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Compact Covering Mappings between Borel Spaces

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1. Introduction

The aim of this paper is to give a general presentation of various results about compact covering mapping between Borel spaces. The material is extracted from [2], [4] and [5]. However this is not a review paper in the strict sense. The style adopted here allows us to sketch the main ideas of the proofs, avoiding both technical and standard arguments.

The starting point of this study goes back to some purely topological problems, but as we showed these problems happen to be intimately related to set theoretical axioms. This is surely not new when dealing with topological problems in large (non separable or non metrizable) spaces. But this is the only example we know of such situation when dealing with a "natural" topological problems inside very nice spaces such as Borel subsets of real line or the Cantor space. The spirit of the proofs and the constructions we make mix intuition from topology and set descriptive theory, and we hope that this synthetic presentation will touch readers from both sides.

In all this paper $f : X \to Y$ denotes a continuous and onto mapping between the spaces X and Y. By space we always mean a separable and metrizable space.

Definitions. The mapping $f : X \to Y$ is said to be:
- **compact covering (CC)** if any compact subset of Y is the direct image of some compact subset of X.
- **inductively perfect (IP)** if there exists a (necessarily closed) subset X' of X such that $f(X') = Y$ and the restriction of f to X' is perfect.

We recall that a mapping is perfect if the inverse image of any compact set is compact. Suppose that f is inductively perfect and let K be any compact subset

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of Y then K is the image of $X' \cap f^{-1}(K)$ which is a compact subset of X', if the restriction of f to X' is perfect onto Y; thus obviously:

$$(\mathbf{IP}) \Rightarrow (\mathbf{CC})$$

The problem whether the converse of this implication holds was first raised, for some particular cases, by E. Michael. In fact:

$$(\mathbf{CC}) \Rightarrow (\mathbf{IP}) \quad \textit{if} \quad \begin{cases} X \textit{ is Polish} \\ \textit{or} \\ Y \textit{ is } \mathbf{K}_\sigma. \end{cases}$$

Under the hypothesis "X is Polish" this was proved independently by J. P. R. Christensen [1] and the second author [11]. Under the hypothesis "Y is $\mathbf{K}_\sigma$" this a much more recent result of A. V. Ostrovsky [10] (also proved by W. Just and H. Wicke [6] when Y is countable). It is clear that none of these two cases can be derived from the other; moreover the methods of their proofs are completely different.

On the other hand, one can construct counter-examples to $(\mathbf{CC}) \Rightarrow (\mathbf{IP})$ using the Axiom of Choice. But of course in such constructions the spaces X and Y have no definability properties.

In [2] we were able – assumig some set theoretical axioms – to construct such counterexamples where both X and Y are coanalytic spaces. In view of this and the previous positive results (when X is Polish or Y is $\mathbf{K}_\sigma$) the following question arises naturally, and was the central motivation of our work:

Does $(\mathbf{CC}) \Rightarrow (\mathbf{IP})$ *if the spaces* X *and* Y *are Borel ?*

2. General notations and terminology

Notation $\mathbb{A}(\mathcal{X}, \mathcal{Y})$.

To lighten the statement of our results, we fix the following two notations in which X and Y denote arbitrary spaces, and $\mathcal{X}$ and $\mathcal{Y}$ denote arbitrary classes of spaces.

$\mathbb{A}(X, Y)$: *"Any compact covering mapping $f : X \to Y$ is inductively perfect"*

$\mathbb{A}(\mathcal{X}, \mathcal{Y})$: $\mathbb{A}(X, Y)\ \forall X \in \mathcal{X}, \forall Y \in \mathcal{Y}$

Classical descriptive classes.

By a *descriptive class*, we mean a class of *subsets of Polish spaces* which is closed by taking inverse images by continuous mappings between Polish spaces. The only classes that we shall consider in this work are the following to which we refer as the *classical descriptive classes.*

$\mathbf{\Delta}^1_1$: *the class of all Borel spaces.*

$\mathbf{\Sigma}^1_1$: *the class of all analytic spaces.*

$\mathbf{\Pi}_1^1$: *the class of all coanalytic spaces.*
$\mathbf{\Sigma}_\xi^0$: *the Baire additive class of order* ξ.
$\mathbf{\Pi}_\xi^0$: *the Baire multiplicative class of order* ξ.

where ξ is any countable ordinal ≥ 3.

Let $\mathscr{X}$ be any of the previous classes. When we say that a space "*X is in $\mathscr{X}$*", we mean that X can be (homeomorphically embedded) in some Polish space P as a $\mathscr{X}$ subset of P. It is well known that this notion is absolute, in the sense that it does not depend on the particular choice of the Polish space P nor on the embedding.

For $\xi = 2$ we shall use the classical notations $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$.

We recall that the previous observation about absoluteness is also valid for the class $\mathbf{G}_\delta$, and it is a classical fact that this class is just the class of all Polish spaces.

However this does not apply to the "class $\mathbf{F}_\sigma$" that we shall not consider in fact as a class of spaces, but only as a class of subsets of some fixed space. A classical way to avoid this problem is to work inside compact (rather than Polish) spaces, in which case the notion of "$\mathbf{F}_\sigma$" would then be absolute and coincide with "σ-compact". But we shall not adopt this solution here, and instead we shall consider a more natural class that we introduce now.

Lemma 2.1. *For a space X, the following are equivalent:*
(i) X is the union of a countable family of closed Polish subspaces.
(ii) X is an $\mathbf{F}_\sigma$ subset of some Polish space.
(iii) X is the difference of two $\mathbf{F}_\sigma$ subsets of some Polish space.
(iv) In any space in which X embeds, X is the difference of two $\mathbf{F}_\sigma$ subsets.

Proof. It is clear that $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$. To prove $(i) \Rightarrow (iv)$ consider some metrizable space E in which X embeds, and let (F_n) be a countable covering of X by relatively closed subsets, such that each F_n is a Polish space. If we denote by $\overline{F_n}$ the closure of F_n in E, we have $F_n = X \cap \overline{F_n}$. Moreover $A = \bigcup_n \overline{F_n}$ is $\mathbf{\Sigma}_2^0$ in E, and since F_n is Polish, $E_n = \overline{F_n} \setminus X = \overline{F_n} \setminus F_n$ is $\mathbf{\Sigma}_2^0$ in $\overline{F_n}$ hence in E. If we put $B = \bigcup_n E_n$, B is $\mathbf{\Sigma}_2^0$ in E, and we have $X = A \setminus B$. Hence X is the difference of two $\mathbf{F}_\sigma$ sets in E. $\Diamond$

The class $\mathbf{P}_\sigma$. We denote by $\mathbf{P}_\sigma$ the class of all spaces X satisfying the conditions of Lemma 2.1, and by $\check{\mathbf{P}}_\sigma$ its dual class inside compact spaces, that is

$$\mathbf{P}_\sigma = \{X = X_0 \cap X_1;\ \text{with } X_0 \mathbf{G}_\delta \text{ and } X_1 \mathbf{K}_\sigma \text{ in some (any) compactification of } X\}$$
$$\check{\mathbf{P}}_\sigma = \{X = X_0 \cup X_1;\ \text{with } X_0 \mathbf{G} \text{ and } X_1 \mathbf{K}_\sigma \text{ in some (any) compactification of } X\}$$

Projection mappings.

When we speak about a **projection mapping** π from X onto Y, we mean that X is a subset of some product space $Y \times Z$, π is the restriction to X of the canonical projection, and $Y = \pi(X)$. If moreover the factor spaces Y and Z are zero-dimensional we shall say that π is a **zero-dimensional projection mapping.**

In fact it will happen frequently that $Y = Z$; however the notation π will always refer to the *projection on the first factor.* Also when several projection mappings are considered with the same factor spaces Y and Z, to avoid ambiguity we shall write π_X for π.

The following simple result reduces the general study of compact covering and inductively perfect mappings to the particular case of zero-dimensional projection mapping.

Lemma 2.2. *Given any continuous and onto mapping $f: X \to Y$, there exists two perfect mappings p and q, and a projection mapping π from a subset X' of $2^\omega \times 2^\omega$ onto $Y' = \pi(X')$ such that the following diagram commutes:*

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and moreover:

a) f is compact covering if and only if $\pi_{X'}$ is compact covering.

b) f is inductively perfect if and only if $\pi_{X'}$ is inductively perfect.

3. When does $\mathbb{A}(\mathscr{X}, \mathscr{Y})$ hold?

In this section we derive $\mathbb{A}(\mathbf{\Pi}_1^1, \mathscr{Y})$ for different classes $\mathscr{Y}$ from different set theoretical axioms. In next section we prove the converse implications.

We denote as usual by Det $(\mathbf{\Sigma}_1^1)$ the following statement:

"Any analytic game on ω is determined"

We recall that one classical (and very weak) consequence of Det $(\mathbf{\Sigma}_1^1)$ is that the set $L \cap \omega^\omega$ of all *constructible reals* is countable. This also holds for all sets $L(\alpha) \cap \omega^\omega$ of all *constructible reals in the parameter* $\alpha \in \omega^\omega$. For more details see for example 9.

Finally a subset A of ω^ω will be said to be $\bigstar$-bounded if there exists $a \in \omega^\omega$ such that

$$\forall x \in A,\ \exists n \in \omega, \text{ such that } \forall m \geq n,\ x(m) \leq a(m)$$

Any countable subset of ω^ω is $\bigstar$-bounded.

Theorem 3.1.

a) If $Det(\mathbf{\Sigma}_1^1)$ then $\mathbb{A}(\mathbf{\Pi}_1^1, \mathbf{\Pi}_1^1)$.

b) If "$\forall \alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ is $\bigstar$-bounded in ω^ω" then $\mathbb{A}(\mathbf{\Pi}_1^1, \mathbf{G}_\delta)$.

c) If "$\forall \alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ is countable" then $\mathbb{A}(\mathbf{\Pi}_1^1, \mathbf{F}_{\sigma\delta})$.

Sketch of proof.

Let π be a projection mapping from $X \subset 2^\omega \times 2^\omega$ onto Y. We suppose that both X and Y are $\mathbf{\Pi}^1_1$ (and Y possibly $\mathbf{G}_\delta$ or $\mathbf{F}_{\sigma\delta}$ when dealing with b) or c)).

Step 1: Introduction of a game.

For each of the three cases a), b), c) one has to introduce a different game. We shall describe the game completely in case a) which is the simplest case, and give some hints for the games involved in b) and c) which are much more complicated that in a).

Case a): The players are asked to choose at each move 0 or 1, producing thus two elements y and z in $\mathbf{2}^\omega$ with y constructed by Player I and z constructed by Player II wins the run if

$$(y \notin Y) \quad \text{or} \quad (H_z \subset X \ \text{ and } \ y \in \pi(H_z))$$

where $z \mapsto H_z$ is some canonical homeomorphism from $\mathbf{2}^\omega$ onto $\mathscr{K}(\mathbf{2}^\omega \times \mathbf{2}^\omega)$ the hyperspace of all nonempty compact subsets of $\mathbf{2}^\omega \times \mathbf{2}^\omega$.

Notice that since X and Y are $\mathbf{\Pi}^1_1$ then the win condition of the game is the difference of two coanalytic sets. But it follows from well known and deep results of Martin and Harrington that such games are also determined under $Det(\mathbf{\Sigma}^1_1)$.

Cases b) and c): First notice that since X is $\mathbf{\Pi}^1_1$ then the set $Z = \{z \in \mathbf{2}^\omega : H_z \subset X\}$ is also $\mathbf{\Pi}^1_1$, hence we can fix a tree T on $\mathbf{2}^\omega \times \omega_1$ such that the projection of $[T]$ (the set of all infinite branches of T) on $\mathbf{2}^\omega$ is exactly Z. In these cases Player I and II construct in each run as in case a), elements y and z in $\mathbf{2}^\omega$; moreover Player II has also to construct some (finite or infinite) sequence α of ordinals in ω_1. One of the main point is that the construction of α is not done coordinate by coordinate but by some limit procedure. The precise rules of "constructing" α are quite technical and different in each of the two cases b) and c). However in both cases the win condition is of the same flavor than in a). Player II wins the run if

$$(y \notin Y) \quad \text{or} \quad ((z, \alpha) \in [T] \ \text{ and } \ y \in \pi(H_z))$$

Notice here that by the choice of T the condition "$(z, \alpha) \in [T]$" implies the condition "$H_z \subset X$" of case a).

The computation of the complexity of the win condition of these games (which involves the rules of "constructing" α) shows that assuming that Y is Borel (what we can do), these games are Borel game on ω_1 hence determined (without any extra assumption to ZF).

Step 2: When II wins the game.

It is clear that in each of these games if Player II has a winning strategy, then this strategy defines canonically a continuous mapping $\phi : Y \to \mathscr{K}(X)$ with $y \in \phi(y)$ for all y. Then it is easy to check that the restriction of π to the set $\bigcup\{\phi(y); y \in Y\}$ is perfect onto Y.

Step 3: When I wins the game.

We now arrive to the heart of the proof of the Theorem. In fact to finish this proof we need to show that (under the various hypothesis a), b), c)) if Player *I* has a winning strategy then there exists some compact subset K of Y which is not the projection of any compact subset of X. The construction of K depends heavily on the rules of the game and is different in each of the three cases.

But again the case a) is very simple. In this case one can take for K the set of all answers y given by a winning strategy of Player *I*, in all possible runs. Because in this case Player *II* has only finitely many choices at each move (in fact two choices: 0 or 1) one easily sees that the space of all possible runs is compact; hence K is compact. On the other hand it is not difficult to check that K is not the projection of a compact subset H of X (otherwise consider the run where *II* plays z such that $H = H_z$).

Notice that because of the parameter α arising in cases b) and c), Player *II* is no more "playing in a compact space" one cannot apply the simple argument above. Also in these cases the compact set K has to be defined in a much more elaborated way. We shall only give here a very vague idea of the construction: The set K is defined as the closure of some subset K_0 of Y, where each element of K_0 is the answer given by the strategy in some particular runs, in fact runs in which Player *II* constructs the parameter α in a very simple way involving only finitely many ordinals. Then assuming the hypothesis in a) and b) one can show by general absoluteness arguments that such a set K_0 is countable. This enables one to "order" in some sense the elements of K_0 and by adequate choices to control all the accumulation points of K_0 so that they all lie in Y. ◇

Remark. It follows from the previous discussion that the main point in the argument in the cases b) and c) of Theorem 3.1 is to find some explicit conditions which insure that an accumulation point of K_0 lies in a Y. This can be managed reasonably when the set Y is $\mathbf{G}_\delta$, but is much more delicate when Y is $\mathbf{F}_{\sigma\delta}$. Also when Y is a Borel set of higher class this procedure becomes very quickly extremely difficult to control. Nevertheless we conjecture the following:

Conjecture 3.2. If "$\forall\alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ is countable", then $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1)$.

We finish this section by the following extension of b):

Theorem 3.3. *If "$\forall\alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ is ★-bounded" then* $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{P}_\sigma)$.

Sketch of Proof.

Let $\pi_X : X \to Y$ $(X \subset Y \times Z)$ be a compact covering zero-dimensional projection where X is a $\mathbf{\Pi}^1_1$ space, and $Y = \pi(X)$ a $\mathbf{P}_\sigma$ space. We embed the spaces Y and Z in ω^ω. Then there exists a $\mathbf{\Pi}^0_2$ subset $\tilde{Y}$ of ω^ω containing Y as a $\mathbf{\Sigma}^0_2$ subset. Consider now the projection mapping $\pi_{\tilde{X}}$ asociated with the set

$$\tilde{X} = X \cup \left((\tilde{Y} \setminus Y) \times \omega^\omega\right) \subset \tilde{Y} \times \omega^\omega$$

then clearly $\tilde{X}$ is $\mathbf{\Pi}_1^1$ and $\pi(\tilde{X}) = \tilde{Y}$ is Polish. Moreover since $X = \tilde{X} \cap (Y \times Z)$, then to prove that π_X is inductively perfect it is enough to prove that $\pi_{\tilde{X}}$ is inductively perfect. But since $\tilde{Y}$ is $\mathbf{G}_\delta$ it is enough by Theorem 3.1b) to prove that $\pi_{\tilde{X}}$ is compact covering.

Now notice that if K is any compact subset of $\tilde{Y}$. Then $Y_0 = K \cap Y$ is σ-compact and the projection π_{X_0} from $X_0 = X \cap (Y_0 \times \omega^\omega)$ onto Y_0 is still compact covering. Then applying Ostrovsky's Theorem to each π_{X_0} one can show that $\pi_{\tilde{X}}$ is compact covering. $\diamond$

4. When $\mathbb{A}(\mathscr{X}, \mathscr{Y})$ does not hold?

In this section we give consequences of $\mathbb{A}(\mathscr{X}, \mathscr{Y})$ for different classes $\mathscr{X}$ and $\mathscr{Y}$. We start by a general result reducing the case $\mathscr{X} = \mathbf{\Pi}_1^1$ to the case $\mathscr{X} = \mathbf{\Delta}_1^1$. Notice that this result does not give any simplification for the proof of Theorem 3.1.

We recall that all the classes we consider contain the class $\mathbf{G}_\delta$.

Theorem 4.1. *For any classical descriptive class $\mathscr{Y} \subset \mathbf{\Delta}_1^1$*

$$\mathbb{A}(\mathbf{\Pi}_1^1, \mathscr{Y}) \Leftrightarrow \mathbb{A}(\mathbf{\Delta}_1^1, \mathscr{Y}) \Leftrightarrow \mathbb{A}(\mathbf{F}_\sigma(\mathscr{Y}), \mathscr{Y})$$

where $\mathbf{F}_\sigma(\mathscr{Y})$ denotes the class of all spaces which can be embedded as an $\mathbf{F}_\sigma$ subset of some space in $\mathscr{Y}$.

Sketch of proof.

Given any mapping $f : X \to Y$ where X is $\mathbf{\Pi}_1^1$ and Y is in $\mathscr{Y}$ we define a mapping spaces $\tilde{X}$ and $\tilde{Y}$, and a mapping $\tilde{f} : \tilde{X} \to \tilde{Y}$ with the following properties:

- $\tilde{X} = X_0 \cup (X_1 \setminus X_2)$ with X_0 a discrete countable subset of X, X_1 in $\mathscr{Y}$ and X_2 $\mathbf{G}_\delta$.
- $\tilde{Y} = Y \cup Y_0$ with Y_0 countable and Y closed in $\tilde{Y}$.
- If f compact covering then $\tilde{f}$ is compact covering.
- If $\tilde{f}$ is inductively perfect then f is inductively perfect. $\diamond$

We now go back to the particular classes $\mathscr{Y}$ considered in Theorem 3.1.

Theorem 4.2. *If $\mathbb{A}(\mathbf{\Pi}_1^1, \mathbf{G}_\delta)$ then "$\forall \alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ is $\bigstar$-bounded".*

Sketch of Proof.

The proof makes substantial use of notions and results from Effective Descriptive Set Theory and also basic properties of the universes $L(\alpha)$. More precisely we use the fact that for any $\alpha \in \omega^\omega$ of there exists a largest $\Pi_1^1(\alpha)$ thin set, that is a set containing no perfect subset; and moreover that the projection of largest $\Pi_1^1(\alpha)$ thin set of the plane $\omega^\omega \times \omega^\omega$ is exactly $\omega^\omega \cap L(\alpha)$.

Fix $\alpha \in \omega^\omega$ and assume that $\omega^\omega \cap L(\alpha)$ is not $\bigstar$-bounded in ω^ω. First, using the largest $\Pi^1_1(\alpha)$ thin set of the plane $\omega^\omega \times \omega^\omega$, we construct a $\Pi^1_1(\alpha)$ thin set $Z \subset \omega^\omega$ satisfying

$$\forall y \in \omega^\omega\ \exists z \in Z \quad \text{such that} \quad \forall n\ \exists m > n,\ y(m) < z(m)$$

Let $Y = \omega^\omega$ and consider the set

$$X = \{(y,z) \in Y \times Z : \forall n\ \exists m > n,\ y(m) < z(m)\}$$

Thus the projection mapping π from X is onto Y; but since any compact subset of ω^ω is bounded in ω^ω (for the pointwise preordering), it follows that π is compact covering. Now using the thinness of X one can show that π is not inductively perfect. $\diamond$

Theorem 4.3. *If* $\mathbb{A}(\mathbf{\Pi}^1_1, \check{\mathbf{P}}_\sigma)$ *then "*$\forall \alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ *is countable".*

Sketch of Proof.
This proof also uses in a fundamental way the existence of thin Π^1_1 sets. The general scheme is the following. Given any $\alpha \in \omega^\omega$ and any (Borel) $\Delta^1_1(\alpha)$ set Y, we construct a projection mapping π from some $\Pi^1_1(\alpha)$ (thin) set onto Y such that:
– If $\omega^\omega \cap L(\alpha)$ is not countable, then π is compact covering.
– If π is inductively perfect then Y is necessarily $\mathbf{P}_\sigma$.
However the construction is more elaborated than in Theorem 4.2. The main difference between the two situations is that in the previous case (where we fixed $Y = \omega^\omega$) the space $\mathcal{K}(Y)$ was $\mathbf{G}_\delta$ and very simply coded by the space ω^ω; whereas in the present context this is not possible since the space $\mathcal{K}(Y)$ can be a true $\mathbf{\Pi}^1_1$. $\diamond$

Theorem 4.4. *If* $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$ *then* $Det(\mathbf{\Sigma}^1_1)$.

Sketch of Proof.
Let $\mathcal{K}(\mathbb{Q})$ be the space of all compact subsets of the space $\mathbb{Q}$ of all rational numbers. We derive from $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$ that the space $\mathcal{K}(\mathbb{Q})$ can be reduced to any $\mathbf{\Pi}^1_1$ non Borel set X. It follows that any such X is $\mathbf{\Pi}^1_1$ complete, and this by a deep result of Harrington implies $Det(\mathbf{\Sigma}^1_1)$. $\diamond$

Corollary 4.5.
a) $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1) \Leftrightarrow Det(\mathbf{\Sigma}^1_1)$
b) $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{F}_{\sigma\delta}) \Leftrightarrow \mathbb{A}(\mathbf{\Delta}^1_1, \check{\mathbf{P}}_\sigma) \Leftrightarrow$ "$\forall \alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ *is countable*"
c) $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{F}_\delta) \Leftrightarrow \mathbb{A}(\mathbf{\Delta}^1_1, \mathbf{P}_\sigma) \Leftrightarrow$ "$\forall \alpha \in \omega^\omega$, $\omega^\omega \cap L(\alpha)$ *is* $\bigstar$*-bounded*"

Notice that by Theorem 4.1 we also have

$$\mathbb{A}(\mathbf{\Delta}^1_1, \mathbf{P}_\sigma) \Leftrightarrow \mathbb{A}(\mathbf{P}_\sigma, \mathbf{P}_\sigma) \quad \text{and} \quad \mathbb{A}(\mathbf{\Delta}^1_1, \check{\mathbf{P}}_\sigma) \Leftrightarrow \mathbb{A}(\mathbf{P}_\sigma \vee \mathbf{P}_\sigma, \check{\mathbf{P}}_\sigma)$$

where $\mathbf{P}_\sigma \vee \mathbf{P}_\sigma$ denotes the class of all spaces which are union of two $\mathbf{P}_\sigma$ sets; but it will follow from Theorem 6.2 b) below that one cannot replace this class by $\mathbf{P}_\sigma$.

In this section all spaces are supposed to be zero-dimensional.

Definition 5.1. *Let $\mathcal{K}(Y)$ denote the space of all compact subsets of a space Y. A subset $\mathcal{A}$ of $\mathcal{K}(Y)$ is said to be a* **cofinal** *if*

$$\forall S \in \mathcal{K}(Y),\ \exists T \in \mathcal{A}, \quad \text{such that} \quad S \subset T$$

Theorem 5.2. *If $\mathcal{Y}$ is any of the classes $\mathbf{G}_\delta$, $\mathbf{F}_{\sigma\delta}$, $\mathbf{\Pi}^1_1$, then the following are equivalent:*

(i) $\mathbb{A}(\mathbf{\Pi}^1_1, \mathcal{Y})$

(ii) For any $\mathbf{\Pi}^1_1$ cofinal subset $\mathcal{A}$ of $\mathcal{K}(Y)$, there exists a continuous mapping $f : Y \to \mathcal{A}$ such that $y \in f(y)$ for all $y \in Y$.

Moreover if $\mathcal{Y}$ is one of the classes $\mathbf{G}_\delta$ or $\mathbf{\Pi}^1_1$ then (i) also equivalent to

(ii)' For any $\mathbf{\Pi}^1_1$ cofinal subset $\mathcal{A}$ of $\mathcal{K}(Y)$, there exists a continuous mapping $F : \mathcal{K}(Y) \to \mathcal{A}$ such that $S \subset F(S)$ for all $S \in \mathcal{K}(Y)$.

Sketch of proof.

We first prove *(i) ⇒ (ii)*. Let $\mathcal{A}$ be a $\mathbf{\Pi}^1_1$ cofinal subset of $\mathcal{K}(Y)$, then the set

$$X = \{(y, T) \in Y \times \mathcal{A} : y \in T\}$$

is also $\mathbf{\Pi}^1_1$. Let π be the projection mapping from X onto Y. If S is any compact subset of Y and $T \in \mathcal{A}$ is such that $S \subset T$ then it is clear that $S \times \{T\} \subset X$. Thus π is projection mapping from $X \subset Y \times U$ (with $Z = \mathcal{K}(Y)$) onto Y with the following strong form of compact covering property:

$$\forall K \text{ compact } \subset Y, \exists z \in Z \text{ such that } K \times \{z\} \subset X$$

We shall then say that π is *strongly compact covering projection.* On the other hand *(ii)* states exactly that π admits a *continuous section* that is a continuous mapping $f : Y \to Z$ with graph in X. So consider the following statement

$\mathbb{A}^*(\mathcal{Y})$: *"Any strongly compact covering projection from any $\mathbf{\Pi}^1_1$ space X onto a space $Y \in \mathcal{Y}$, admits a continuous section"*

We recall that in this section by "space" we mean a "*zero-dimensional space*". It follows then from the previous discussion that the implication *(i) ⇒ (ii)* follows from the implication

$$\mathbb{A}(\mathbf{\Pi}^1_1, Y) \Rightarrow \mathbb{A}^*(\mathcal{Y})$$

that we will prove now. For simplicity write (respectively):

$\mathbb{A}^*$ for $\mathbb{A}^*(\mathbf{\Pi}^1_1)$; $\mathbb{A}^*(\mathbf{F}_{\sigma\delta})$; $\mathbb{A}^*(\mathbf{G}_\delta)$

$\mathbb{A}$ for $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$; $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{F}_{\sigma\delta})$; $\mathbb{A}(\mathbf{\Pi}^1_1, \mathbf{G}_\delta)$

$\mathbb{B}$ for $Det(\mathbf{\Sigma}^1_1)$; "$\forall\alpha$, $\omega^\omega \cap L(\alpha)$ is countable"; "$\forall\alpha$, $\omega^\omega \cap L(\alpha)$ is ★-bounded in ω^ω"

Thus by Theorems 4.4, 4.3, 4.2, we have $\mathbb{A} \Rightarrow \mathbb{B}$. On the other hand a simple inspection of Theorem 3.1 shows that $\mathbb{B} \Rightarrow \mathbb{A}^*$; hence $\mathbb{A} \Rightarrow \mathbb{A}^*$.

This finishes the proof of *(i)* $\Rightarrow$ *(ii)*. The converse implication needs more work. The main idea that we shall not develop is the following: given any compact covering mapping from X onto Y, one can "embed" (in some sense) the space $\mathscr{K}(X)$ in $\mathscr{K}(Y)$ as a cofinal subset.

Finally *(ii)'* $\Rightarrow$ *(ii)* is obvious; and it remains to prove *(i)* $\Rightarrow$ *(ii)'* when $\mathscr{Y} = \mathbf{G}_\delta$, or $\mathbf{\Pi}^1_1$. For this consider the set

$$\tilde{X} = \{(S,T) \in \mathscr{K}(Y) \times \mathscr{A} : S \subset T\}$$

which is also $\mathbf{\Pi}^1_1$ and let $\tilde{\pi}$ be the projection mapping from $\tilde{X}$ onto $\tilde{Y} = \mathscr{K}(Y)$. Since $\mathscr{Y} = \mathbf{G}_\delta$, or $\mathbf{\Pi}^1_1$, then $\tilde{Y} \in \mathscr{Y}$ (and this not true for another class $\mathscr{Y} \subset \mathbf{\Pi}^1_1$). It is easy to see that $\tilde{\pi}$ is strongly compact covering: If $\mathscr{C}$ is any compact subset of $\mathscr{K}(Y)$ then $S = \bigcup \mathscr{C}$ is a compact subset of Y, and if $T \in \mathscr{A}$ is such that $S \subset T$ then as above $S \times \{T\} \subset \tilde{X}$. The rest of the argument is as in *(i)* $\Rightarrow$ *(ii)* above. ◇

Remark. In the previous result when the class $\mathscr{Y}$ is not one of the classes $\mathbf{G}_\delta$ or $\mathbf{\Pi}^1_1$, the equivalence of (i) and (iii) is false. The most striking situation is the case where $\mathscr{Y} = \mathbb{Q}$ the space of all rational numbers. In fact in this case it follows from Just-Wicke and Ostrovsky Theorem (mentioned in the introduction) that (i) holds in ZF, with the following strong form (no descriptive restriction on the cofinal set):

For any cofinal subset $\mathscr{A}$ of $\mathscr{K}(\mathbb{Q})$, there exists a continuous mapping $f : \mathbb{Q} \to \mathscr{A}$ such that $y \in f(y)$ for all $y \in \mathbb{Q}$.

But the proof (that we did not develop) of Theorem 4.4 shows that the following statement:

For any $\mathbf{\Pi}^1_1$ cofinal subset $\mathscr{A}$ of $\mathscr{K}(\mathbb{Q})$, there exists a continuous mapping $F : \mathscr{K}(\mathbb{Q}) \to \mathscr{A}$ such that $S \subset F(S)$ for all $S \in \mathscr{K}(\mathbb{Q})$.

is equivalent to $Det(\mathbf{\Sigma}^1_1)$.

6. Compact covering images of Borel sets

We discuss in this last section another problem concerning compact covering mappings on Borel spaces. However unlike the other problems considered previously we do not know whether this one has a (positive or negative) solution in ZFC.

We start by some results from [2] that we state without proof.

Theorem 6.1. *Suppose that $f : X \to Y$ is inductively perfect, and let $\xi \geq 2$.*
(a) If X is $\mathbf{\Pi}^1_1$ then Y is $\mathbf{\Pi}^1_1$
(b) If X is $\mathbf{\Pi}^0_\xi$ then Y is $\mathbf{\Pi}^0_\xi$
(c) If X is $\mathbf{\Sigma}^0_\xi$ then Y is $\mathbf{\Sigma}^0_\xi$

Theorem 6.2. *Suppose that $f : X \to Y$ is compact covering, and let $\xi \geq 3$.*
a) If X is $\mathbf{G}_\delta$ then Y is $\mathbf{G}_\delta$
b) If X is $\mathbf{P}_\sigma$ then Y is $\mathbf{P}_\sigma$
c) If X is $\mathbf{\Pi}^0_\xi$ and Y is Borel then Y is $\mathbf{\Pi}^0_\xi$
d) If X is $\mathbf{\Sigma}^0_\xi$ and Y is Borel then Y is $\mathbf{\Sigma}^0_\xi$

Open problem 6.3. *Is the image of a Borel set by a compact covering mapping also Borel?*

Remarks. Notice that by Theorem 6.1 the answer is positive if we assume $Det(\mathbf{\Sigma}^1_1)$. In fact we proved in [2] that the the answer is also positive under the weaker assumption:

$$\text{“}\forall \alpha \in \omega^\omega,\ \omega^\omega \cap L(\alpha) \text{ is countable”}$$

The results above suggest to consider the following statement, in which no restriction is a priori imposed on the range space of the mapping:

$\mathbb{A}(\mathscr{X})$: *Any compact covering mapping defined on a space $X \in \mathscr{X}$ is inductively perfect.*

It follows then from the previous study that:
a) $\mathbb{A}(\mathbf{G}_\delta)$ *holds in ZFC.*
b) $\mathbb{A}(\mathbf{P}_\sigma) \Leftrightarrow$ "$\forall \alpha \in \omega^\omega, \omega^\omega \cap L(\alpha)$ *is* $\bigstar$*-bounded*"
c) $\mathbb{A}(\mathbf{\Delta}^1_1) \Rightarrow$ "$\forall \alpha \in \omega^\omega, \omega^\omega \cap L(\alpha)$ *is countable*"

Notice that Theorem 3.1 gives particular cases for which the converse implication of c) above holds. In fact it is easy to see that Conjecture 3.2 can be restated in the following equivalent form:

Conjecture 6.4. $\mathbb{A}(\mathbf{\Delta}^1_1) \Leftrightarrow$ "$\forall \alpha \in \omega^\omega, \omega^\omega \cap L(\alpha)$ *is countable*"

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