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Differentiability Points of a Distance Function

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Let \( K \subset [0, 1] \) be the usual Cantor set, and let \( A \overset{\text{def}}{=} \{ f \in C(K) : 0 \in \text{Range}(f) \} \). Its distance function \( \varphi : C(K) \to \mathbb{R} \) is defined by \( \varphi(f) \overset{\text{def}}{=} \text{dist}(f, A) \).

In this note we characterize the set of the points of the Gâteaux differentiability of this function \( \varphi \). We prove that, \( \varphi \) is not Gâteaux differentiable at a function \( f \) iff \( Z_f = \{ x \in K : f(x) = 0 \} \) can be covered by disjoint open sets \( U_1, U_2, \ldots, U_m \) for which there exist non-zero constants \( c_1, c_2, \ldots, c_m \) such that 0 is a porosity point of the set \( \bigcup_{n=1}^m c_n \text{Range}(f|_{U_n}) \).

During the attempts to answer the question whether the \( \sigma \) ideal of Aronszajn null sets and Gaussian null sets coincide in a separable Banach space \( E \) (see [1], [2]), it was important to study the following strange set:

Let \( K \subset [0, 1] \) be the usual Cantor set, and let

\[
A \overset{\text{def}}{=} \{ f \in C(K) : 0 \in \text{Range}(f) \}.
\] (1)

It is clear that \( A \) is a closed subset of \( C(K) \). It turned out that \( A \) contains a cube, that is, there is a system of functions of dense span \( f_0, f_1, f_2, \ldots \in C(K) \) for which \( \sum_{i=1}^{\infty} \| f_i \| < \infty \) and \( f_0 + \sum_{i=1}^{\infty} r_i f_i \in A \) for every sequence \( r_1, r_2, \ldots \in [0, 1] \). This surprising fact developed into the idea to look for ‘a nearly cube’ inside any non-Aronszajn null set \( A \), more precisely, to find an appropriate cube \( x_0 + \sum_{i=1}^{\infty} r_i x_i \) (where \( r_i \in [0, 1], x_1, x_2, \ldots \) is a sequence of the points of \( E \) of dense span, and \( \sum_{i=1}^{\infty} \| x_i \| < \infty \)) such that \( A \) is large in this cube, i.e. the Lebesgue measure of the set \( \{ (r_1, r_2, \ldots) \in [0, 1]^N : x_0 + \sum_{i=1}^{\infty} r_i x_i \in A \} \) is large.

On the other hand, since the set \( A \) defined by (1) is not Aronszajn null, it must contain points of Gâteaux differentiability of any Lipschitz function, in particular of its distance function \( \varphi : C(K) \to \mathbb{R} \) defined by

\[
\varphi(f) \overset{\text{def}}{=} \text{dist}(f, A).
\]

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In this note we characterize the set of the points of the differentiability of this function $\varphi$. This turned out to be interesting in itself, because of its connection to porosity properties.

Since $\varphi$ is non-negative, if it is Gâteaux differentiable at a point of $A$, then its derivative must be 0. It is easy to see that

$$\varphi(f) = \inf |f| .$$

Indeed, $\varphi(f) \geq \inf |f|$ is trivial, and for the continuous real function

$$h_f(x) = \begin{cases} 0 & \text{if } |x| < \inf |f| \\ 2(x - \inf |f|) & \text{if } \inf |f| \leq |x| < 2\inf |f| \\ x & \text{if } 2\inf |f| \leq |x| \end{cases}$$

we have $h_f \circ f \in A$ and $\|h_f \circ f - f\| = \inf |f|$.

Thus, $\varphi$ is differentiable at $f \in A$ iff

$$\lim_{t \to 0^+} \frac{\varphi(f - tg) - \varphi(f)}{t} = \lim_{t \to 0^+} \frac{\inf |f - tg|}{t} = 0 \quad (\ast)$$

holds for every $g \in C(K)$.

**Lemma.** If for a sequence $x_n$ and a function $g \in C(K)$ we have $x_n \to x$, $f(x_n) \to f(x) = 0$, $\frac{f(x_n)}{f(x_{n+1})} \to 1$ and $\text{sgn } g(x) = \text{sgn } f(x_n) \neq 0$ for every $n$, then $\varphi$ is differentiable at $f$ in the direction of $g$, that is, $(\ast)$ holds for $f$ and $g$.

**Proof.** Suppose indirectly that there exists a sequence $t_n \searrow 0$ and $\varepsilon > 0$ for which $\frac{|f - t_ng|}{t_n} > \varepsilon$. Now, for every $k$ and $n$ we have

$$\frac{|f(x_k) - t_ng(x_k)|}{t_n} = |f(x_k)| \left| \frac{1}{t_n} - \frac{g(x_k)}{f(x_k)} \right| > \varepsilon .$$

Since $g$ is continuous, we have $\text{sgn } g(x_k) = \text{sgn } g(x) = \text{sgn } f(x_k) \neq 0$ if $k$ is large, thus by $g(x_k) \to g(x) \neq 0$ and $f(x_k) \to 0$ we have $\lim_{k \to \infty} \frac{g(x_k)}{f(x_k)} = +\infty$. If $n$ is large enough then we can choose a $k = k(n)$ for which

$$\frac{g(x_k)}{f(x_k)} \leq \frac{1}{t_n} < \frac{g(x_{k+1})}{f(x_{k+1})} ,$$

and for this $k$ have

$$\frac{g(x_{k+1})}{f(x_{k+1})} - \frac{g(x_k)}{f(x_k)} > \frac{1}{t_n} - \frac{g(x_k)}{f(x_k)} > \frac{\varepsilon}{|f(x_k)|} ,$$

that is

$$\frac{\left| f(x_k) \right|}{f(x_{k+1})} g(x_{k+1}) - \frac{\left| f(x_k) \right|}{f(x_k)} g(x_k) > \varepsilon$$

for every $k = k(n)$. Now, if $n \to \infty$ then $k(n) \to \infty$ and the left hand side of the inequality above tends to 0. The obtained contradiction proves the Lemma. ■

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For a given function $f \in C(K)$ let $Z_f \overset{\text{def}}{=} \{x : f(x) = 0\}$. It is easy to see that if 0 is a porosity point of Range$(f)$ then either for $g \equiv 1$ or $g \equiv -1$, 0 can not be the limit value in $(*)$. In the case $|Z_f| = 1$ we prove the reverse implication, but in the general case the truth is a bit more complicated.

**Theorem 1.** If for a function $f$ we have $|Z_f| = 1$, then $\varphi$ is Gâteaux differentiable at $f$ if and only if 0 is not a porosity point of Range$(f)$.

**Proof.** We have seen that if 0 is a porosity point of Range$(f)$ then $\varphi$ is not differentiable. On the other hand, if 0 is not a porosity point of Range$(f)$ then, we can choose sequences $x_n$ and $x_n^*$ for which $f(x_n) \to f(x) = 0$, $f(x_n) > 0$, $f(x_n^*) \to 1$ and $f(x_n^*) \to f(x) = 0$, $f(x_n^*) < 0$, $f(x_n) \to 1$. Now, applying our Lemma, $\varphi$ is differentiable at $f$ in the direction $g$ whenever $g(x) > 0$ or $g(x) < 0$. Finally, for functions $g$ with $g(x) = 0$ we have $\varphi(f - tg) - \varphi(f) \equiv 0$, thus the differentiability is trivial.

Now we consider the case $|Z_f| = 2$, say $Z_f = \{x, y\}$. Let $U$ and $V$ be disjoint open neighbourhoods of $x$ and $y$. Since $K$ is the Cantor set, we can assume that these open neighbourhoods are closed. Let

$$P_f \overset{\text{def}}{=} (\text{Range}(f|_U) \times \mathbb{R}) \cup (\mathbb{R} \times \text{Range}(f|_V)) \subset \mathbb{R}^2.$$  

**Theorem 2.** If $|Z_f| = 2$ then $\varphi$ is Gâteaux differentiable at $f$ iff for every line $l$ on the plane different from the axes for which $0 \in l$ the point 0 is not a (linear) porosity point of $l \cap P_f$. That is, $\varphi$ is differentiable at $f$ if and only if for every non-zero constants $c_1, c_2$, the value 0 is not a porosity point of the set $c_1 \text{Range}(f|_U) \cup c_2 \text{Range}(f|_V)$.

**Proof.** First we prove that if 0 is not a porosity point of the sets $c_1 \text{Range}(f|_U) \cup c_2 \text{Range}(f|_V)$ then $(*)$ holds for every $g$.

This is clear if $g(x) = 0$ or $g(y) = 0$, because then $\varphi(f - tg) - \varphi(f) \equiv 0$. In the other case we choose $l$ to be the line of slope $\frac{g(x)}{g(y)}$, that is we choose $c_1$ and $c_2$ such that $c_2 : c_1 = g(x) : g(y)$. Then we choose a ‘thick’ sequence from $l \cap P_f$: we choose a sequence $\{d(k)f(x_k)\}_{k=1}^\infty$ where

$$d(k) = \begin{cases} c_1 & \text{if } x_k \in U \\ c_2 & \text{if } x_k \in V \end{cases},$$

such that $f(x_k) \to 0$,

$$\frac{d(k)f(x_k)}{d(k+1)f(x_{k+1})} \to 1,$$

and

$$\text{sgn } f(x_k) = \begin{cases} \text{sgn } g(x) & \text{if } x_k \in U \\ \text{sgn } g(y) & \text{if } x_k \in V \end{cases}.$$
This last assumption means that we choose our points from one of the two half lines of $l$.

Now, suppose indirectly that $(\ast)$ doesn’t hold. We know that $\frac{g(x_i)}{f(x_i)} \to +\infty$ (signs are OK). Then, similarly to the proof of the Lemma, there exists an $\varepsilon > 0$ and a sequence $t_n \setminus 0$ for which
\[ |f(x_k)| \frac{1}{t_n} - \frac{g(x_k)}{f(x_k)} > \varepsilon, \]
and for an $n$ large enough and suitable $k = k(n)$ we have
\[ \frac{|f(x_k)|}{f(x_{k+1})} g(x_{k+1}) - \frac{|f(x_k)|}{f(x_k)} g(x_k) > \varepsilon. \]

Thus
\[ |g(x_{k+1})| \left| \frac{f(x_k)}{f(x_{k+1})} - \frac{g(x_k)}{g(x_{k+1})} \right| > \varepsilon. \]

We choose a subsequence $n_m$ such that either all the points $x_{k(n_m)}$ are in $U$ or all of them are in $V$, and either all the points $x_{k(n_m)+1}$ are in $U$ or all of them are in $V$. Now, if $m \to \infty$ then $\frac{f(x_{k(n_m)})}{f(x_{k(n_m)+1})}$ and $\frac{g(x_{k(n_m)+1})}{g(x_{k(n_m)+1})}$ tend to the same number (to $\frac{c_i}{c_j}$ for some $i, j \in \{1, 2\}$), thus the limit of the left hand side of the inequality above is 0, which is a contradiction.

Now we suppose that 0 is a porosity point of the set $c_1 \text{Range}(f|_U) \cup c_2 \text{Range}(f|_V)$ for some $c_1, c_2$. Then there exist an $\varepsilon > 0$ and a sequence $t_n \setminus 0$ for which
\[ \inf \frac{|t_n - c_1 f|_U}{t_n} > \varepsilon \]
and
\[ \inf \frac{|t_n - c_2 f|_V}{t_n} > \varepsilon. \]

Let $g$ be a continuous function for which $g(z) = 1/c_1$ for every $z \in U$ and $g(z) = 1/c_2$ for every $z \in V$.

For $t = t_n$, we have
\[ \inf_U \left| \frac{f - tg}{t} \right| = \inf_U \left| \frac{f}{t} - \frac{1}{c_1} \right| = \frac{1}{|c_1|} \inf_U \left| \frac{c_1 f - t}{t} \right| > \frac{\varepsilon}{|c_1|}, \]
and similarly
\[ \inf_V \left| \frac{f - tg}{t} \right| > \frac{\varepsilon}{|c_2|}. \]

Finally
\[ \inf_{K \setminus (U \cup V)} \frac{|f - tg|}{t} \geq \inf_{K \setminus (U \cup V)} \frac{|f|}{t} - \max |g|, \]

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and this tends to $\infty$ if $t \to 0$. Hence $(\ast)$ doesn’t hold, thus $\varphi$ is not Gâteaux differentiable at $f$, as required.

It is easy to see that in the case $|Z_f| = N < \infty$ the result and its proof is similar. Now we consider the general case.

**Theorem 3.** The function $\varphi$ is not Gâteaux differentiable iff $Z_f$ can be covered by disjoint open sets $U_1, U_2, \ldots, U_m$ for which there exist non-zero constants $c_1, c_2, \ldots, c_m$ such that $0$ is a porosity point of the set

$$
\bigcup_{n=1}^{m} c_n \text{Range}(f|_{U_n}).
$$

**Proof.** Assume that for some $U_1, U_2, \ldots, U_m$ and $c_1, c_2, \ldots, c_m$, zero is a porosity point of the above union. We can assume that our disjoint open sets $U_1, U_2, \ldots, U_m$ are closed, and we choose a continuous function $g$ for which $g(z) = 1/c_i$ for every $z \in U_i$. By a way similar to that of the proof of Theorem 2 we have that $f$ is not a Gâteaux differentiability point of $\varphi$.

Now we assume that $(\ast)$ doesn’t hold. Then there exist a function $g$, an $\varepsilon > 0$ and a sequence $t_n \searrow 0$ for which

$$
\inf \frac{|f - t_n g|}{t_n} > \varepsilon,
$$

that is, $\varphi$ is not differentiable in the direction of $g$. For every $x \in Z_f$ we choose a small neighbourhood $U_x$. We can assume that $U_x$ is a set of form $K \cap \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]$. For every $\delta > 0$ we can choose $U_x$ so small that the oscillation of $g$ on $U_x$ is less than $\delta$. Moreover, assuming $Z_f \cap Z_g = \emptyset$ (in the other case $\varphi$ would trivially be differentiable), we choose $U_x$ satisfying $U_x \cap Z_g = \emptyset$.

Since $K$ is compact we can choose a finite covering $U_1, U_2, \ldots, U_m \subset \{U_x : x \in Z_f\}$, and we can also assume that the sets $U_i$ are pairwise disjoint. We fix a point $z_i \in Z_f \cap U_i$ for every $1 \leq i \leq m$, and we consider the line $c_m : c_{m-1} : \ldots : c_1 = g(z_1) : g(z_2) : \ldots : g(z_m)$ (we know that $g(z_i) \neq 0$).

Suppose indirectly that $0$ is not a porosity point of $\bigcup_{n=1}^{m} c_n \text{Range}(f|_{U_n})$, then we can choose a ‘thick’ sequence on the half line determined by $\text{sgn} g(z_i) = \text{sgn} f(x_i)$ for $x_i \in U_i$. Now we have

$$
\varepsilon < \left| \frac{f(x_{k(n)})}{f(x_{k(n)} + 1)} g(x_{k(n)} + 1) - g(x_{k(n)}) \right|.
$$

Choosing a subsequence $n_m$ for which the points $x_{k(n_m)}$ are in the same set $U_i$ and the points $x_{k(n_m) + 1}$ are in the same set $U_j$ the limes superior of the right hand side of the inequality above is at most

$$
\limsup \left| \frac{c_j}{c_i} (g(z_j) + g(x_{k(n)} + 1) - g(z_i)) - g(z_j) + g(z_i) - g(x_{k(n)}) \right|,
$$

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and \((c_j/c_i) g(z_j) - g(z_i) = 0\), thus we have the upper bound
\[
\limsup \left| \frac{c_j}{c_i} (g(x_{k(n)+1}) - g(z_j)) + g(z_i) - g(x_{k(n)}) \right| \leq \delta \left( \left| \frac{c_j}{c_i} \right| + 1 \right) \leq \left( \frac{\max_{z \in Z} |g|}{\min_{z \in Z} |g|} + 1 \right).
\]
For \(\delta\) small enough this is a contradiction.

References