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On the Splitting Number and Mazurkiewicz’s Theorem

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We give a new proof of Mazurkiewicz’s theorem about bounded sequences of Borel functions. In this proof we use Shoenfield’s absoluteness theorem for $\Sigma^1_2$-sentences and one characterization of some class of sequentially compact topological spaces which involves the splitting number.

1. Introduction

We use standard set theoretical notation. By $\omega$ we denote the set of natural numbers. By $[X]^\omega$ we denote the family of all infinite subsets of a set $X$. The cardinality of a set $X$ we denote by $|X|$. By $\kappa$, $\lambda$ we always denote infinite cardinal numbers.

By $[0,1]$ we denote the unit interval of the real line. For an infinite cardinal number $\kappa$ by $\{0,1\}^\kappa$ we denote the generalized Cantor set of length $\kappa$. Similarly $[0,1]^\kappa$ denotes the generalized Tichonov cube of length $\kappa$. By $\text{Perf}([0,1])$ we denote the Polish space of all non-empty perfect subsets of the interval $[0,1]$ with the Hausdorff metric. We treat the set $[\omega]^\omega$ as a Polish space, since we may identify this set with a $G_\delta$ subset of the classical Cantor set $\{0,1\}^\omega$.

Let us recall that the splitting number $s$ (see [4]) is the least cardinal number such that there exists a family $\mathcal{F}$ of infinite subsets of $\omega$ such that $(\forall A \in [\omega]^\omega) \ (\exists S \in \mathcal{F}) \ (|A \cap S| = |A \setminus S| = \omega)$. It is well known that $\omega < s \leq 2^\omega$ and that $s$ is a relatively small cardinal number. Moreover, Martin’s Axiom implies that $s = 2^\omega$. This implies that each transitive model of the theory $\text{ZFC}$ can be extended, via a forcing extension, to a model of theory $\text{ZFC} \cap \{s > \aleph_1\}$.

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A topological space is *sequentially compact* if each sequence of its elements has a convergent subsequence. It is known that a countable product of sequentially compact spaces is sequentially compact and that a continuous image of sequentially compact space is sequentially compact, too (see [1]). We shall use the following characterization of the cardinal number $s$ (see [4])

$$s = \min\{\kappa : \{0,1\}^\kappa \text{ is not sequentially compact}\}.$$

By a canonical Polish space we understood a countable product of spaces $\{0,1\}^\omega$, $[0,1]$, $\text{Perf}([0,1])$ and so on. A sentence $\varphi$ is a $\Sigma^1_2$-sentence if for some canonical Polish spaces $X$, $Y$ and some Borel $B \subseteq X \times Y$ we have $\varphi = (\exists x \in X)(\forall y \in Y)((x,y) \in B)$. Spaces $X$, $Y$ and the set $B$ are called “parameters” of the sentence $\varphi$. We shall use the following classical theorem about absoluteness of $\Sigma^1_2$-sentences (see [3]):

**Theorem 1.** (Shoenfield) Suppose that $M \subseteq N$ are transitive models of the theory $ZF$ such that $\omega^N_1 \subseteq M$. Let $\varphi$ be $\Sigma^1_2$-sentences with parameters from the model $M$. Then $\varphi$ holds in the model $M$ if and only if $\varphi$ holds in the model $N$.

Notice that if the model $N$ is a generic extension of the transitive model $M$ then both models $M$ and $N$ have the same ordinal numbers, so the inclusion $\omega^N_1 \subseteq M$ trivially holds.

### 2. Proof of Mazurkiewicz’s theorem

We start our consideration with one probably well known characterization of the splitting number.

**Lemma 1.** The following three cardinal numbers are the same:

1. $s = \min\{\kappa : \{0,1\}^\kappa \text{ is not sequentially compact}\}$,
2. $s' = \min\{\kappa : \{0,1\}^{\omega^\kappa} \text{ is not sequentially compact}\}$,
3. $s'' = \min\{\kappa : \{0,1\}^\kappa \text{ is not sequentially compact}\}$.

**Proof.** Suppose that $\lambda < s$. Then the space $\{0,1\}^\lambda$ is sequentially compact. But the space $\{\{0,1\}^{\omega^\lambda}\}^{\lambda} \simeq \{\{0,1\}^{\omega^\lambda}\}^{\lambda}$ is a product of countably many sequentially compact spaces, so it is sequentially compact, too. This shows that $s \leq s'$. Suppose now that $\lambda < s'$. Then the space $\{\{0,1\}^{\omega^\lambda}\}^{\lambda}$ is sequentially compact. Therefore the space $[0,1]^{\lambda}$, as a continuous image of the space $\{\{0,1\}^{\omega^\lambda}\}^{\lambda}$, is sequentially compact, too. This shows that $s' \leq s''$. Finally, notice that if $\{f_n\}_{n \in \omega}$ is a sequence of elements of the space $\{0,1\}^\kappa$ without any convergent subsequence then the same sequence $\{f_n\}_{n \in \omega}$, treated as a sequence of element of the space $[0,1]^\kappa$, has not any convergent subsequence. This shows that $s'' \leq s$. □

Now we formulate and give a new proof of one theorem about sequences of bounded Borel functions proved in 1932 by Mazurkiewicz in [2].

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**Theorem 2.** (Mazurkiewicz) Let \( \{f_n\}_{n \in \omega} \) be a sequence of Borel functions from \([0, 1]\) to \([0, 1]\). Then there exists a non-empty perfect subset \( P \) of \([0, 1]\) and a subsequence \( \{f_{n_k}\}_{k \in \omega} \) which is pointwise convergent on the set \( P \).

**Proof.** Let \( \{f_n\}_{n \in \omega} \) be a sequence of Borel functions from \([0, 1]\) to \([0, 1]\). Let \( V' \) be a generic extension of the universe \( V \) such that \( V' = (s > \aleph_1) \).

For a while we shall work in the universe \( V' \). We choose an arbitrary set \( T \subset [0, 1] \) of cardinality \( \aleph_1 \) and consider the sequence \( \{f_n \upharpoonright T\}_{n \in \omega} \). Since the inequality \( |T| < s \) holds, by Lemma 1, there exists an infinite set \( A \subset \omega \) such that the sequence \( \{f_n \upharpoonright T\}_{n \in A} \) is pointwise convergent on the whole set \( T \). Notice that the set

\[
C = \{x \in [0, 1] : \{f_n(x)\}_{n \in A} \text{ is convergent}\}
\]

is Borel and contains the set \( T \). But \( T \) is uncountable, therefore the set \( C \) contains a non-empty perfect set. Therefore the sentence

\[
\varphi = (\exists A \in [\omega]^{\omega})(\exists P \in Perf([0, 1])) (\forall x \in P) (\{f_n(x)\}_{n \in A} \text{ is convergent})
\]

holds in the universe \( V' \). But \( \varphi \) is a \( \Sigma^1_2 \)-sentence with parameters from the universe \( V \) and so, by Shoenfield’s absoluteness theorem, \( \varphi \) holds in the universe \( V \), too.

\[\square\]

**References**


