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Application of Base Tree Theorem

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We consider combinatorical facts on $[\omega]^\omega$ which walk back and forth around Base Tree Theorem. Ideals $\mathcal{K}^\kappa$ are introduced and their cardinal invariants are estimated. Known facts about $\beta\mathbb{N}$ are adopted for $[\omega]^\omega$.

1. Introduction. A family of infinite subsets of natural numbers is *almost disjoint* if each two its elements have finite intersection. An infinite family consisting of almost disjoint sets is called a *maximal almost disjoint family*, whenever any infinite subset of natural numbers has infinite intersection with some element of this family. Following shortened characters will be used: AD-family instead of almost disjoint family; MAD-family instead of maximal infinite almost disjoint family; $A \in [X]^\omega$ instead of A is a infinite subset of X; and *A meets B* instead of A has infinite intersection with B. Thus ω denotes the set of all natural numbers; and $[\omega]^\omega$ denotes the family of all infinite subset of natural numbers. For AD-families $\mathcal{U}$ and $\mathcal{V}$ we say that $\mathcal{U}$ refines $\mathcal{V}$, whenever any element of $\mathcal{U}$ meets at most one element of $\mathcal{V}$. But for MAD-families $\mathcal{U}$ *refines* $\mathcal{V}$ if and only, if any element of $\mathcal{U}$ is almost contained in some element of $\mathcal{V}$ – recall that X is *almost contained* in Y, whenever the difference $X \setminus Y$ is finite. We assume that our readers are familiar with standard notions of set theory, i.e. with ordinal and cardinal numbers. We need following less known facts from this theory.

Base Tree Theorem. *There exists a family $\Theta = \{\mathcal{D}_\alpha : \alpha < h\}$ with the following properties: every $\mathcal{D}_\alpha$ is MAD-family; if $\alpha < \beta < h$, then $\mathcal{D}_\beta$ refines $\mathcal{D}_\alpha$; for any $X \in [\omega]^\omega$ there exists an ordinal $\alpha < h$ such that X almost contains continuum elements of $\mathcal{D}_\alpha$; if $\alpha < \beta < h$, then every element of $\mathcal{D}_\alpha$ meets continuum elements of $\mathcal{D}_\beta$.* □

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Base Tree Theorem was stated in B. Balcar, J. Pelant, P. Simon [2]. It had been using in B. Balcar, J. Dockalkova, P. Simon [1], B. Balcar, P. Simon [3], B. Balcar, P. Vojtas [4], A. Dow [6] and [7], R. Frankiewicz, P. Zbierski [9], Sz. Plewik [11], S. Shelah, O. Spinas [12]. Assume that h is the minimal ordinal for which Base Tree Theorem is valid, so h is a regular uncountable cardinal. In [2]: see Lemma 2.6, there was stated the following.

Lemma. *If $\mathscr{U}$ has cardinality less than h and $\mathscr{U}$ consists of MAD-families, then there exists a MAD-family which refines every family belonging to $\mathscr{U}$.* □

2. Ideals $\mathscr{K}^\kappa$. Suppose $\mathscr{A}$ is some AD-family and κ is a cardinal number such that $2 \leq \kappa \leq \mathfrak{c}$, where $\mathfrak{c}$ stands for the cardinal 2^ω: this cardinal is called continuum. Put

$$J^\kappa(\mathscr{A}) = \{X \in [\omega]^\omega : X \text{ meets at least } \kappa \text{ elements of } \mathscr{A}\}$$

and let $\mathscr{K}^\kappa$ be the ideal on $[\omega]^\omega$ generated by the family of sets

$$\{J^\kappa(\mathscr{A}) : \mathscr{A} \text{ is AD-family}\}.$$

Since in ZFC every infinite AD-family is contained in some MAD-family, one could say that $\mathscr{K}^\kappa$ is generated by the family of sets $\{J^\kappa(\mathscr{A}) : \mathscr{A}$ is MAD-family$\}$.

Lemma 1. *If $2 \leq \kappa \leq \mathfrak{c}$, and $\lambda < h$, and a family $\{\mathscr{A}_\alpha : \alpha < \lambda\}$ consists of MAD-families, then there exists some MAD-family $\mathscr{B}$ such that*

$$\bigcup\{J^\kappa(\mathscr{A}_\alpha) : \alpha < \lambda\} \subseteq J^\kappa(\mathscr{B}).$$

Proof. One could use Lemma from the introduction and consider some MAD-family $\mathscr{B}$ which refines every family $\mathscr{A}_\alpha$. □

Note that $\mathscr{K}^2$ is exactly the ideal of nowhere Ramsey sets, see Lemma 3 in [11] or compare Claim on p. 352 in [3]. On the other hand $\mathscr{K}^\mathfrak{c}$ is exactly the ideal of all sets which have ADR. Indeed, following [1], [3] or [4] we say that a family $\mathscr{U} \subset [\omega]^\omega$ has ADR, whenever there is some AD-family $\mathscr{A}$ such that for any $U \in \mathscr{U}$ there is some $A \in \mathscr{A}$ with $A \subseteq U$.

Theorem 1. *A family of subsets of natural numbers has ADR if and only, if it belongs to $\mathscr{K}^\mathfrak{c}$.*

Proof. Let $\mathscr{A}$ be some MAD-family. For any $U \in J^\mathfrak{c}(\mathscr{A})$ choose $\varphi(U) \in \mathscr{A}$ such that $\varphi(U)$ meets U and $\varphi : J^\mathfrak{c}(\mathscr{A}) \to \mathscr{A}$ is some one-to-one function. The family

$$\{U \cap \varphi(U) : U \in J^\mathfrak{c}(\mathscr{A})\}$$

is some AD-family which shows – since the intersection $U \cap \varphi(U)$ is always contained in U, that $J^\mathfrak{c}(\mathscr{A})$ has ADR. Because of the definition every element of $\mathscr{K}^\mathfrak{c}$ has to have ADR.

Let $\mathcal{A}$ be AD-family which shows that a family $\mathcal{U}$ has ADR. Split any element of $\mathcal{A}$ onto continuum almost disjoint and infinite pieces and denote the family of those pieces by $\mathcal{A}^*$. We have $U \in J^{\mathfrak{c}}(\mathcal{A}^*)$, i.e. $U \in \mathcal{K}^{\mathfrak{c}}$. □

Directly from the definition one concludes the following inclusions

$$\mathcal{K}^2 \supseteq \mathcal{K}^3 \supseteq \dots \supseteq \mathcal{K}^{\omega} \supseteq \dots \supseteq \mathcal{K}^{\mathfrak{c}}.$$

Some of them are proper.

Theorem 2. *If n and m are different natural numbers, then* $\mathcal{K}^n \neq \mathcal{K}^m$.

Proof. Let $2 \leq m < n < \omega$. Since $\mathcal{K}^n \subset \mathcal{K}^m$, we shall show that the family $J^m(\mathcal{A})$ does not belong to $\mathcal{K}^n$ for every MAD-family $\mathcal{A}$. Suppose $\mathcal{B}$ is some MAD-family. Choose sets $A_1, A_2, \dots, A_m$ which belong to $\mathcal{A}$ and sets $B_1, B_2, \dots, B_m$ which belong to $\mathcal{B}$ such that A_k meets B_k, whenever $1 \leq k \leq m$. The union

$$A_1 \cap B_1 \cup A_2 \cap B_2 \cup \dots \cup A_m \cap B_m$$

belongs to $J^m(\mathcal{A})$ – because it meets any set $A_1, A_2, \dots, A_m$ – and does not belong to $J^n(\mathcal{B})$ – because it meets less than n elements of $\mathcal{B}$. By the definition of $\mathcal{K}^{\kappa}$ one concludes that $\mathcal{K}^m$ is not contained in $\mathcal{K}^n$. □

Theorem 2 implies that $\mathcal{K}^{\omega}$ is a proper subfamily of any $\mathcal{K}^n$, where n is some natural number. In [3] – see Theorem 4.18, there was given set-theoretical assumptions which imply $\mathcal{K}^{\omega} = \mathcal{K}^{\mathfrak{c}}$. However the validity of this equality remains still open, compare also [1] p. 82. Note that we have showed the following: *If* $2 \leq n < \omega$ *and* $\mathcal{A}$ *is some MAD-family, then* $J^n(\mathcal{A}) \setminus J^{\mathfrak{c}}(\mathcal{A})$ *has not ADR.* So, we have obtained some examples which were in search by S. H. Hechler [10] p. 109.

3. Additivity and covering numbers for $\mathcal{K}^{\kappa}$. If S is a set, then $[S]$ denotes its cardinality. Recall that the *additivity number* of family $\mathcal{A}$ is the cardinal

$$add(\mathcal{A}) = min\{|\mathcal{S}|: \mathcal{S} \subseteq \mathcal{A} \text{ and } \bigcup \mathcal{S} \notin \mathcal{A}\};$$

but the *covering number* is defined by

$$cov(\mathcal{A}) = min\{|\mathcal{S}|: \mathcal{S} \subseteq \mathcal{A} \text{ and } \bigcup \mathcal{A} = \bigcup \mathcal{S}\}.$$

For every non-empty family $\mathcal{A}$ the covering number $cov(\mathcal{A})$ is always well defined But additivity number $add(\mathcal{A})$ is well defined, if $\bigcup \mathcal{A}$ does not belong to $\mathcal{A}$. Directly from the definitions it follows that for $2 \leq \kappa \leq \mathfrak{c}$ the family of all infinite subset of natural numbers does not belong to $\mathcal{K}^{\kappa}$, i.e. $[\omega]^{\omega} \notin \mathcal{K}^{\kappa}$. So, cardinal numbers $add(\mathcal{K}^{\kappa})$ and $cov(\mathcal{K}^{\kappa})$ are well defined. In [11] – compare [3] p. 352 – there was observed that $add(\mathcal{K}^2) = cov(\mathcal{K}^2) = h$. Let us generalize those facts.

Lemma 2. *If* $2 \leq \kappa \leq \mathfrak{c}$, *then* $add(\mathcal{K}^{\kappa}) \geq h$.

Proof. Consider some family

$$\{J^\kappa(\mathscr{A}_\alpha) : \alpha < \lambda\}.$$

If $\lambda < h$, then – by the Lemma from Introduction – there is a MAD-family $\mathscr{A}$ which refines every family $\mathscr{A}_\alpha$. By the definition we have

$$\bigcup\{J^\kappa(\mathscr{A}_\alpha) : \alpha < \lambda\} \subseteq J^\kappa(\mathscr{A}).$$

This means that every family of less that h elements of $\mathscr{K}^\kappa$ has union which has to belong to $\mathscr{K}^\kappa$. □

Lemma 3. *If* $2 \leq \kappa \leq \mathfrak{c}$, *then* $cov(\mathscr{K}^\kappa) \leq h$.

Proof. Consider some family $\Theta = \{\mathscr{D}_\alpha : \alpha < h\}$ of MAD-families with properties as in Base Tree Theorem. Since, for any $X \in [\omega]^\omega$ there exists an ordinal $\alpha < h$ such that X almost contains continuum elements of $\mathscr{D}_\alpha$ and by the definitions one concludes that

$$\bigcup\{J^\kappa(\mathscr{D}_\alpha) : \alpha < h\} = [\omega]^\omega,$$

and the family $\{J^\kappa(\mathscr{D}_\alpha) : \alpha < h\}$ consists of elements of $\mathscr{K}^\kappa$. □

The next theorem generalizes [10] p. 97 Theorem 2.8, and answers the problem 4, see [10] p. 109.

Theorem 3. *If* $2 \leq \kappa \leq \mathfrak{c}$, *then* $cov(\mathscr{K}^\kappa) = add(\mathscr{K}^\kappa) = h$.

Proof. Since $[\omega]^\omega \notin \mathscr{K}^\kappa$ one concludes that $add(\mathscr{K}^\kappa) \leq cov(\mathscr{K}^\kappa)$. By Lemmas 4 and 5 one infers

$$h \leq add(\mathscr{K}^\kappa) \leq cov(\mathscr{K}^\kappa) \leq h.$$

This means that $add(\mathscr{K}^\kappa) = cov(\mathscr{K}^\kappa) = h$. □

4. Cofinality number for $\mathscr{K}^\kappa$. Recall that for a family $\mathscr{A}$ the *cofinality number* $cof(\mathscr{A})$ is the least cardinal $|\mathscr{S}|$ for families $\mathscr{S} \subseteq \mathscr{A}$ which fulfill the following condition: *for any* $A \in \mathscr{A}$ *there exists* $S \in \mathscr{S}$ *such that* $A \subseteq S$.

Theorem 4. *If* $2 \leq \kappa \leq \mathfrak{c}$, *then* $cof(\mathscr{K}^\kappa) > \mathfrak{c}$.

Proof. Suppose $\{\mathscr{A}_\alpha : \alpha < \mathfrak{c}\}$ are MAD-families and let $\mathscr{A}_0 = \{V_\alpha : \alpha < \mathfrak{c}\}$. For every ordinal $\alpha < \mathfrak{c}$ choose some $B_\alpha \in \mathscr{A}_\alpha$ which meets V_α. Let $\{C_\beta : \beta < \mathfrak{c}\}$ be some AD-family which consists of subsets contained in $B_\alpha \cap V_\alpha$. If $\mathscr{A}$ is a MAD-family which contains all above defined families $\{C_\beta : \beta < \mathfrak{c}\}$, then $J^\kappa(\mathscr{A})$ is contained in no $J^\kappa(\mathscr{A}_\alpha)$: in fact

$$B_\alpha \cap V_\alpha \in J^\kappa(\mathscr{A}) \setminus J^\kappa(\mathscr{A}_\alpha).$$

This implies that no family of cardinality $\mathfrak{c}$ which consists of elements of $\mathscr{K}^\kappa$ could be considered in the definition of $cof(\mathscr{K}^\kappa)$. □

Theorem 5. *If $\mathscr{U}$ contains no AD-family of cardinality $\mathfrak{c}$, then $\mathscr{U} \in \mathscr{K}^2$.*

Proof. For any $A \in [\omega]^\omega$ there is $V_A \subseteq A$ such that V_A almost contains no element of $\mathscr{U}$. Indeed, if $\{C_\alpha : \alpha < \mathfrak{c}\}$ is some AD-family consisting of subset of A, then some C_α one could take as V_A. In the opposite case, for every $\alpha < \mathfrak{c}$ one takes some element of $\mathscr{U}$ which is almost contained in C_α. By this way one would choose AD-family which could not exist because of the assumptions. If $\mathscr{B}$ is a MAD-family which consists of subsets of sets V_A – where $A \in [\omega]^\omega$ – then $\mathscr{U} \subseteq J^2(\mathscr{B})$. □

We do not know if the above theorem holds for some $\mathscr{K}^\kappa$, where $\kappa \neq 2$. In [3]: Theorem 4.16, there was stated that a union of less than continuum ultrafilters has ADR. This fact follows that any set of cardinality less than continuum belongs to $\mathscr{K}^\kappa$, in fact has ADR.

5. $J^\kappa(\mathscr{A})$ and AD-families of large cardinality. Consider some AD-family $\mathscr{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$. For every ordinal $\alpha < \mathfrak{c}$ put

$$B_\alpha = \bigcup\{\{m\} \times \{0, 1, \dots, m\} : m \in A_\alpha\}.$$

Lemma 4. *The family $\{B_\alpha : \alpha < \mathfrak{c}\} \subset [\omega \times \omega]^\omega$ consists of almost disjoint sets and any set B_α meets each set $\omega \times \{n\}$.*

Proof. By the definition B_α is some infinite union of non-empty pairwise disjoint sets, so every B_α is infinite. Also

$$B_\alpha \cap B_\beta = \bigcup\{\{m\} \times \{0, 1, \dots, m\} : m \in A_\alpha \cap A_\beta\}.$$

If $\alpha \neq \beta$, then $B_\alpha \cap B_\beta$ has to be finite because of $A_\alpha \cap A_\beta$ is finite. Since

$$B_\alpha \cap (\omega \times \{n\}) = \{(m, n) : n \leq m \in A_\alpha\},$$

then this intersection has to be infinite. □

Theorem 6. *If $\mathscr{A}$ is infinite AD-family, then $J^\omega(\mathscr{A})$ contains some AD-family of cardinality $\mathfrak{c}$.*

Proof. Take different sets $A_0, A_1, A_2, \dots$ which belong to $\mathscr{A}$. Let

$$f_n : \omega \times \{n\} \to A_n \setminus (A_0 \cup A_1 \cup A_2 \cup \dots \cup A_{n-1})$$

be one-to-one functions and put $f_0 \cup f_1 \cup \dots = F$. If $\{B_\alpha : \alpha < \mathfrak{c}\}$ is a family as in Lemma 4, then $F(B_\alpha) \in J^\omega(\mathscr{A})$ for every $\alpha < \mathfrak{c}$. Therefore the family of images $\{F(B_\alpha) : \alpha < \mathfrak{c}\}$ is a desired one. □

6. Sets which have to belong to $\mathscr{K}^\mathfrak{c}$. For some infinite and countable AD-family $\{R_n : n < \omega\}$ denote by $\mathscr{F}_R$ the filter which is generated by sets $\omega \setminus (R_0 \cup R_1 \cup \dots \cup R_n)$, and put

$$I(\mathscr{F}_R) = J^\omega(\{R_n : n < \omega\}).$$

Recall that $\mathcal{F} \subset [\omega]^\omega$ is a *filter*, whenever: – it is closed under finite intersection, i.e. $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; – if A is almost contained in $B \subseteq \omega$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$. A family $\mathcal{U}$ consists of generators of a filter $\mathcal{F}$, if $\mathcal{F}$ is the intersections of all filters which contains $\mathcal{U}$. A filter $\mathcal{F}$ is *countably generated*, if there exist sets $F_0, F_1, F_2, \ldots$ such that $\mathcal{F}$ is generated by those sets and $F_0 \supset F_1 \supset F_2 \supset \ldots$, and $F_{n+1} \setminus F_n$ are always infinite. Next lemmas explain when $J^\omega(\mathcal{A}) = J^\omega(\mathcal{B})$, for infinite and countably AD-families $\mathcal{A}$ and $\mathcal{B}$.

Lemma 5. *If $F_0 \supset F_1 \supset F_2 \supset \ldots$ are generators of a filter $\mathcal{F}$ such that $F_{n+1} \setminus F_n$ is always infinite, then*

$$J^\omega(\{F_0 \setminus F_1, F_1 \setminus F_2, F_2 \setminus F_3, \ldots\}) = I(\mathcal{F}).$$

Proof. Suppose that $H_0, H_1, H_2, \ldots$ and $G_0, G_1, G_2 \ldots$ are two collections of generators of $\mathcal{F}$ such that for each natural number k there hold: G_k almost contains H_k; and H_k almost contains G_{k+1}; and $G_k \setminus H_k$ is infinite; and $H_k \setminus G_{k+1}$. This follows that $H_k \setminus H_{k+m}$ is almost contained in $G_k \setminus G_{k+m-1}$. To obtain

$$J^\omega(\{R_n : n < \omega\}) \subseteq J^\omega(\{F_0 \setminus F_1, F_1 \setminus F_2, F_2 \setminus F_3, \ldots\})$$

one could consider generators H_k on the form $\omega \setminus (R_0 \cup R_1 \cup \ldots \cup R_n)$, and generators G_k on the form F_n. But to obtain

$$J^\omega(\{R_n : n < \omega\}) \supseteq J^\omega(\{F_0 \setminus F_1, F_1 \setminus F_2, F_2 \setminus F_3, \ldots\})$$

one should consider generators G_k in the form $\omega \setminus (R_0 \cup R_1 \cup R_2 \ldots \cup R_n)$, and generators H_k in the form F_n. □

Lemma 6. *If $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ is a sequence of countably generated filter and always $M \in I(\mathcal{F}_n)$, then M belongs to $I(\bigcup_n\{\mathcal{F}_n : n < \omega\})$.*

Proof. This is immediately consequence of the following property: *If $M \in I(\mathcal{F})$, then for any $G \in \mathcal{F}$ there is $\mathcal{H} \in \mathcal{F}$ such that M meets $G \setminus H$.* One concludes this property directly for the definition of $I(\mathcal{F})$. □

Let $\{g^\kappa : \kappa < b\}$ be some fixed, unbounded and increasing family of sequences of natural number. This means that: $g^\kappa = \{g_0^\kappa, g_1^\kappa, \ldots\}$ for every ordinal κ; if $\beta < \kappa < b$, then $g_n^\beta < g_n^\kappa$ for all but finite many $n < \omega$; no sequence of natural number $f_0, f_1, \ldots$ fulfills $g_n^\beta < f_n$, for all but finite many $n < \omega$ and for every $\beta < b$. Assume that the cardinal b is minimal ordinal for which there exists unbounded and increasing family of sequences of natural number. More details about b one can find in [5].

Lemma 7. *Let $\mathcal{F}$ be some countably generated filter. There exists a family $\{\mathcal{F}_\alpha : \alpha < b\}$ consisting of countably generated filter such that: $\mathcal{F} \subset \mathcal{F}_\alpha$ for every ordinal α; if $\alpha \neq \beta$, then there are $A \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$ such that A does not meet B; if $M \in I(\mathcal{F})$, then $M \in I(\mathcal{F}_\alpha) \cap I(\mathcal{F}_\beta)$ for some $\alpha \neq \beta$.*

Proof. Let $F_0 \supset F_1 \supset F_2 \dots$ be some generators of $\mathscr{F}$ such that $F_n \setminus F_{n+1}$ is always infinite. For any ordinal $\kappa < b$ put

$$Y(\mathscr{F}, \kappa) = \bigcup \{\{n \in F_m \setminus F_{m+1} : n < g_m^\kappa\} : m < \omega\}.$$

Let $\mathscr{F}_\alpha$ be filters generated by families

$$\mathscr{F} \cup \{Y(\mathscr{F}, \alpha) \setminus Y(\mathscr{F}, \zeta_n) : \lim_{n\to\infty} \zeta_n = \alpha\},$$

where all sets $Y(\mathscr{F}, \zeta_{n+1}) \setminus Y(\mathscr{F}, \zeta_n)$ are always infinite.

If $M \in I(\mathscr{F})$, then there are different filters $\mathscr{G}$ and $\mathscr{H}$ which have been defined by the above formula, and $M \in I(\mathscr{G}) \cap I(\mathscr{H})$. Indeed, put $\zeta_0 = 0$, and suppose that we have defined ζ_n. Since $M \in I(\mathscr{F})$ there exists an increasing sequence $m_0, m_1, m_2, \dots$ such that $M \cap F_{m_j} \setminus F_{m_{j+1}}$ is always infinite. For each $j < \omega$ choose $k_j \in M \cap F_{m_j} \setminus F_{m_{j+1}}$ such that $g_{m_j}^{\zeta_n} < k_j$. Consider the sequence of natural number $f_0, f_1, \dots$ such that: $f_i = k_0$ whenever $i \leqslant m_0$; and $f_i = k_j$ whenever $m_{j-1} < i \leqslant m_j$. Since $\{g^\kappa : \kappa < b\}$ is unbounded one could take an ordinal $\zeta_{n+1} > \zeta_n$ such that $f_i < g_i^{\zeta_{n+1}}$ for infinitely many $i < \omega$. If $m_{j-1} < i \leqslant m_j$ and $f_i < g_i^{\zeta_{n+1}}$, then $k_j = f_i < g_i^{\zeta_{n+1}} \leqslant g_{m_j}^{\zeta_{n+1}}$, i.e. $k_j < g_{m_j}^{\zeta_{n+1}}$, because of the sequence $g^{\zeta_{n+1}}$ is increasing. Therefore the set $M \cap Y(\mathscr{F}, \zeta_{n+1}) \setminus Y(\mathscr{F}, \zeta_n)$ is always infinite. Put $\eta = \sup\{\zeta_n : n < \omega\}$. This is possible since b is a regular cardinal number. The filter $\mathscr{G}$ is generated by the family

$$\mathscr{F} \cup \{Y(\mathscr{F}, \eta) \setminus Y(\mathscr{F}, \zeta_n) : \lim_{n\to\infty} \zeta_n = \eta\},$$

such that $M \in I(\mathscr{G})$. A next filter $\mathscr{H}$ one defines similarly, but with the starting point $\zeta_0 = \eta$. In fact one could define filters $\mathscr{F}_\alpha$ such that $M \in I(\mathscr{F}_\alpha)$ for b many ordinals, where $\alpha < b$ because of b is a regular cardinal. □

Theorem 7. *If a family $\{R_n : n < \omega\}$ consists of infinite and parwise disjoint sets of natural numbers, then $J^\omega(\{R_n : n < \omega\})$ belongs to $\mathscr{K}^c$.*

Proof. We construct a tree $T_0 \cup T_1 \cup T_2 \dots$ — where T_n denotes the n-th level of the tree — of height ω consisting of countably generated filters. Let $T_0 = \{\mathscr{F}_R\}$, i.e. it consists of the filter generated by sets $\omega \setminus (R_0 \cup R_1 \cup \dots R_n)$. Suppose that the level T_n has been defined. If $\mathscr{F} \in T_n$, then the immediately successors of $\mathscr{F}$ could be filters which exist by Lemma 7. For any $M \in I(\mathscr{F}_R)$ choose some filter

$$\mathscr{G}_M = \bigcup \{\mathscr{F}_n : n < \omega\},$$

where $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \dots$, such that: always $\mathscr{F}_k \in T_k$; and always $M \in I(\mathscr{F}_k)$; and if $N \neq M$, then $\mathscr{G}_N \neq \mathscr{G}_M$. This is possible because of by Lemma 6 for any M one could choose $\mathscr{G}_M$ between continuum filters. For every filter $\mathscr{G}_M$ fix a sequence $F_0 \supset F_1 \supset F_2 \supset \dots$ such that M always meets $F_n \setminus F_{n+1}$: this is possible because of Lemma 6. Choose some $m_k \in M \cap F_n \setminus \Gamma_{n+1}$ and put $\mathscr{A}(M) = \{m_0, m_1, m_2, \dots\}$. The family $\{\mathscr{A}(M): M \in J^{\omega}(\{R_0, R_1, R_2, \ldots\}) = I(\mathscr{F}_R)\}$ is AD-family: by the definition $\mathscr{A}(M)$ is almost contained in any element of $\mathscr{G}_M$; and if $N \neq M$, then there are $G \in \mathscr{G}_N$ and $H \in \mathscr{G}_M$ such that $G \cap H$ is finite. We have proved that the family $J^{\omega}(\{R_n : n < \omega\})$ has ADR. It has to be $J^{\omega}(\{R_n : n < \omega\}) \in \mathscr{K}^c$ because of Theorem 1. □

Theorem 7 or Lemma 7 are combinatorical roots which had been considered in [1]: Lemma 2.1, in [3]: Lemma 4.15, in [4]: Theorem A, in R. Frankiewicz [8]: Lemma 2.2, and in [9]: Lemma 3.2 on p. 101. Our proof of Lemma 7 does not use Base Tree Theorem, but in quoted papers this theorem was used.

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