On Some Characterizations of Quasiregularity

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In the present paper we give a survey and extend a result in [5] which we call "a Morera type criterion for quasiregularity". Throughout the paper the following basic notations will be used. $\mathscr{D}$ will stand for a domain (i.e. open connected set) in $\mathbb{R}^n$, $n \geq 2$. By $|\cdot|$ we also denote the outer Lebesgue measure. Sets of the form $\sigma = [a_1, b_1] \times \ldots \times [a_n, b_n]$ will be called intervals and by $\partial\sigma$ we denote the boundary of σ oriented by the exterior normal. Cubic intervals are denoted by Q. Given a mapping $f = (f_1, \ldots, f_n) : \mathscr{D} \to \mathbb{R}^n$ and a subset $E \subset \mathscr{D}$, we let $N(f\,|E, \cdot)$ be the multiplicity function of $f\,|E$ and $\|N(f\,|E)\| := \sup N(f\,|E, \cdot)$. We associate to $f = (f_1, \ldots, f_n)$ the differential forms

$$\omega_{ij} = f_i\,\mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_j} \wedge \ldots \wedge \mathrm{d}x_n\,, \qquad 1 \leq i,j \leq n\,. \tag{1}$$

In this paper we also use the notion of the topological degree $\deg(y, f, \bar{\Omega})$ (of f at the point y with respect to $\bar{\Omega}$) where Ω is an open set, $\Omega \subset\subset \mathscr{D}$.

Definition 1. ([1]). A continuous mapping $f = (f_1, \ldots, f_n) : \mathscr{D} \to \mathbb{R}^n$ is called quasiregular (or a mapping with bounded distortion) if the following conditions are fulfilled:

(a) $f \in W^{1,n}_{\mathrm{loc}}(\mathscr{D})$;
(b) either $J(f, \cdot) := \det(\partial f_i/\partial x_j(\cdot)) \geq 0$ a.e. in $\mathscr{D}$ or $J(f, \cdot) \leq 0$ a.e. in $\mathscr{D}$;
(c) there exists a constant $K \geq 1$ so that

$$\|f'(x)\|^n \leq K\,|J(f, x)|$$

for almost all $x \in \mathscr{D}$, where $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear operator whose matrix in the standard basis of $\mathbb{R}^n$ is $(\partial f_i/\partial x_j(x))$.

It was shown in [1] that each non-constant quasiregular mapping is open, discrete, a.e. differentiable and satisfies the Lusin's condition (N) (see e.g. [2] for a survey of quasiregular mappings).

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We consider first a sufficient condition for a mapping to be in the Sobolev class $W^{1,p}_{\text{loc}}(\mathscr{D})$, $p > 1$.

Theorem 1. *Let $f = (f_1, \ldots, f_n) : \mathscr{D} \to \mathbb{R}^n$ be a continuous mapping satisfying the following conditions. There exist a number $p > 1$ and a finite subadditive function (cf. [3], p. 206) of interval $\Phi \geq 0$ so that*

$$\forall i, j \; \forall Q \subset \mathscr{D} : \left| \int_{\partial Q} \omega_{ij} \right| \leq \Phi(Q)^{\frac{1}{p}} \, |Q|^{\frac{p-1}{p}} . \tag{2}$$

Then $f \in W^{1,p}_{\text{loc}}(\mathscr{D})$.

Proof. Consider the function of interval

$$F_{ij}(\sigma) := \int_{\partial \sigma} \omega_{ij} , \qquad \sigma \in \mathscr{D} , \; 1 \leq i, j \leq n , \tag{3}$$

which is obviously additive. Since each σ with rational edges can be written in the form

$$\sigma = \bigcup_{s=1}^{m} Q_s , \qquad \operatorname{Int} Q_s \cap \operatorname{Int} Q_r = \emptyset , \; s \neq r , \tag{4}$$

we deduce from (2), (3), (4) that

$$|F_{ij}(\sigma)| \leq \sum_{s=1}^{m} \Phi(Q_s)^{\frac{1}{p}} \, |Q_s|^{\frac{p-1}{p}} \leq \Phi(\sigma)^{\frac{1}{p}} \, |\sigma|^{\frac{p-1}{p}} . \tag{5}$$

Given an arbitrary interval σ we may find a sequence $\{\sigma_\nu\}$ of concentric intervals of the type (4) so that $\sigma_\nu \subset \sigma_{\nu+1}$, $\operatorname{Int} \sigma = \lim \sigma_\nu$. Then from (3), (5) we get by continuity of f that

$$\forall \sigma \subset \mathscr{D} : |F_{ij}(\sigma)| \leq \Phi(\sigma)^{\frac{1}{p}} \, |\sigma|^{\frac{p-1}{p}} \tag{6}$$

Now let $\sigma_0 \subset \mathscr{D}$ be a fixed interval and $\{\sigma_s\}$, $1 \leq s \leq m$, an arbitrary system of intervals in σ_0 without common interior points. Then by (6) we have

$$\sum_{s=1}^{m} |F_{ij}(\sigma_s)| \leq \Phi(\sigma_0)^{\frac{1}{p}} \left(\sum_{s=1}^{m} |\sigma_s| \right)^{\frac{p-1}{p}} , \tag{7}$$

which means that F_{ij} is an absolutely continuous function of interval in σ_0. This permits to conclude (cf. [4], Th. 7.4, Ch. 4) that there exists a function $h_{ij} \in L_{\text{loc}}(\mathscr{D}, \mathbb{R})$ such that

$$\forall \sigma \subset \mathscr{D} : F_{ij}(\sigma) = \int_{\sigma} h_{ij} \, \mathrm{d}x \tag{8}$$

Since Φ is subadditive and finite we have (cf. [3], pp. 205—207) that for almost all $x \in \mathscr{D}$ there exists a finite derivative

$$\Phi'(x) = \lim \Phi(Q)/|Q|, \ \operatorname{diam} Q \to 0, \ x \in Q, \tag{9}$$

and moreover,

$$\forall \sigma \subset \mathscr{D}: \int_\sigma \Phi'(x)\, \mathrm{d}x \leq \Phi(\sigma) < \infty . \tag{10}$$

From (6), (8) we get for each cube $Q \subset \mathscr{D}$

$$|Q|^{-1} \left| \int_Q h_{ij}\, \mathrm{d}x \right| \leq (\Phi(Q)/|Q|)^{\frac{1}{p}} . \tag{11}$$

Let $x \in \mathscr{D}$ be a Lebesgue point of h_{ij}, $1 \leq i, j \leq n$, at which $\Phi'(x)$ exists. Then letting $\operatorname{diam} Q \to 0$, $x \in Q$, in (11) we obtain that

$$|h_{ij}(x)| \leq \Phi'(x)^{\frac{1}{p}}, \ 1 \leq i, j \leq n, \tag{12}$$

holds for almost all $x \in \mathscr{D}$. Now fix an interval $\sigma_0 \subset \mathscr{D}$ and for each $0 < \varepsilon < \operatorname{dist}(\sigma_0, \partial \mathscr{D})$, $x \in \sigma_0$, let

$$f_{i\varepsilon}(x) = |B(x, \varepsilon)|^{-1} \int_{B(x,\varepsilon)} f_i(y)\, \mathrm{d}y ,$$

$$h_{ij\varepsilon}(x) = |B(x, \varepsilon)|^{-1} \int_{B(x,\varepsilon)} h_{ij}(y)\, \mathrm{d}y ,$$

where $B(x, \varepsilon) = \{y : \|y - x\| < \varepsilon\}$, and

$$\omega_{ij\varepsilon} = f_{i\varepsilon}\, \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_j} \wedge \ldots \wedge \mathrm{d}x_n .$$

It is well-known that $f_{i\varepsilon} \in C^1(\sigma_0)$, $h_{ij\varepsilon} \in C(\sigma_0)$. Combining (3), (8) and applying Fubini's theorem we get

$$\forall \sigma \subset \sigma_0 : \int_{\partial\sigma} \omega_{ij\varepsilon} = \int_\sigma h_{ij\varepsilon}\, \mathrm{d}x ,$$

whence by Stokes' theorem

$$\forall \sigma \subset \sigma_0 : \int_\sigma \partial f_{i\varepsilon}/\partial x_j\, \mathrm{d}x = (-1)^{j+1} \int_\sigma h_{ij\varepsilon}\, \mathrm{d}x .$$

By continuity of integrands we infer that the equality

$$\partial f_{i\varepsilon}/\partial x_j = (-1)^{j+1} h_{ij\varepsilon}, \ 1 \leq i, j \leq n , \tag{13}$$

holds everywhere in σ_0. Obviously $f_{i\varepsilon} \in W^{1,p}(\sigma_0)$ and since $f_{i\varepsilon} \to f_i$ in $C(\sigma_0)$ and $h_{ij\varepsilon} \to h_{ij}$ in $L(\sigma_0)$ as $\varepsilon \to 0$, we conclude immediately by (10), (12), (13) that $f | \sigma_0 \in W^{1,p}(\sigma_0)$ which completes the proof. Note that $\partial f_i/\partial x_j = (-1)^{j+1} h_{ij}$ a.e. in $\mathscr{D}$ and that

$$\forall \sigma \subset \mathscr{D}: \int_{\partial\sigma} \omega_{ij} = (-1)^{j+1} \int_{\sigma} \partial f_i/\partial x_j \, dx . \tag{14}$$

This assertion is used in proofs of theorems 2, 4.

Theorem 2. ([5]). *Let $f = (f_1, \ldots, f_n): \mathscr{D} \to \mathbb{R}^n$ be a continuous open discrete mapping. Then f is quasiregular if and only if there exists a constant $M > 0$ such that the following condition holds:*

$$\forall Q \subset \mathscr{D} \, \forall i, j: \left| \int_{\partial Q} \omega_{ij} \right| \leq M(\|N(f|Q)\| \cdot |f(Q)| \cdot |Q|^{n-1})^{\frac{1}{n}}. \tag{15}$$

Sketch of the proof. The necessity of (15) is almost straighforward. To this end we use Stokes' theorem and the properties of quasiregular mappings that have been mentioned at the beginning. We also make use of the relation

$$\forall \sigma \subset \mathscr{D}: \int_{\sigma} |J(f, x)| \, dx = \int_{\mathbb{R}^n} N(f|\sigma, y) \, dy \tag{16}$$

which holds for continuous a.e. differentiable mappings satisfying Lusin's condition (N) and which is important in getting the estimate (15). It is also important to note that given a continuous open discrete mapping $f: \mathscr{D} \to \mathbb{R}^n$, one has $\dim \beta(f) \leq n - 2$, where $\beta(f)$ is the set of branch points of f (cf. [6], [7]). From this fact it can be deduced that the topological index $\imath(f, x)$ (i.e. the local topological degree of f) has no zeros and is of the same sign in the domain $\mathscr{D}$. This allows to prove that

$$\forall \Omega \subset\subset \mathscr{D}: \|N(f|\Omega)\| < \infty$$

so that the right-hand part in (15) is always finite. The proof of sufficiency of (15) begins by showing that $f \in W^{1,n}_{\text{loc}}(\mathscr{D})$. This follows by Theorem 1 applied to each domain $\Omega \subset\subset \mathscr{D}$ if we let $p = n$ and define the subadditive function of interval by

$$\Phi(\sigma) = M^n \|N(f|\Omega)\| \, V(f, \operatorname{Int} \sigma) .$$

Here $V(f, \operatorname{Int} \sigma)$ denotes the variation (in the Banach sense) of f on $\operatorname{Int} \sigma$ (cf. [3], pp. 202, 279). So we have $f|\Omega \in W^{1,n}(\Omega)$ and thus f satisfies condition (a) for quasiregularity, i.e. $f \in W^{1,n}_{\text{loc}}(\mathscr{D})$. Since f is open, this implies that f is differentiable a.e. in $\mathscr{D}$ [1] (see [5] for an alternative proof). Now again fix any open set $\Omega \subset\subset \mathscr{D}$. By (14), (15) we may write

$$\forall Q \subset \Omega: \left| \int_{\partial Q} \omega_{ij} \right| = \left| \int_{Q} \partial f_i/\partial x_j \, dx \right| \leq M\big(\|N(f|\Omega)\| \cdot |f(Q)| \cdot |Q|^{n-1}\big)^{\frac{1}{n}}. \tag{17}$$

Let $x \in \Omega$ be a Lebesgue point of all $\partial f_i/\partial x_j$ and f differentiable at x. Dividing (17) by $|Q|$ and letting $\operatorname{diam} Q \to 0$, $x \in Q$, we obtain that

$$|\partial f_i/\partial x_j(x)| \leq M \|N(f|\Omega)\|^{\frac{1}{n}} \cdot |J(f, x)|^{\frac{1}{n}} \tag{18}$$

holds for almost all $x \in \Omega$. Since the topological index $\imath(f, x)$ preserves its sign in $\mathscr{D}$ it follows that sign $\imath(f, x) = $ sign $J(f, x)$ at each point x where f is differentiable and $J(f, x) \neq 0$. Thus f satisfies condition (b), i.e. $J(f, \cdot)$ does not change its sign. From (18) we obtain that $f|\Omega$ is quasiregular whence $|\beta(f) \cap \Omega| = 0$ (see [1]). Since $\Omega \subset\subset \mathscr{D}$ was chosen arbitrarily we conclude that $|\beta(f)| = 0$. Finally, letting Ω in (18) be any open set off $\beta(f)$ such that $f|\Omega$ is injective, we get $\|N(f|\Omega)\| = 1$ whence

$$|\partial f_i/\partial x_j(x)| \leq M|J(f, x)|^{\frac{1}{n}} \tag{19}$$

for almost all $x \in \mathscr{D}$. This completes the proof since (19) is obviously equivalent to the condition (c) for quasiregularity.

In order to establish the next result (Theorem 4) we shall make use of the following notion analogous to that given in [8].

Definition 2. A continuous mapping is called pseudomonotone if

$$\exists C \geq 1 \; \forall Q \subset \mathscr{D}: \operatorname{diam} f(Q) \leq C \operatorname{diam} f(\partial Q).$$

We should note that in [8] this definition was formulated considering balls instead of cubes.

Theorem 3 ([8]). *Let $f: \mathscr{D} \to \mathbb{R}^n$ be a continuous pseudomonotone mapping of the class $W^{1,n}_{\mathrm{loc}}(\mathscr{D})$. Then f satisfies Lusin's condition (N).*

Remark. Though this assertion was proved in the case of the pseudomonotonicity defined by balls, the analysis of the proof shows that the result is valid if balls are replaced by cubes. The required inequality in the oscillation lemma ([8], 2.1) also holds for cubes:

$$\operatorname{diam} f(\partial Q)^n \leq A_n r \int_{\partial Q} \|f'(y)\|^n \, \mathrm{d}S_y \tag{20}$$

where r is the length of the edge of Q and A_n is a constant depending on n. This inequality can be readily deduced for instance from Lemma 6.3 in [1], Ch. 2, § 6.

Definition 3. A continuous mapping $f: \mathscr{D} \to \mathbb{R}^n$ is orientation preserving (resp. reversing) if

$$\forall Q \subset \mathscr{D} \; \forall y \notin f(\partial Q): \deg(y, f, Q) \geq 0 \quad (\text{resp. } \deg(y, f, Q) \leq 0)\,. \tag{21}$$

Theorem 4. *Let $f: \mathscr{D} \to \mathbb{R}^n$ be a continuous pseudomonotone and orientation preserving (or reversing) mapping. Then f is quasiregular if and only if there exists a constant $M > 0$ such that*

$$\forall Q \subset \mathscr{D} \; \forall i, j: \int_{\partial Q} \omega_{ij} \leq M \left(\int_{\mathbb{R}^n} N(f|Q, y) \, \mathrm{d}y \right)^{\frac{1}{n}} |Q|^{\frac{n-1}{n}} < \infty\,. \tag{22}$$

Proof follows mainly the same pattern as in the case of Theorem 2, so that we shall single out only new points in it. To fix ideas, assume that f is an orientation preserving mapping. The necessity of (22) can be shown in the same way as (15) (cf. [5]). Note that the right-hand part in (15) is merely an upper estimate of the middle term in (22). Now we pass to the sufficiency of (22). Letting $p = n$ and defining

$$\Phi(\sigma) = M^n \int_{\mathbb{R}^n} N(f \,|\, \mathrm{Int}\,\sigma, y)\,\mathrm{d}y \tag{23}$$

we may apply Theorem 1. Thus we get from (22), (23) that $f \in W^{1,n}_{\mathrm{loc}}(\mathscr{D})$. But since f is pseudomonotone, this implies that f is differentiable a.e. in $\mathscr{D}$. The proof is the same as for monotone (e.g. open) mappings. Moreover, by Theorem 3 the mapping f satisfies Lusin's condition (N). Hence we may write (cf. [3], Th. 2, p. 363)

$$\forall Q \subset \mathscr{D}: \int_Q J(f, x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \deg(y, f, Q)\,\mathrm{d}y\,.$$

This yields, since f is orientation preserving, that $J(f, x) \geq 0$ for amost all $x \in \mathscr{D}$. Furthermore, using (16) and Stokes' theorem we may rewrite (22) in the form

$$\overline{|Q|^{-1} \int_Q \partial f_i/\partial x_j \,\mathrm{d}x} \leq M \left(\int_Q J(f, x)\,\mathrm{d}x \right)^{\frac{1}{n}}.$$

Let $x \in \mathscr{D}$ be a Lebesgue point of $J(f, \cdot)$ and all $\partial f_i/\partial x_j$. Letting diam $Q \to 0$, $x \in Q$, we obtain that

$$|\partial f_i/\partial x_j(x)| \leq M J(f, x)^{\frac{1}{n}}$$

holds for almost all $x \in \mathscr{D}$ which clearly completes the proof.

Remarks. (i) Theorem 4 remains valid if instead of (21) we assume for instance that f is a.e. approximately differentiable and det f'_{ap} does not change its sign.
(ii) Theorems 2, 4 remain also valid if in (15) and (22) we consider only cubes of the form $Q_k(x) = x + Q_k$, $x \in \mathscr{D}$, (i.e. translates of Q_k) where $\{Q_k\}$ is a fixed sequence of cubes centered at $x = 0$, diam $Q_k \to 0$.

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