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Differentiability Points of a Distance Function

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Let $K \subset [0, 1]$ be the usual Cantor set, and let $A \stackrel{\text{def}}{=} \{f \in C(K) : 0 \in \text{Range}(f)\}$. Its distance function $\varphi : C(K) \to \mathbf{R}$ is defined by $\varphi(f) \stackrel{\text{def}}{=} \text{dist}(f, A)$.

In this note we characterize the set of the points of the Gâteaux differentiability of this function φ. We prove that, φ is not Gâteaux differentiable at a function f iff $Z_f = \{x \in K : f(x) = 0\}$ can be covered by disjoint open sets $U_1, U_2, \ldots, U_m$ for which there exist non-zero constants $c_1, c_2, \ldots, c_m$ such that 0 is a porosity point of the set $\bigcup_{n=1}^m c_n \,\text{Range}(f|_{U_n})$.

During the attempts to answer the question whether the σ ideal of Aronszajn null sets and Gaussian null sets coincide in a separable Banach space E (see [1], [2]), it was important to study the following strange set:

Let $K \subset [0, 1]$ be the usual Cantor set, and let

$$A \stackrel{\text{def}}{=} \{f \in C(K) : 0 \in \text{Range}(f)\}. \tag{1}$$

It is clear that A is a closed subset of $C(K)$. It turned out that A contains a cube, that is, there is a system of functions of dense span $f_0, f_1, f_2, \ldots \in C(K)$ for which $\sum_{i=1}^\infty \|f_i\| < \infty$ and $f_0 + \sum_{i=1}^\infty r_i f_i \in A$ for every sequence $r_1, r_2, \ldots \in [0, 1]$. This surprising fact developed into the idea to look for 'a nearly cube' inside any non-Aronszajn null set A, more precisely, to find an appropriate cube $x_0 + \sum_{i=1}^\infty r_i x_i$ (where $r_i \in [0, 1]$, $x_1, x_2 \ldots$ is a sequence of the points of E of dense span, and $\sum_{i=1}^\infty \|x_i\| < \infty$) such that A is large in this cube, i.e. the Lebesgue measure of the set $\{(r_1, r_2, \ldots) \in [0, 1]^{\mathbf{N}} : x_0 + \sum_{i=1}^\infty r_i x_i \in A\}$ is large.

On the other hand, since the set A defined by (1) is not Aronszajn null, it must contain points of Gâteaux differentiability of any Lipschitz function, in particular of its distance function $\varphi : C(K) \to \mathbf{R}$ defined by

$$\varphi(f) \stackrel{\text{def}}{=} \text{dist}(f, A).$$

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In this note we characterize the set of the points of the differentiability of this function φ. This turned out to be interesting in itself, because of its connection to porosity properties.

Since φ is non-negative, if it is Gâteaux differentiable at a point of A, then its derivative must be 0. It is easy to see that

$$\varphi(f) = \inf |f| \, .$$

Indeed, $\varphi(f) \geq \inf |f|$ is trivial, and for the continuous real function

$$h_f(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad |x| < \inf |f| \\ 2(x - \inf |f|) & \text{if} \quad \inf |f| \leq |x| < 2 \inf |f| \\ x & \text{if} \quad 2 \inf |f| \leq |x| \end{cases}$$

we have $h_f \circ f \in A$ and $\|h_f \circ f - f\| = \inf |f|$.

Thus, φ is differentiable at $f \in A$ iff

$$\lim_{t \to 0+} \frac{\varphi(f - tg) - \varphi(f)}{t} = \lim_{t \to 0+} \frac{\inf |f - tg|}{t} = 0 \qquad (*)$$

holds for every $g \in C(K)$.

Lemma. If for a sequence x_n and a function $g \in C(K)$ we have $x_n \to x$, $f(x_n) \to f(x) = 0$, $\frac{f(x_n)}{f(x_{n+1})} \to 1$ and $\operatorname{sgn} g(x) = \operatorname{sgn} f(x_n) \neq 0$ for every n, then φ is differentiable at f in the direction of g, that is, $(*)$ holds for f and g.

Proof. Suppose indirectly that there exists a sequence $t_n \searrow 0$ and $\varepsilon > 0$ for which $\frac{|f - t_n g|}{t_n} > \varepsilon$. Now, for every k and n we have

$$\frac{|f(x_k) - t_n g(x_k)|}{t_n} = |f(x_k)| \left| \frac{1}{t_n} - \frac{g(x_k)}{f(x_k)} \right| > \varepsilon \, .$$

Since g is continuous, we have $\operatorname{sgn} g(x_k) = \operatorname{sgn} g(x) = \operatorname{sgn} f(x_k) \neq 0$ if k is large, thus by $g(x_k) \to g(x) \neq 0$ and $f(x_k) \to 0$ we have $\lim_{k \to \infty} \frac{g(x_k)}{f(x_k)} = +\infty$. If n is large enough then we can choose a $k = k(n)$ for which

$$\frac{g(x_k)}{f(x_k)} \leq \frac{1}{t_n} < \frac{g(x_{k+1})}{f(x_{k+1})},$$

and for this k have

$$\frac{g(x_{k+1})}{f(x_{k+1})} - \frac{g(x_k)}{f(x_k)} > \frac{1}{t_n} - \frac{g(x_k)}{f(x_k)} > \frac{\varepsilon}{|f(x_k)|},$$

that is

$$\frac{|f(x_k)|}{f(x_{k+1})} g(x_{k+1}) - \frac{|f(x_k)|}{f(x_k)} g(x_k) > \varepsilon$$

for every $k = k(n)$. Now, if $n \to \infty$ then $k(n) \to \infty$ and the left hand side of the inequality above tends to 0. The obtained contradiction proves the Lemma. ■

For a given function $f \in C(K)$ let $Z_f \stackrel{\text{def}}{=} \{x : f(x) = 0\}$.

It is easy to see that if 0 is a porosity point of Range(f) then either for $g \equiv 1$ or $g \equiv -1$, 0 can not be the limit value in $(*)$.

In the case $|Z_f| = 1$ we prove the reverse implication, but in the general case the truth is a bit more complicated.

Theorem 1. If for a function f we have $|Z_f| = 1$, then φ is Gâteaux differentiable at f if and only if 0 is not a porosity point of Range(f).

Proof. We have seen that if 0 is a porosity point of Range(f) then φ is not differentiable. On the other hand, if 0 is not a porosity point of Range(f) then, we can choose sequences x_n and x_n^* for which $f(x_n) \to f(x) = 0$, $f(x_n) > 0$, $\frac{f(x_n)}{f(x_{n+1})} \to 1$ and $f(x_n^*) \to f(x) = 0$, $f(x_n^*) < 0$, $\frac{f(x_n^*)}{f(x_{n+1}^*)} \to 1$. Now, applying our Lemma, φ is differentiable at f in the direction g whenever $g(x) > 0$ or $g(x) < 0$. Finally, for functions g with $g(x) = 0$ we have $\varphi(f - tg) - \varphi(f) \equiv 0$, thus the differentiability is trivial. ∎

Now we consider the case $|Z_f| = 2$, say $Z_f = \{x, y\}$. Let U and V be disjoint open neighbourhoods of x and y. Since K is the Cantor set, we can assume that these open neighbourhoods are closed. Let

$$P_f \stackrel{\text{def}}{=} (\text{Range}(f|_U) \times \mathbf{R}) \cup (\mathbf{R} \times \text{Range}(f|_V)) \subset \mathbf{R}^2 .$$

Theorem 2. If $|Z_f| = 2$ then φ is Gâteaux differentiable at f iff for every line l on the plane different from the axes for which $0 \in l$ the point 0 is not a (linear) porosity point of $l \cap P_f$. That is, φ is differentiable at f if and only if for every non-zero constants c_1, c_2, the value 0 is not a porosity point of the set $c_1 \text{Range}(f|_U) \cup c_2 \text{Range}(f|_V)$.

Proof. First we prove that if 0 is not a porosity point of the sets $c_1 \text{Range}(f|_U) \cup c_2 \text{Range}(f|_V)$ then $(*)$ holds for every g.

This is clear if $g(x) = 0$ or $g(y) = 0$, because then $\varphi(f - tg) - \varphi(f) \equiv 0$. In the other case we choose l to be the line of slope $\frac{g(x)}{g(y)}$, that is we choose c_1 and c_2 such that $c_2 : c_1 = g(x) : g(y)$. Then we choose a 'thick' sequence from $l \cap P_f$: we choose a sequence $\{d(k) f(x_k)\}_{k=1}^{\infty}$ where

$$d(k) = \begin{cases} c_1 & \text{if } x_k \in U \\ c_2 & \text{if } x_k \in V \end{cases},$$

such that $f(x_k) \to 0$,

$$\frac{d(k) f(x_k)}{d(k+1) f(x_{k+1})} \to 1 ,$$

and

$$\operatorname{sgn} f(x_k) = \begin{cases} \operatorname{sgn} g(x) & \text{if } x_k \in U \\ \operatorname{sgn} g(y) & \text{if } x_k \in V \end{cases}.$$

This last assumption means that we choose our points from one of the two half lines of l.

Now, suppose indirectly that $(*)$ doesn't hold. We know that $\frac{g(x_k)}{f(x_k)} \to +\infty$ (signs are OK). Then, similarly to the proof of the Lemma, there exists an $\varepsilon > 0$ and a sequence $t_n \searrow 0$ for which

$$|f(x_k)| \left| \frac{1}{t_n} - \frac{g(x_k)}{f(x_k)} \right| > \varepsilon ,$$

and for an n large enough and suitable $k = k(n)$ we have

$$\frac{|f(x_k)|}{f(x_{k+1})} g(x_{k+1}) - \frac{|f(x_k)|}{f(x_k)} g(x_k) > \varepsilon .$$

Thus

$$|g(x_{k+1})| \left| \frac{f(x_k)}{f(x_{k+1})} - \frac{g(x_k)}{g(x_{k+1})} \right| > \varepsilon .$$

We choose a subsequence n_m such that either all the points $x_{k(n_m)}$ are in U or all of them are in V, and either all the points $x_{k(n_m)+1}$ are in U or all of them are in V. Now, if $m \to \infty$ then $\frac{f(x_{k(n_m)})}{f(x_{k(n_m)+1})}$ and $\frac{g(x_{k(n_m)})}{g(x_{k(n_m)+1})}$ tend to the same number (to $\frac{c_i}{c_j}$ for some $i, j \in \{1, 2\}$), thus the limit of the left hand side of the inequality above is 0, which is a contradiction.

Now we suppose that 0 is a porosity point of the set $c_1 \operatorname{Range}(f|_U) \cup c_2 \operatorname{Range}(f|_V)$ for some c_1, c_2. Then there exist an $\varepsilon > 0$ and a sequence $t_n \searrow 0$ for which

$$\frac{\inf |t_n - c_1 f|_U|}{t_n} > \varepsilon$$

and

$$\frac{\inf |t_n - c_2 f|_V|}{t_n} > \varepsilon .$$

Let g be a continuous function for which $g(z) = 1/c_1$ for every $z \in U$ and $g(z) = 1/c_2$ for every $z \in V$.

For $t = t_n$ we have

$$\inf_U \frac{|f - tg|}{t} = \inf_U \left| \frac{f}{t} - \frac{1}{c_1} \right| = \frac{1}{|c_1|} \inf_U \left| \frac{|c_1 f - t|}{t} \right| > \frac{\varepsilon}{|c_1|} ,$$

and similarly

$$\inf_V \frac{|f - tg|}{t} > \frac{\varepsilon}{|c_2|} .$$

Finally

$$\inf_{K \setminus (U \cup V)} \frac{|f - tg|}{t} \geq \frac{\inf_{K \setminus (U \cup V)} |f|}{t} - \max |g| ,$$

and this tends to ∞ if $t \to 0$. Hence $(*)$ doesn't hold, thus φ is not Gâteaux differentiable at f, as required. ■

It is easy to see that in the case $|Z_f| = N < \infty$ the result and its proof is similar. Now we consider the general case.

Theorem 3. The function φ is not Gâteaux differentiable iff Z_f can be covered by disjoint open sets $U_1, U_2, \ldots, U_m$ for which there exist non-zero constants $c_1, c_2, \ldots, c_m$ such that 0 is a porosity point of the set

$$\bigcup_{n=1}^{m} c_n \operatorname{Range}(f|_{U_n}).$$

Proof. Assume that for some $U_1, U_2, \ldots, U_m$ and $c_1, c_2, \ldots, c_m$, zero is a porosity point of the above union. We can assume that our disjoint open sets $U_1, U_2, \ldots, U_m$ are closed, and we choose a continuous function g for which $g(z) = 1/c_i$ for every $z \in U_i$. By a way similar to that of the proof of Theorem 2 we have that f is not a Gâteaux differentiability point of φ.

Now we assume that $(*)$ doesn't hold. Then there exist a function g, an $\varepsilon > 0$ and a sequence $t_n \searrow 0$ for which

$$\inf \frac{|f - t_n g|}{t_n} > \varepsilon,$$

that is, φ is not differentiable in the direction of g. For every $x \in Z_f$ we choose a small neighbourhood U_x. We can assume that U_x is a set of form $K \cap \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]$. For every $\delta > 0$ we can choose U_x so small that the oscillation of g on U_x is less than δ. Moreover, assuming $Z_f \cap Z_g = \emptyset$ (in the other case φ would trivially be differentiable), we choose U_x satisfying $U_x \cap Z_g = \emptyset$.

Since K is compact we can choose a finite covering $U_1, U_2, \ldots, U_m \subset \{U_x : x \in Z_f\}$, and we can also assume that the sets U_i are pairwise disjoint. We fix a point $z_i \in Z_f \cap U_i$ for every $1 \le i \le m$, and we consider the line $c_m : c_{m-1} : \ldots : c_1 = g(z_1) : g(z_2) : \ldots : g(z_m)$ (we know that $g(z_i) \neq 0$).

Suppose indirectly that 0 is not a porosity point of $\bigcup_{n=1}^{m} c_n \operatorname{Range}(f|_{U_n})$, then we can choose a 'thick' sequence on the half line determined by $\operatorname{sgn} g(z_i) = \operatorname{sgn} f(x_i)$ for $x_i \in U_i$. Now we have

$$\varepsilon < \left| \frac{f(x_{k(n)})}{f(x_{k(n)+1})} g(x_{k(n)+1}) - g(x_{k(n)}) \right|.$$

Choosing a subsequence n_m for which the points $x_{k(n_m)}$ are in the same set U_i and the points $x_{k(n_m)+1}$ are in the same set U_j the limes superior of the right hand side of the inequality above is at most

$$\limsup \left| \frac{c_j}{c_i} (g(z_j) + g(x_{k(n)+1}) - g(z_j)) - g(z_i) + g(z_i) - g(x_{k(n)}) \right|,$$

and $(c_j/c_i)\, g(z_j) - g(z_i) = 0$, thus we have the upper bound

$$\limsup \left| \frac{c_j}{c_i} \left(g(x_{k(n)+1}) - g(z_j)\right) + g(z_i) - g(x_{k(n)}) \right| \leq \delta \left(\left| \frac{c_j}{c_i} \right| + 1 \right) \leq \left(\frac{\max_{Z_f} |g|}{\min_{Z_f} |g|} + 1 \right).$$

For δ small enough this is a contradiction. ■

References

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