERTEL F., *Operator ideals and the principle of local reflexivity*; Acta Universitatis Carolinae – Mathematica et Physica 33, No. 2 (1992), 115–120.

[11] OERTEL F., *Local properties of accessible injective operator ideals*, will appear in Czech. J. Math., 1996.

[12] PIETSCH A., *Operator ideals*, North-Holland Amsterdam, London, New York, Tokio 1980.

[13] PIETSCH A., *Eigenvalues and s-numbers*, Cambridge studies in advanced mathematics 13 (1987).

[14] TOMCZAK-JAEGERMANN N., *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monogr. and Surv. Pure and Appl. Math. 38, 1989.