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BV-Sets, Functions and Integrals

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In this survey type talk the main topic is bounded variation (BV). After reviewing some classical concepts and results we turn to the more recent concept of *BV*-integrals and announce the program made related to the multiplier problem of these Riemann type integrals.

1. *BV*-Sets

We work in the m-dimensional space $\mathbf{R}^m$. The ball centered at x and of radius r will be denoted by $B(x, r)$. The closure, the interior and the exterior of a set A is denoted by cl A, int A, and ext A, respectively. We denote by $|A|$ the Lebesgue measure of $A \subset \mathbf{R}^m$. In this talk we consider only measurable subsets of $\mathbf{R}^m$.

Definition. Given a set $A \subset \mathbf{R}^m$ the point $x \in \mathbf{R}^m$ is a *density* point of A when

$$\lim_{r \to 0+} \frac{|B(x, r) \cap A|}{|B(x, r)|} = 1 .$$

The set of all density points of A is its *essential interior*, denoted by int* A. The *essential exterior* of A, ext* A, equals the essential interior of $\mathbf{R}^m \setminus A$. The *essential closure* of A, cl* A, equals $\mathbf{R}^m \setminus$ ext* A. Finally $\partial^* A$ denotes the *essential boundary* of A which is $\mathbf{R}^m \setminus (\text{int}^* A \cup \text{ext}^* A)$. By Lebesgue's density theorem almost every point of the (measurable) set A belongs to int* A, and almost every point of $\mathbf{R}^m \setminus A$ belongs to ext* A.

We denote the s-dimensional Hausdorff measure by $\mathscr{H}^s$, in the special case when $s = m - 1$ we just simply write $\mathscr{H}$, omitting the superscript.

Definition. The perimeter (surface area) of $H \subset \mathbf{R}^m$ is $\|H\| \stackrel{\text{def}}{=} \mathscr{H}(\partial^* H)$. Sets of finite perimeter are called sets of bounded variation (BV sets, or Caccioppoli sets). We say that $A \in \mathscr{BV}$ if $A \in BV$ and cl* $A = A$.

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Clearly $\mathrm{cl}^* A \in \mathscr{BV}$ for every BV set A. Denoting by $A \triangle B$ the symmetric difference of the sets A and B from the Lebesgue density theorem we infer $|A \triangle \mathrm{cl}^* A| = 0$.

We say that two BV-sets A and B are *nonoverlapping* if $|A \cap B| = 0$.

For a BV set A a unit exterior normal ν_A can be defined $\mathscr{H}$-almost everywhere on A such that

$$\int_A \operatorname{div} v \, d\lambda = \int_{\partial^* A} v \cdot \nu_A \, d\mathscr{H} \tag{1.1}$$

holds for every vector field v which is continuously differentiable in a neighborhood of cl A [EG, Sections 5.1 and 5.8].

Next we turn to some results which can help to understand the structure of BV-sets. First we discuss the approximation property of BV sets.

Sets of the form $\times_{i=1}^m [a_i, b_i]$ are called intervals. *Figures* are finite unions of nondegenerate intervals. The class of figures in $\mathbf{R}^m$ will be denoted by $\mathscr{F}$. By [BuP, Proposition 1.1] BV-sets can be approximated by figures:

Theorem 1.1. *Given a BV set $A \subset \mathbf{R}^m$ there exists a sequence of figures A_n such that*

i) $\lim |A_n \triangle A| = 0$;

ii) $\sup \|A_n\| \leq c_m \|A\|$, *where the constant c_m depends only on the dimension;*

iii) diam $A_n \leq$ diam A for all n.

It is clear that $\mathscr{F}$ is a subclass of $\mathscr{BV}$.

The regularity of the BV-set A is $r(A) \stackrel{\text{def}}{=} |A|/\operatorname{diam}(A) \cdot \|A\|$ when $|A| > 0$, otherwise $r(a) = 0$. Given a number $r > 0$ we say that A is r-regular when $r(A) > r$. The higher the regularity constant the "closer" the set A to a ball.

The following theorem, giving some information about the structure of BV-sets, is due to J. Mařík [Ma, 33. Theorem]. He used a different definition for a class of sets which equals the class BV.

We say that the sets $A_1, A_2 \subset \mathbf{R}$ are equivalent if the one-dimensional measure of $A_1 \triangle A_2$ equals zero. For a set $A \subset \mathbf{R}^m$ and a point $x \in \mathbf{R}^{m-1}$ we denote $A_x = \{t \in \mathbf{R} : (x_1, \ldots, x_{m-1}, t) \in A\}$, that is, A_x is the "vertical" section of A.

Theorem 1.2. *Given a set $A \in BV$ there exists a set $E \subset \mathbf{R}^{m-1}$ such that*

i) $\mathscr{H}(\mathbf{R}^{m-1} \setminus E) = 0$;

ii) For every $x \in E$ there exists a non-negative integer $n(x)$ and real numbers $a_1(x) < b_1(x) < \ldots < a_{n(x)}(x) < b_{n(x)}(x)$ such that A_x is equivalent to $\bigcup_{j=1}^{n(x)} (a_j(x), b_j(x))$;

iii)

$$2 \int_{\mathbf{R}^{m-1}} n(x) \, dx \leq \|A\| .$$

The above result roughly states that almost every "vertical section" of a BV set is equivalent to the union of finitely many intervals. The integral of the number of these intervals is not greater than half times the perimeter of A. This statement reminds to a well-known theorem of Banach [S, Ch. IX. (6.4) Theorem], which is usually among the very first theorems one learns about bounded variation:

Theorem 1.3. *Let f be a continuous function on the interval $I = [a, b]$ and let $n(t)$ denote the number (finite or infinite) of the points of I at which f assumes the value t. Then $\int_{-\infty}^{\infty} n(t)\,\mathrm{d}t$ equals the variation of f on I, namely, f is a function of bounded variation whenever this integral is finite.*

We can also given a slightly different interpretation to the previous theorem. Recall that $\mathscr{H}^0(A)$ equals the number of the elements of A. Observe that if $A \subset \mathbf{R}$ is a one dimensional BV set then $\|A\| = \mathscr{H}^0(\partial^* A) < \infty$, that is, $\partial^* A$ is finite and A is equivalent to a finite union of intervals (note that these one dimensional BV sets appear in Theorem 1.2 as "vertical sections"). Let $E_t \overset{\text{def}}{=} \{x \in [a, b] : f(x) > t\}$. Then $n(t)$ for almost every t equals $\mathscr{H}^0(\partial^* E_t)$, that is, the variation of f equals $\int_{-\infty}^{\infty} \mathscr{H}^0(\partial^* E_t)\,\mathrm{d}t$.

This leads us to the second topic of this talk.

2. *BV* Functions

Given an open set Ω we denote by $C_c^1(\Omega; \mathbf{R}^m)$ the class of continuously differentiable $\Omega \to \mathbf{R}^m$ maps with compact support.

Definition. The integrable function $f : \Omega \to \mathbf{R}$ is of *bounded variation* in Ω, that is, $f \in BV(\Omega)$ if $f \in L^1(\Omega)$ and

$$\sup \left\{ \int_\Omega f \operatorname{div} \varphi : \varphi \in C_c^1(\Omega; \mathbf{R}^m), |\varphi| \le 1 \right\} < \infty .$$

For a detailed treatment of BV functions we recommend reading [EG, Chapter 5] or [Z, Chapter 5]. We just mention that the weak partial derivatives of BV functions are Radon measures, and a set $A \subset \mathbf{R}^m$ is a BV-set if and only if its characteristic function χ_A is a BV function. When one deals with generalized integrals it is much easier to think of BV functions by using their characterization obtained from the Coarea Formula [EG, Section 5.5]. For a given function $f : \Omega \to \mathbf{R}$ and a $t \in \mathbf{R}$ we denote the upper level set $\{x \in \Omega : f(x) > t\}$ by E_t.

Theorem 2.1. *Assume that the function f is integrable on the open set $\Omega \subset \mathbf{R}^m$. Then $f \in BV(\Omega)$ if and only if $\|Df\|(\Omega) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \mathscr{H}(\partial^*(E_t) \cap \Omega)\,\mathrm{d}t$ is finite.*

This implies that for BV functions almost every upper level set, E_t, is of finite perimeter in Ω. It is also clear that Theorem 2.1 is a generalization of the classical result in Theorem 1.3.

The BV integral is a multidimensional Henstock-Kurzweil type non-absolute integration procedure. We refer to the monograph [P2] for the history and details of the theory of generalized Riemann type integrals.

We discuss two integration procedures the first, the $\mathscr{F}$-integral, deals with figures while the $\mathscr{BV}$-integral with $\mathscr{BV}$-sets. Since the definitions are similar $\mathscr{A}$ will denote either the class $\mathscr{F}$, or $\mathscr{BV}$.

Definition. The function $F : \mathscr{A} \to \mathbf{R}$ is a *charge* when

i) F is additive, that is, $F(A \cup B) = F(A) + F(B)$ when $A, B \in \mathscr{A}$ are non-overlapping;

ii) F is continuous, that is, for every $\varepsilon > 0$ there exists $\eta > 0$ such that $|F(A)| < \varepsilon$ for each $A \in \mathscr{A}$ with $A \subset B(\mathbf{0}, 1/\varepsilon)$, $\|A\| < 1/\varepsilon$ and $|A| < \eta$.

Charges are the possible "indefinite" integrals. For example, $F(A) = (Lebesgue) \int_A f$ for any locally integrable function f is a charge. The other type of standard example of charges is the BV-flux $F(A) = \int_{\partial^* A} \operatorname{div} v \cdot \nu_A \, d\mathscr{H}$ for continuous vector fields $v : \mathbf{R}^m \to \mathbf{R}^m$ and bounded BV-sets A.

A set in $\mathbf{R}^m$ is *thin* when it is of sigma finite $\mathscr{H}$ measure. If $A \subset \mathbf{R}^m$ then $\delta : A \to [0, \infty)$ is a *gage function* on A if its null set $\{x : \delta(x) = 0\}$ is thin.

The collection $\{(A_i, x_i)\}_{i=1}^p$ is an $\mathscr{A}$*-partition* in A when $x_i \in A_i \subset A$ holds for all i and the sets $A_i \in \mathscr{A}$ are non-overlapping. Given a gage function the above partition is δ-fine when $A_i \subset B(x_i, \delta(x_i))$ for each i. Finally if the regularity of each A_i is bigger than $r > 0$ then the partition is called r-regular.

Definition. If $A \in \mathscr{A}$, then $f : A \to \mathbf{R}$ is $\mathscr{A}$-integrable on A if there exists a charge F such that for all $\varepsilon > 0$, there exists a gage δ on A satisfying

$$\sum_{i=1}^{p} |f(x_i)\, |A_i| - F(A_i)| < \varepsilon$$

for each ε-regular δ-fine $\mathscr{A}$-partition $\{(A_i, x_i)\}_{i=1}^p$ in A. Then $(\mathscr{A}) \int_A f \stackrel{\text{def}}{=} F(A)$.

Using the $\mathscr{BV}$-integral a very general divergence theorem (generalization of formula (1.1)) can be stated, for the details see [P1], [BuP].

One natural question in this field is whether the classes of $\mathscr{F}$- and $\mathscr{BV}$-integrable functions are different. W. F. Pfeffer in [P1] proved the following.

Theorem 3.1. *If K is a figure then f is $\mathscr{F}$-integrable on K iff it is $\mathscr{BV}$-integrable on K.*

On the other hand an unpublished example of the present author (for details see, [P1]) implies that there exists $A \in \mathscr{BV}$ and an $f : A \to \mathbf{R}$ which is $\mathscr{BV}$-integrable on A but has no $\mathscr{F}$-integrable extension onto a figure containing A.

Given a bounded BV-set A and a function $f : A \to \mathbf{R}$ denote by $\bar{f}$ its extension which equals f on A and 0 or $\mathbf{R}^m \setminus A$. The function f is R-integrable on A if $\bar{f}$ is

$\mathscr{BV}$-integrable on any subfigure of $\mathbf{R}^m$ (in view of Theorem 3.1 we could assume $\mathscr{F}$-integrability as well). The class of R-integrable functions on A is denoted by $R(A)$ and $(R)\int_A f \overset{\text{def}}{=} (\mathscr{BV})\int_K f$ where K is a figure containing A (it is easy to see that the value of the integral does not depend on the choice of K).

The *multiplier problem* for the R-integrable functions is the following:

Classify the class M of those functions $g : A \to \mathbf{R}$ for which from $f \in R(A)$ it follows that $fg \in R(A)$ as well.

The 1-dimensional case of this problem was solved by Bongiorno and Skvortsov in [BS], by showing that multipliers are the functions of bounded variation. On the other hand the higher dimensional case turned out to be more difficult. Mortensen and Pfeffer in [MP] verified that all Lipschitz functions are multipliers. Later Pfeffer [P4] showed that characteristic functions of BV-sets are multipliers and each multiplier is a bounded BV function on A ($BV^\infty(A)$ function).

In [D] De Pauw related the multiplier problem to the description of the dual space of the R-intregrable functions endowed with a suitable topology.

Finally in the recent paper [BDP] Buczolich, De Pauw and Pfeffer prove that the class of multipliers M equals the class $BV^\infty(A)$.

We remark that in the paper [CLL] the multiplier problem is considered for double Henstock integrals.

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