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Wolfgang Lusky

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On the Isomorphic Classification of Weighted Spaces of Holomorphic Functions

W. LUSKY

Paderborn

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We show that there are only two isomorphism classes for weighted spaces of holomorphic functions on the unit disk with moderately decreasing weights. In particular a space of holomorphic functions with a weighted sup-norm here is either isomorphic to l_∞ or to H_∞ depending on special properties of the weight which can be easily checked.

1 Introduction

We deal with Banach spaces of holomorphic functions on

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

For $0 < r$ and $1 \leq p < \infty$ put

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta\right)^{1/p}$$

and $M_\infty(f, r) = \sup_{|z|=r} |f(z)|$.

We study holomorphic functions f on D where $M_p(f, r)$ grows in a controlled way as $r \to 1$ according to a given weight measure μ. So, let μ be a positive bounded Borel measure on $[0, 1]$ and put, for $1 \leq p \leq \infty$,

$$|f\|_{p,q} = \left(\int_0^1 M_p^q(f, r) \, d\mu(r)\right)^{1/q} \text{ if } 1 \leq q < \infty$$

and $\|f\|_{p,\infty} = \sup_{0 \leq r < 1}(M_p(f, r) \, \mu([r, 1]))$. Define

$$B_{p,q}(\mu) = \{f : D \to \mathbb{C} : f \text{ holomorphic}, \ \|f\|_{p,q} < \infty\}$$

Fachbereich 17, Universität Gesamthochschule, Warburger Strasse 100, D-33098 Paderborn, Germany.
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and

$$B_{p,0}(\mu) = \{f \in B_{p,\infty}(\mu) : \lim_{r\to 1} M_p(f, r)\,\mu([r, 1]) = 0\}.$$

The assumption of boundedness of μ ensures that these spaces contain all polynomials. The $B_{p,q}(\mu)$ are Banach spaces under the given norms $\|\cdot\|_{p,q}$ (see [13]). We want to assume that μ satisfies

$$\lim_{r\to 1} \mu([r, 1]) = 0. \tag{1.1}$$

(If $\mu(\{1\}) > 0$ then we would obtain, for example, that $B_{p,p}(\mu)$ is isomorphic to H_p.) Moreover we want to assume that

$$0 < \mu([r, 1]) \quad \text{for each} \quad r < 1. \tag{1.2}$$

((1.2) is not really a restriction. If supp $\mu \subset [0, a]$ for some $a < 1$ then we could replace, $[0, 1]$ by $[0, a]$ and use substitution to reduce everything to the case $a = 1$.)

So from now on we assume (1.1) and (1.2). Note that we obtain, for a holomorphic function $f : D \to \mathbb{C}$,

$$f \in B_{p,\infty}(\mu) \quad \text{if and only if} \quad M_p(f, r) = O\left(\frac{1}{\mu([r, 1])}\right) \quad \text{as } r \to 1$$

while

$$f \in B_{p,0}(\mu) \quad \text{if and only if} \quad M_p(f, r) = o\left(\frac{1}{\mu([r, 1])}\right) \quad \text{as } r \to 1.$$

$B_{\infty,0}(\mu)$ and $B_{\infty,\infty}(\mu)$ have been studied by Shields and Williams ([19], [20]) and by many other authors.

Similarly, the elements in $B_{p,q}(\mu)$ for $1 \leq q < \infty$ are characterized by average growth conditions for $M_p(f, r)$.

Example. Let $d\mu(r) = 2\pi r\, dr$. Then

$$\|f\|_{p,p} = \left(\iint_D |f(x + iy)|^p \,dx\,dy\right)^{1/p}$$

and $B_{p,p}(\mu)$ is the classical Bergman space.

The aim of this paper is to finish the isomorphic classification of $B_{p,q}(\mu)$ for moderately decreasing μ which was started in [12] and [13].

1.1. Definition. Let μ be a bounded Borel measure on $[0, 1]$ satisfying (1.1) and (1.2). We consider the following conditions

$$(*) \qquad \sup_n \frac{\mu([1 - 2^{-n}, 1])}{\mu([1 - 2^{-n-1}, 1])} < \infty \qquad \text{and}$$

$$(**) \qquad \inf_{k=1,2\ldots} \limsup_{n\to\infty} \frac{\mu([1 - 2^{-n-k}, 1])}{\mu([1 - 2^{-n}, 1])} < 1$$

For further characterizations of the conditions $(*)$ and $(**)$ see [4].

Examples. $d\mu_1(r) = (1 - r)^\alpha\, dr$ for some $\alpha > -1$ and $d\mu_2(r) = r^\beta\, dr$ for some $\beta > -1$ satisfy $(*)$ and $(**)$. (This includes the Bergman spaces.) On the other hand,

$$\mu_3 = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \delta_{1-2^{-k}}$$

and

$$d\mu_4(r) = \frac{dr}{(1 - r)\log^\gamma (e/(1 - r))} \quad \text{for some } \gamma > 1$$

fulfill $(*)$ but not $(**)$.

In [13] it was shown that

$B_{p,q}(\mu)$ is isomorphic to $(\sum_n \oplus l_p^n)_{(q)}$ for any q if $1 < p < \infty$ provided that μ satisfies $(*)$.

(For Banach spaces X_n we put

$$\Big(\sum_n \oplus X_n\Big)_{(q)} = \Big\{(x_n) : x_n \in X_n \text{ for all } n, \Big(\sum_n \|x_n\|^q\Big)^{1/q} < \infty\Big\} \quad \text{if } 1 \le q$$

$$\Big(\sum_n \oplus X_n\Big)_{(\infty)} = \Big\{(x_n) : x_n \in X_n \text{ for all } n, \sup_n \|x_n\| < \infty\Big\} \quad \text{and}$$

$$\Big(\sum_n \oplus X_n\Big)_{(0)} = \Big\{(x_n) \in \Big(\sum_n \oplus X_n\Big)_{(\infty)} : \lim_{n\to\infty} \|x_n\| = 0\Big\})$$

Now we clarify the remaining cases. Let $A_p^n = \operatorname{span}\{1, z, z^2, \dots, z^n\}$ be endowed with the norm $M_p(f, 1)$. Then we have

1.2. Theorem. *Let μ satisfy $(*)$. Assume that $p \in \{1, \infty\}$.*

If μ satisfies $(**)$ then $B_{p,q}(\mu)$ is isomorphic to $(\sum_n \oplus l_p^n)_{(q)}$ for arbitrary q.

If μ does not satisfy $(**)$ then $B_{p,q}(\mu)$ is isomorphic to $(\sum_n \oplus A_p^n)_{(q)}$ for arbitrary q.

The first part of the theorem was already proved in [13], Corollary 2.7. We prove the remaining part in section 3.

1.3. Corollary. *Let μ satisfy $(*)$. If μ also satisfies $(**)$ then $B_{\infty,\infty}(\mu)$ is isomorphic to l_∞. If μ does not satisfy $(**)$ then $B_{\infty,\infty}(\mu)$ is isomorphic to H_∞.*

Proof. If μ satisfies $(**)$ then $B_{\infty,\infty}(\mu)$ is isomorphic to $(\sum_n \oplus l_\infty^n)_{(\infty)}$ which is l_∞. Otherwise $B_{\infty,\infty}(\mu)$ is isomorphic to $(\sum_n \oplus A_\infty^n)_{(\infty)}$ which itself is isomorphic to H_∞ ([22], III E 18). □

Problem. Does Theorem 1.2. remain true if μ does not satisfy $(*)$?

In [13] also the corresponding spaces $b_{p,q}(\mu)$ of harmonic functions were investigated. It turned out that, in contrast to $B_{p,q}(\mu)$, $b_{p,q}(\mu)$ is always isomorphic to $(\sum_n \oplus l_p^n)_{(q)}$ if μ satisfies $(*)$.

This is no longer true if we drop the assumption $(*)$: In [14] an example was constructed where both spaces, $B_{\infty,\infty}(\mu)$ and $b_{\infty,\infty}(\mu)$ are not isomorphic to l_∞. On the other hand, if $\mu([r,1)] = \exp(-1/(1-r))$, then μ does not satisfy $(*)$. But here $B_{\infty,\infty}(\mu)$ and $b_{\infty,\infty}(\mu)$ are isomorphic to l_∞ (see [15]). So, also in the case where $(*)$ does not hold, there are at least two different isomorphism classes of $B_{\infty,\infty}(\mu)$.

In the following, if not noted otherwise, p is always a fixed element of $[1,\infty]$ and q is a fixed element of $\{0\} \cup [1,\infty]$.

2 The spaces $(\sum_n \oplus A_p^n)_{(q)}$

For $f(r\,e^{i\theta}) = \sum_{k\geq 0} \alpha_k r^k\, e^{ik\theta}$ put

$$(2.1) \qquad (\sigma_n f)(r\,e^{i\theta}) = \sum_{k=0}^{n} \alpha_k \frac{n-k}{n} \alpha_k r^k\, e^{ik\theta}$$

It is well-known that σ_n is contractive with respect to the norms $M_p(f,r)$ (for fixed r), see for example [10].

2.1. Lemma. *Let n_1 and n_2 be positive integers. If $m \leq \min(n_1, n_2)$ then there is an isometry $i: A_p^m \to (A_p^{n_1} \oplus A_p^{n_2})_{(q)}$ and a projection $P: (A_p^{n_1} \oplus A_p^{n_2})_{(q)} \to i(A_p^m)$ with $\|P\| \leq 2$ and*

$$(2.2) \qquad P(z^k, 0) = 0 = P(0, z^k) \qquad \textit{if} \quad k > m.$$

Proof. Put $(Uf)(z) = z^m f(\bar{z})$. Define

$$i\left(\sum_{k=0}^{m} \alpha_k z^k\right) = \sum_{k=0}^{m} \alpha_k \frac{1}{2^{1/q}} (z^k, z^{m-k})$$

which is easily checked to be an isometry. (Recall, we consider the norms $M_p(\cdot, 1)$.) Then take $P: (A_p^{n_1} \oplus A_p^{n_2})_{(q)} \to i(A_p^m)$ with

$$P(f,g) = (\sigma_m f + U\sigma_m g, U\sigma_m f + \sigma_m g).$$

Hence

$$P(z^k, 0) = \begin{cases} \frac{m-k}{m}(z^k, z^{m-k}) & \text{if } k \leq m, \\ 0 & \text{else} \end{cases} \quad \text{and}$$

$$P(0, z^k) = \begin{cases} \frac{m-k}{m}(z^{m-k}, z^k) & \text{if } k \leq m, \\ 0 & \text{else} \end{cases}$$

This shows in particular that P is a projection. We have $\|P\| \leq 2$. □

2.2. Lemma. *Let (n_k) be a sequence of positive integers such that* $\sup_k n_k = \infty$. *Then*

$$\Big(\sum_n \oplus A_p^{n_k}\Big)_{(q)} \quad \textit{and} \quad \Big((\sum_n \oplus A_p^n)_{(q)} \oplus (\sum_n \oplus A_p^n)_{(q)} \oplus \dots\Big)_{(q)}$$

are isomorphic to $\big(\sum_n \oplus A_p^n\big)_{(q)}$.

Proof. Put $X = \big(\sum_n \oplus A_p^n\big)_{(q)}$ and $Y = \big(X \oplus X \oplus \dots\big)_{(q)}$. Clearly, by counting all positive integers infinitely many times we see that Y is of the form $\big(\sum_k \oplus A_p^{n_k}\big)_{(q)}$ for suitable n_k. Using Lemma 2.1. we see that $\big(\sum_k \oplus A_p^{n_k}\big)_{(q)}$ is isomorphic to a complemented subspace of X. Moreover, by Lemma 2.1. for suitable pairs of components, $\big(A_p^{n_k}, A_p^{n_{k'}}\big)$, we obtain that X is isomorphic to a complemented subspace of $\big(\sum_k \oplus A_p^{n_k}\big)_{(q)}$. Since this is true in particular for $\big(\sum_k \oplus A_p^{n_k}\big)_{(q)} = Y$, Pelczynski's decomposition method yields that Y is isomorphic to X and then, that $\big(\sum_k \oplus A_p^{n_k}\big)_{(q)}$ in general is isomorphic to X. □

3 Some convolution operators

For $f(z) = \sum_{k \geq 0} \alpha_k z^k$ put

$$(3.1) \qquad (R_n f)(z) = \sum_{k=0}^{2^n} \alpha_k z^k + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k z^k$$

Then we have (see (2.1)) $R_n = 2\sigma_{2^{n+1}} - \sigma_{2^n}$. Hence $M_p(R_n f, r) \leq 3M_p(f, r)$ for any p and any $r > 0$.

Moreover define

$$(3.2) \qquad (P_m f)(z) = \sum_{j \geq 0} \alpha_{2^m j} z^{2^m j}$$

P_m is a projection and we have $M_p(P_m f, r) \leq M_p(f, r)$ for all p and $r > 0$. This follows from the fact that

$$(P_m f)(z) = \frac{1}{2^m} \sum_{j=0}^{2^m - 1} f\left(\exp\left(\frac{2\pi j}{2^m} i\right) z\right)$$

since, for any integer k,

$$\frac{1}{2^m} \sum_{j=0}^{2^m - 1} \exp\left(\frac{2\pi k j}{2^m} i\right) = \begin{cases} 1 & \text{if } k \in 2^m \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

3.1. Lemma. *Let* $n_1 < n_2$ *and* $n_3 < n_4$ *be positive integers and put* $X = span\{z^{2^{n_1}+1}, z^{2^{n_1}+2}, \dots, z^{2^{n_2+1}-1}\}$ $Y = span\{z^{2^{n_3}+1}, z^{2^{n_3}+2}, \dots, z^{2^{n_4+1}-1}\}$. *Fix some radii* $r > 0$ *and* $s > 0$ *and some constants* $c > 0$ *and* $d > 0$. *Consider the norms* $M_p(f, r)\, c$ *on* X *and* $M_p(g, s)\, d$ *on* Y. *Let* $m = \min(2^{n_2 - n_1 - 1}, 2^{n_4 - n_3 - 1})$.

Then there is an isometry $i : A_p^m \to (X \oplus Y)_{(q)}$ *and a projection* $Q : (X \oplus Y)_{(q)} \to i(A_p^m)$ *with* $\|Q\| \leq 2$ *such that*

(3.3) $$((R_{n_2}-R_{n_1})f,(R_{n_4}-R_{n_3})g)=(f,g)$$

whenever $(f,g)\in i(A_p^m)$.

Proof. Recall that, for $f(z)=\sum_{k\geq 0}\alpha_k z^k$ we obtain

$$((R_{n_2}-R_{n_1})f)(z)=\sum_{k=2^{n_1}+1}^{2^{n_2+1}}\alpha_k\frac{k-2^{n_1}}{2^{n_1}}z^k+\sum_{k=2^{n_1+1}+1}^{2^{n_2}}\alpha_k z^k+\sum_{k=2^{n_2}+1}^{2^{n_2+1}}\alpha_k\frac{2^{n_2+1}-k}{2^{n_2}}z^k$$

(3.4)

in view of (3.1). Hence

$$\begin{aligned}(P_{n_1+1}(R_{n_2}-R_{n_1})f)(z)&=((R_{n_2}-R_{n_1})P_{n_1+1}f)(z)\\&=\sum_{j=1}^{2^{n_2-n_1-1}}\alpha_{j2^{n_1+1}}z^{j2^{n_1+1}}+\\&\sum_{j=2^{n_2-n_1-1}+1}^{2^{n_2-n_1}-1}\alpha_{j2^{n_1+1}}\frac{2^{n_1+1}-j2^{n_1+1}}{2^{n_2}}z^{j2^{n_1+1}}\end{aligned}$$

X is isometric to $Z=\operatorname{span}\{z^{2^{n_1+1}},z^{2^{n_1+2}},\dots,z^{2^{n_2+1}-1}\}$ endowed with $M_p(\cdot,1)$ as norm. Let $T:X\to Z$ be the canonical isometry. Hence $P_{n_1+1}X$ is isometric to

$$TP_{n_1+1}X=\operatorname{span}\{z^{j2^{n_1+1}}:j=1,\dots,2^{n_2-n_1}-1\}\subset Z.$$

Now, for $f\in A_p^{2^{n_2-n_1-1}}$ put $(Sf)(z)=f(z^{2^{n_1+1}})$. Then S is an isometry from $A_p^{2^{n_2-n_1-1}}$ onto $TP_{n_1+1}X$. This shows that $P_{n_1+1}X$ is isometric to $A_p^{2^{n_2-n_1-1}}$.

Similarly, $P_{n_3+1}Y$ is isometric to $A_p^{2^{n_4-n_3-1}}$. Hence $((P_{n_1+1}X)\oplus(P_{n_3+1}Y))_{(q)}$ is isometric to $(A_p^{2^{n_2-n_1-1}}\oplus A_p^{2^{n_4-n_3-1}})_{(q)}$. Let $m=\min(2^{n_2-n_1-1},2^{n_4-n_3-1})$ and apply Lemma 2.1. to find an isometric copy $i(A_p^m)$ of A_p^m in

(3.5) $$\operatorname{span}\{z^{j2^{n_1+1}}:j=1,\dots,2^{n_2-n_1}-1\}\oplus\operatorname{span}\{z^{j2^{n_3+1}}:j=1,\dots,2^{n_4-n_3}-1\}$$

which is complemented in $((P_{n_1+1}X)\oplus(P_{n_3+1}Y))_{(q)}$ by a projection $\tilde{Q}$ with $\|\tilde{Q}\|\leq 2$ satisfying (2.2). Define

$$Q(f,g)=\tilde{Q}(P_{n_1+1}f,P_{n_3+1}g)\quad\text{for all}\quad (f,g)\in(X\oplus Y)_{(q)}.$$

(3.4) and the choice of m yield $(R_{n_2}-R_{n_1})f=f$ whenever there is g with $(f,g)\in i(A_p^m)$. Similarly we have $(R_{n_4}-R_{n_3})g=g$. □

In [13], Theorem 2.5., the following proposition was proved.

3.2. Proposition. *Assume that μ satisfies $(*)$. Put $m_1=1$ and let m_{k+1} be the smallest integer larger than m_k with*

$$\mu\left(\left[1-\frac{1}{2^{m_k}},1\right]\right)\geq 3\mu\left(\left[1-\frac{1}{2^{m_{k+1}}},1\right]\right).$$

Then there are constants $a>0$ and $b>0$ such that, for every $f\in B_{p,q}(\mu)$,

$$\text{(3.6)} \qquad a\left(\sum_k M_p^q((R_{m_k} - R_{m_{k-1}})f, 1)\,\mu\left(\left[1 - \frac{1}{2^{m_k}}, 1 - \frac{1}{2^{m_{k+1}}}\right[\right)\right)^{1/q} \le$$

$$\|f\|_{p,q} \le b\left(\sum_k M_p^q((R_{m_k} - R_{m_{k-1}})f, 1)\,\mu\left(\left[1 - \frac{1}{2^{m_k}}, 1 - \frac{1}{2^{m_{k+1}}}\right[\right)\right)^{1/q}$$

if $1 \le q < \infty$ *and*

$$\text{(3.7)} \qquad a \sup_k (M_p((R_{m_k} - R_{m_{k-1}})f, 1)\,\mu\left(\left[1 - \frac{1}{2^{m_k}}, 1\right]\right) \le \|f\|_{p,q} \le$$

$$b \sup_k (M_p((R_{m_k} - R_{m_{k-1}})f, 1)\,\mu\left(\left[1 - \frac{1}{2^{m_k}}, 1\right]\right)$$

if $q = 0$ *or* $q = \infty$.

If $(**)$ *is not satisfied and* $p = 1$ *or* $p = \infty$ *then we have* $\sup_k (m_k - m_{k-1}) = \infty$.

It is easily seen that the polynomials are dense in $B_{p,q}(\mu)$ if $q = 0$ or $1 \le q < \infty$ (see [13], Proposition 2.1.). In particular, for these q, this implies that

$$f = \sum_n (R_n - R_{n-1})f \quad \text{if} \quad f \in B_{p,q}(\mu).$$

(3.7) shows that $B_{p,0}(\mu)$ is isomorphic to a subspace of $(\sum_n \oplus E_n)_{(0)}$ for some finite dimensional spaces E_n. We derive easily that, with the natural embedding, $B_{p,0}(\mu)^{**} = B_{p,\infty}(\mu)$. This is even true if μ does not satisfy $(*)$, see [13], Corollary 2.3.)

We retain the notation of Proposition 3.2. In the following put

$$\alpha_k = \begin{cases} \mu\left(\left[1 - \frac{1}{2^{m_k}}, 1 - \frac{1}{2^{m_{k+1}}}\right[\right) & \text{if} \quad 1 \le q < \infty \\ \mu\left(\left[1 - \frac{1}{2^{m_k}}, 1\right]\right) & \text{if} \quad q = 0 \text{ or } q = \infty \end{cases}$$

We have

$$\mu\left(\left[1 - \frac{1}{2^{m_k}}, 1 - \frac{1}{2^{m_{k+1}}}\right[\right) = \mu\left(\left[1 - \frac{1}{2^{m_k}}, 1\right]\right) - \mu\left(\left[1 - \frac{1}{2^{m_{k+1}}}, 1\right]\right)$$

and, by construction,

$$\mu\left(\left[1 - \frac{1}{2^{m_k}}, 1\right]\right) < 3\mu\left(\left[1 - \frac{1}{2^{m_{k+1}-1}}, 1\right]\right).$$

From this in combination with condition $(*)$ we derive

$$0 < \inf_k\left(\frac{\alpha_k}{\alpha_{k+1}}\right) \le \sup_k\left(\frac{\alpha_k}{\alpha_{k+1}}\right) < \infty$$

(see [13], Lemma 5.1.)

Now we have

3.3. Lemma. *Let* μ satisfy $(*)$. *Then* $B_{p,q}(\mu)$ *is isomorphic to a complemented subspace of* $(\sum_n \oplus A_p^n)_{(q)}$.

Proof. This is essentially the proof of [13], Lemma 4.3. We prove the case $q \neq 0, \infty$. The proof for the case $q = 0$ is identical, while the proof for $q = \infty$ follows from the biduality.

We have (in view of (3.1)), for any holomorphic $f: D \to \mathbb{C}$,

$$(R_{m_k} - R_{m_{k-1}}) f \in \operatorname{span}\{1, z, \ldots, z^{2^{m_k+1}}\}.$$

Let $X_k = \operatorname{span}\{1, z, \ldots, z^{2^{m_k+1}}\}$ be endowed with $M_p(f, 1)\, \alpha_k^{1/q}$ as norm. Then, of course, X_k is isometric to $A_p^{2^{m_k+1}}$. Define $T: B_{p,q}(\mu) \to (\sum_k \oplus X_k)_{(q)}$ by $Tf = ((R_{m_k} - R_{m_{k-1}}) f)$. By (3.6), T is an isomorphism. Moreover, define $S: (\sum_k \oplus X_k)_{(q)} \to B_{p,q}(\mu)$ by

$$S((g_k)) = \sum_k (R_{m_k+1} - R_{m_{k-1}-1})\, g_k$$

whenever $g_k \in X_k$. We obtain $STf = f$ for every $f \in B_{p,q}(\mu)$ which follows from the fact that

$$(R_{m_k+1} - R_{m_{k-1}-1})(R_{m_k} - R_{m_{k-1}}) f = (R_{m_k} - R_{m_{k-1}}) f$$

and $f = \sum_k (R_{m_k} - R_{m_{k-1}}) f$. Moreover, we have, with the constant b of (3.6),

$$\begin{aligned}
\|S(g_k)\|_{p,q} &\le b \left(\sum_j M_p^q\Big((R_{m_j} - R_{m_{j-1}}) \sum_k (R_{m_k+1} - R_{m_{k-1}-1})\, g_k, 1\Big)\, \alpha_j \right)^{1/q} \\
&\le c_1 \left(\sum_j \sum_{k=j-2}^{j+2} M_p^q\big((R_{m_j} - R_{m_{j-1}})(R_{m_k+1} - R_{m_{k-1}-1})\, g_k, 1\big)\, \alpha_j \right)^{1/q} \\
&\le c_2 \left(\sum_k M_p^q(g_k, 1)\, \alpha_k \right)^{1/q} \\
&= c_2 \|(g_k)\|
\end{aligned}$$

for some universal constants $c_1 > 0$ and $c_2 > 0$. Here we used the facts that $(R_{m_j} - R_{m_{j-1}})(R_{m_k+1} - R_{m_{k-1}-1}) = 0$ if $k < j - 2$ or $k > j + 2$ (see (3.1)), that the R_m are uniformly bounded and that

$$0 < \inf_k \left(\frac{\alpha_k}{\alpha_{k+1}} \right) \le \sup_k \left(\frac{\alpha_k}{\alpha_{k+1}} \right) < \infty .$$

Hence TS is a bounded projection from $(\sum_k \oplus X_k)_{(q)}$ onto $TB_{p,q}(\mu)$. □

Finally we obtain

3.4. Lemma. *Let* μ *satisfy* $(*)$ *and assume that* $p = 1$ *or* $p = \infty$. *If* $(**)$ *does not hold then* $B_{p,q}(\mu)$ *contains a complemented subspace which is isomorphic to* $(\sum_n \oplus A_p^n)_{(q)}$.

Proof. It suffices to assume $q = 0$ or $1 \leq q < \infty$. The case $q = \infty$ then follows in view of $B_{p,0}(\mu)^{**} = B_{p,\infty}(\mu)$.

If $m_{k-1} + 1 < m_k - 1$ we have

$$(3.9) \quad (R_{m_j} - R_{m_{j-1}})(R_{m_k-1} - R_{m_{k-1}+1}) = \begin{cases} R_{m_k-1} - R_{m_{k-1}+1} & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Put, for these k,

$$X_k = (R_{m_k-1} - R_{m_{k-1}+1})\, B_{p,q}(\mu) = \operatorname{span}\{z^{2^{m_{k-1}+1}+1}, \dots, z^{2^{m_k}-1}\}.$$

By (3.6), (3.7) and (3.9) the norm $\|\cdot\|_{p,q}$ on X_k is equivalent to $M_p(\cdot, 1)\, \alpha_k^{1/q}$ if $1 \leq q < \infty$ and to $M_p(\cdot, 1)\, \alpha_k$ if $q = 0$. Since $\sup_k(m_k - m_{k-1}) = \infty$ we have $\sup_k \dim X_k = \infty$. The space $X = \text{closed span}(\bigcup_k X_k) \subset B_{p,q}(\mu)$ is isomorphic to $(\sum_k \oplus X_k)_{(q)}$.

For $f \in B_{p,q}(\mu)$ and some subsequence (n_k) of the indices put

$$Tf = \sum_k (R_{m_{n_k}-1} - R_{m_{n_{k-1}}+1})\, f$$

Then, in view of the fact that the polynomials are dense in $B_{p,q}(\mu)$, according to (3.6) and (3.7), T is well-defined and bounded.

Using Lemma 3.1. we find indices $1 \leq n_1 \leq n_2 \leq \dots$ such that $(\sum_m \oplus A_p^m)_{(q)}$ is isometric to a subspace Y of $\tilde{X} = \text{closed span}(\bigcup_k X_{n_k})$ and there is a bounded projection $\tilde{Q} : \tilde{X} \to Y$ with

$$(3.10) \quad (R_{m_{n_k}-1} - R_{m_{n_{k-1}}+1})\, f_k = f_k$$

whenever $f_k \in X_{n_k}$ and $\sum_k f_k \in Y$.

Define, for $f \in B_{p,q}(\mu)$,

$$Qf = \tilde{Q} \sum_k (R_{m_{n_k}-1} - R_{m_{n_{k-1}}+1})\, f.$$

Then, by (3.6) and (3.7), Q is bounded. Using (3.9), (3.10) we see that Q is a projection onto Y.

The Lemmas 3.3 and 3.4. together with Pelczynski's decomposition method prove that $B_{p,q}(\mu)$ is isomorphic to $(\sum_n \oplus A_p^n)_{(q)}$ if $p = 1$ or $p = \infty$ and μ satisfies $(*)$.

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