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Location of Min-Max Critical Points for Multivalued Functionals

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In this paper we show how quantitative deformation lemma (for continuous functionals) can be used to obtain location of min-max critical points for multivalued functionals with closed graph. Finally, we obtain mountain pass type results for multivalued functionals, using the suitable compactness condition.

1 Introduction

Brezis and Nirenberg [1], Ghoussoub [8] and Willem [17], using the Ekeland's variational principle and the deformation arguments for "homotopy stable family with boundary", obtained general location results for C^1-functionals. Same results in non-smooth case have been obtained by Ribarska-Tsachev-Krastanov [15], [16]. Our main goal in this paper is to get some information about the location of the critical points for multivalued functionals, using the notion of invariance with respect to deformation for a family of sets, which generalizes the notion of "homotopy stable family with boundary".

A critical point theory for multivalued functionals with closed graph is developed by M. Frigon in the paper [7].

First we recall some definitions and results from this paper and from [4], [5]. Let (X, d) be a metric space and let $F : X \to \bar{\mathbb{R}}$ be a multivalued mapping with closed graph and nonempty values. We denote by

$$\operatorname{graph} F = \{(u,c) \in X \times \mathbb{R} \mid c \in F(u)\}.$$

The set graph F is a metric space endowed with the metric

$$d_g((u, b), (v, c)) = \sqrt{d^2(u, v) + |b - c|^2}.$$

Now, we recall the definition of *weak slope* for F, see [7].

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Definition 1.1. *Let* $F : X \to \bar{\mathbb{R}}$ *be a multivalued mapping with closed graph and let* $(u, b) \in \operatorname{graph} F$ *be a point. The* **weak slope** *of* F *at* (u, b), *denoted by* $|dF|(u, b)$ *is the supremum of* $\sigma \in [0, \infty[$ *such that there exists* $\delta > 0$ *and a continuous function*

$$\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2); B((u, b), \delta) \times [0, \delta] \to \operatorname{graph} F,$$

(where $B((u, b), \delta)$ *is the open ball in* graph F *centered at* (u, b) *of radius* δ*) such that*

$$d_g(\mathscr{H}((v, c), t), (v, c)) \leq t\sqrt{1 + \sigma^2}; \tag{1.0a}$$

$$\mathscr{H}_2((v, c), t) \leq c - \sigma t. \tag{1.0b}$$

Definition 1.2. *Let* (X, d) *be a complete metric space and let* $f : X \to \mathbb{R}$ *be a continuous function and* $u \in X$ *a fixed element. We denote by* $|df|(u)$ *the supremum of the* $\sigma \in [0, \infty[$ *such that there exist* $\sigma > 0$ *and a continuous map*

$$\mathscr{H} : B(u, \delta) \times [0, \delta] \to X$$

such that $\forall v \in B(u, \delta)$ *for all* $t \in [0, \delta]$ *we have*

(*a*) $d(\mathscr{H}(v, t), v) \leq t$

(*b*) $f(\mathscr{H}(v, t)) \leq f(v) - \sigma t$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

In the case where $F(u) = \{f(u)\}$ is a continuous single-valued function then $|dF|(u, f(u)) = |df|(u)$, see [7, p. 737], and it coincides with the norm of the derivative when f is of class C^1 defined on a Finsler manifold of class C^1.

We define the function $\mathscr{G}_F : \operatorname{graph} F \to \mathbb{R}$ given by $\mathscr{G}_F(u, c) = c$, where $(u, c) \in \operatorname{graph} F$.

Remark 1.3. [7] For $(u, c) \in \operatorname{graph} F$

$$|dF|(u, c) = \begin{cases} \dfrac{|d\mathscr{G}_F|(u, c)}{\sqrt{1 - |d\mathscr{G}_F|^2(u, c)}}, & |d\mathscr{G}_F|(u, c) < 1 \\ \infty, & |d\mathscr{G}_F|(u, c) = 1 \end{cases}$$

Definition 1.4. *Let* $F : X \to \bar{\mathbb{R}}$ *be a multivalued mapping with closed graph, and let* $c \in \mathbb{R}$. *We say that* $u \in X$ *is a critical point of* F *at level,* c, *if* $c \in F(u)$ *and* $|dF|(u, c) = 0$. *The set of critical points of* F *at level* c *will be denoted by* K_c. *We say that* c *is a critical value of* F *if* $K_c \neq \emptyset$, *i.e.* (u, c) *is a critical element of* F *for some* u.

Definition 1.5. *We say that the multivalued function* $F : X \to \bar{\mathbb{R}}$ *satisfies the Palais-Smale condition at level* c *(short* $(PS)_c$*), if every sequence* $(u_k) \subset X$ *for which* $c_n \in F(x_n)$ *with* $c_n \to c$ *and* $|dF|(u_n, c_n) \to 0$, *has a convergent subsequence in* X.

Remark 1.6. The multivalued function $F : X \to \bar{\mathbb{R}}$ satisfies the condition $(PS)_c$ if and only if the function $\mathcal{G}_F$ satisfies the Palais-Smale condition at level c.

Remark 1.7. The element $(u, c) \in$ graph F is a critical point for $\mathcal{G}_F$ if and only if u is a critical point of F.

We introduce the following notations

$$f^c = \{x \in X \mid f(x) \le c\}$$
$$f_c = \{x \in X \mid f(x) \ge c\}$$
$$C_\delta := \{x \in X \mid d(x, C) \le \delta\}, \delta > 0.$$

To prove the quantitative deformation lemma for continuous functionals, we need the following results and notions.

Definition 1.8. *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function. We define the function*

$$\mathcal{G}_f : epi(f) \to \mathbb{R}$$

putting

$$epi(f) = \{(u, \xi) \in X \times \mathbb{R} : f(u) \le \xi\} \quad \textit{and} \quad \mathcal{G}_f(u, \xi) = \xi .$$

In the following $epi(f)$ will be endowed with the metric

$$d_{ep}((u, \xi), (v, \mu)) = (d(u, v)^2 + (\xi - \mu)^2)^{\frac{1}{2}} .$$

Of course $epi(f)$ is closed in $X \times \mathbb{R}$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Consequently $|d\mathcal{G}_f|(u, \xi) \le 1$ for every $(u, \xi) \in epi(f)$.

Proposition 1.9. ([5, Proposition 2.3]) *Let $f : X \to \mathbb{R}$ be a continuous function and let $(u, \xi) \in epi(f)$. Then*

$$|d\mathcal{G}_f|(u, \xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1 + |df|(u)^2}}, & \textit{if } f(u) = \xi \textit{ and } |df|(u) < \infty , \\ 1, & \textit{if } f(u) < \xi \textit{ or } |df|(u) = \infty . \end{cases}$$

Theorem 1.10. ([4, Theorem 2.11]) *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function, C a closed subset of X and $\delta, \sigma > 0$ such that*

$$d(u, C) \le \delta \Rightarrow |df|(u) > \sigma .$$

Then there exists a continuous map $\eta : X \times [0, \delta] \to X$ such that

1) $d(\eta(u, t)) \le t,$
2) $f(\eta(u, t)) \le f(u)$
3) $d(u, C) \ge \delta \Rightarrow \eta(u, t) = u,$
4) $u \in C \Rightarrow f(\eta(u, t) \le f(u) - \sigma t.$

Theorem 1.11. *Let (X, d) be a complete metric space, $f : X \to \mathbb{R}$ a continuous function, C a closed subset of X, $c \in \mathbb{R}$ and $\varepsilon, \lambda > 0$. Suppose that*

(1.0) $$C \cap f^{c+\varepsilon'} \cap f_{c-\varepsilon'} \neq \emptyset, \text{ where } \varepsilon' = \frac{\varepsilon \min\{\varepsilon, \lambda\}}{2\sqrt{1+\varepsilon^2}} \text{ and}$$

(1.1) $$\forall u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} \Rightarrow |df|(u) > \varepsilon .$$

Then there exists a continuous map $\eta : X \times [0,1] \to X$ such that:

a) $d(\eta(u,t), u) \le \lambda t$, $\forall t \in [0,1]$, $\forall u \in X$,

b) $f(\eta(u,t)) \le f(u)$, $\forall t \in [0,1]$, $\forall u \in X$,

c) if $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \eta(u,t) = u$, $\forall t \in [0,1]$

d) $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$,

e) $\forall t \in]0,1]$ *and* $\forall u \in f^c \cap C$ *we have* $f(\eta(u,t)) < c$.

Proof. First, we suppose that the function $f : X \to \mathbb{R}$ is Lipschitz continuous with constant 1. We consider the set:

(1.2) $$C^* := \{u \in X \mid c - \varepsilon \le f(u) \le c + \varepsilon, d(u, C) \le \varepsilon\}.$$

Obviously the set C^* is a closed subset of X, and is not empty from (1.0). We observe that $d(u, C^*) \le \varepsilon$ implies $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$.

Indeed, let $d(u, C^*) \le \varepsilon$. Then $u \in C_{2\varepsilon}$ by triangle inequality and $u \in f^{-1}([c-\varepsilon, c+2\varepsilon])$ as f is 1-Lipschitz.

Because $\varepsilon > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ from the above we obtain $|df|(u) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$, for all u s.t. $d(u, C^*) \le \varepsilon$.

Now we can apply Theorem 1.10, for $C := C^*$, $\delta := \varepsilon$, $\sigma := \dfrac{\varepsilon}{\sqrt{1+{}^2}}$. We get a continuous function $\eta' : X \times [0, \varepsilon] \to X$ which satisfies the conditions 1)–4) from Theorem 1.10. Let $\lambda_1 := \min\{\lambda, \varepsilon\}$ and define the function $\eta : X \times [0,1] \to X$ by $\eta(u,t) = \eta'(u, \lambda_1 t)$. The properties a) and b) are obvious. Using 3) from Theorem 1.10 and the above reason, we get $\eta(u,t) = u$.

For the proof of d) we distinguish two cases:

(1.3) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) \ge c - \varepsilon'$ it follows that $u \in C^*$, hence we have

$$f(\eta(u,1)) = f(\eta'(u,\lambda_1)) \le f(u) - \frac{\varepsilon\lambda_1}{\sqrt{1+\varepsilon^2}} \le c + \varepsilon' - \frac{\varepsilon\lambda_1}{\sqrt{1+\varepsilon^2}} = c - \varepsilon' .$$

(1.4) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) < c - \varepsilon'$, then from b) we get

$$f(\eta(u,1)) \le f(u) < c - \varepsilon' .$$

For the proof of e) we use also the 4) from Theorem 1.10.

Now we consider the general case. For this let $C^{**} = \{(u,\xi) \in epi(f) \mid u \in C\}$. The set $epi(f)$ is closed in $X \times \mathbb{R}$ and it follows that $epi(f)$ is a complete metric space with the metric d_{ep}. In the next we prove that for every $(u,\xi) \in epi(f)$ with $(u,\xi) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C^{**}_{2\varepsilon}$, we have $|d\mathcal{G}_f|(u,\xi) > \dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

We distinguish two cases:

I) Let $f(u) = \xi$. In this case we have two subcases.

a) $|df|(u) < \infty$. If $(u, f(u)) \in \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}^{**}$, then we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon])$ and $d_{ep}((u, f(u)), C^{**}) \leq 2\varepsilon$. Since $d(u, C) \leq d_{ep}((u, f(u)), C^{**}) \leq 2\varepsilon$ we get $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}$ and using (1.1) it follows that $|df|(u) < \varepsilon$. Since $|df|(u) < \infty$ from Proposition 1.9 we have $|d\mathcal{G}_f|(u, f(u)) = \dfrac{|df|(u)}{\sqrt{1 + |df|^2(u)}}$ and using the fact that the function $x \mapsto \dfrac{x}{\sqrt{1 + x^2}}$ is increasing we have $|d\mathcal{G}_f|(u, f(u)) > \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}$.

b) If $|df|(u) = \infty$ using Proposition 1.9 we get $|d\mathcal{G}_f|(u, f(u)) = 1 > \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}$.

II) If $f(u) < \xi$, then from Proposition 1.9 we have $|d\mathcal{G}_f|(u, \xi) = 1 > \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}$ also.

From these we get that if $(u, \xi) \in \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}^{**}$ then $|d\mathcal{G}_f|(u, \xi) > \dfrac{\varepsilon}{\sqrt{1 + \varepsilon^2}}$.

We apply the previous step for $X := epi(f)$, $f := \mathcal{G}_f$ and $C := C^{**}$, using the fact that $\mathcal{G}_f$ is Lipschitz continuous with constant 1. Of course $C^{**} \cap \mathcal{G}_f^{c+\varepsilon'} \cap (\mathcal{G}_f)_{c-\varepsilon'} \neq \emptyset$. Then there exists a continious mapping $\overline{\eta} := (\overline{\eta_1}, \overline{\eta_2}) : epi(f) \times [0, 1] \to epi(f)$ such that the following hold:

(1.5) $d_{ep}((\overline{\eta}(u, \xi), t), (u, \xi)) \leq \lambda t, \ \forall (u, \xi) \in epi(f), \ \forall t \in [0, 1]$;

(1.6) $\mathcal{G}_f(\overline{\eta}(u, \xi), t) = \overline{\eta}_2((u, \xi), t) \leq \xi = \mathcal{G}_f(u, \xi)$, for all $(u, \xi) \in epi(f)$, and $\forall t \in [0, 1]$;

(1.7) $\overline{\eta}((u, \xi), t) = (u, \xi)$ for every $(u, \xi) \in epi(f), t \in [0, 1]$ with $(u, \xi) \notin \mathcal{G}_f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}^{**}$;

(1.8) $\overline{\eta}(\mathcal{G}_f^{c+\varepsilon'} \cap C^{**}, 1) \subset \mathcal{G}_f^{c-\varepsilon'}$;

(1.9) $\mathcal{G}_f(\overline{\eta}((u, \xi), t)) < c$ for every $t \in]0, 1]$ and $\forall (u, \xi) \in \mathcal{G}_f^c \cap C^{**}$.

We define the function $\eta : X \times [0, 1] \to X$ by

(1.10) $\eta(u, t) = \overline{\eta}_1((u, f(u)), t)$.

Because $\overline{\eta}$ takes its values in $epi(f)$, we have

(1.11) $f(\overline{\eta}_1((u, f(u)), t)) \leq \overline{\eta}_2((u, f(u)), t)$.

From (1.5) we have:

$$\begin{aligned} d(\eta(u, t), u) &= d(\overline{\eta}_1((u, f(u)), t), u) \leq \\ &\leq [d^2((\overline{\eta}_1(u, f(u)), t), u) + (\overline{\eta}_2((u, f(u)), t) - f(u))^2]^{\frac{1}{2}} = \\ &= d_{ep}(\overline{\eta}((u, f(u)), t), (u, f(u))) \leq \lambda t. \end{aligned}$$

From the relations (1.6) and (1.11) we get

$$f(\eta(u, t)) = f(\overline{\eta}_1((u, f(u)), t)) \leq \overline{\eta}_2((u, f(u)), t) \leq f(u).$$

From $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$ we have $(u, f(u)) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}^{**}$. Therefore if $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$, then from (1.7) we get $\eta(u, t) = \overline{\eta}_1((u, f(u)), t) = u$.

If $f(u) \leq c + \varepsilon'$ then from (1.8) and (1.11) we get

$$f(\eta(u, 1)) = f(\overline{\eta}_1(u, f(u)), 1) \leq \overline{\eta}_2((u, f(u)), 1) \leq c - \varepsilon' .$$

From (1.9) and (1.11) we get the relation e).

Remark 1.12. If the function f is of class C^1, we obtain the "Quantitative deformation lemma" of Willem, see [17]. In the non-smooth case, similar results have been obtained by Ribarska-Tsachev-Krastanov [15], [16].

In the next we use the following remark.

Remark 1.13. If (X, d) is a metric space and A is a subset of X, then we have the following relation $d(x, A) = d(x, \overline{A})$.

The following result represent the multivalued version of the "Quantitative deformation lemma".

Theorem 1.14. *Let (X, d) be a complete metric space and let $F : X \to \mathbb{R}$ be a multivalued functional with closed graph and nonempty values. Let C a subset of* graph F, *$c \in \mathbb{R}$ and $\lambda, \varepsilon > 0$. Suppose that $C \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$, where $\varepsilon' = \dfrac{\varepsilon \min\{\varepsilon, \lambda\}}{2\sqrt{1+\varepsilon^2}}$ and*

$$\forall (u, b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} \Rightarrow |dF|(u, b) > \varepsilon .$$

Then there exists a continuous map $\eta = (\eta_1, \eta_2) : \text{graph } F \times [0, 1] \to \text{graph } F$ such that:

1) $d_g((\eta(u, b), t), (u, b)) \leq \lambda t;$
2) $\eta_2((u, b), t) \leq b$, $\forall t \in [0, 1]$ *and* $\forall (u, b) \in \text{graph } F;$
3) *if* $(u, b) \notin \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$, *then* $\eta((u, b), t) = (u, b)$, $\forall t \in [0, 1];$
4) $\eta(C \cap (X \times (-\infty, c+\varepsilon'], 1) \subset X \times (-\infty, c-\varepsilon'];$
5) $\forall t \in]0, 1]$ *and* $\forall (u, b) \in C \cap (X \times (-\infty, c])$ *we have* $\eta_2((u, b), t) < c$.

Proof. We have that $X \times \mathbb{R}$ is a complete metric space with the metric d_g defined on $(X \times \mathbb{R})$ by $d_g((u, b), (v, c)) = \sqrt{d^2(u, v) + |b-c|^2}$ for every $(u, b), (v, c) \in X \times \mathbb{R}$. Since graph F is a closed subset of $X \times \mathbb{R}$, we have that (graph F, d_g) is a complete metric space. For $(u, b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$, we have $|d\mathcal{G}_F|(u, b) > \dfrac{\varepsilon}{\sqrt{\varepsilon^2+1}}$. Indeed, if $|d\mathcal{G}_F|(u, b) = 1$, the above is trivial. Otherwise, if $|d\mathcal{G}_F|(u, b) < 1$, then from Remark 1.3 we have $|d\mathcal{G}_F|(u, b) = \dfrac{|dF|(u, b)}{\sqrt{|dF|^2(u, b) + 1}}$. Now, we apply the first step from the proof of Theorem 1.11 with $X :=$ graph F, $f := \mathcal{G}_F$ and we get the assertion.

2 Location theorem and minmax principle

In this section we prove a minmax result in the case of multivalued functionals. The main tool used for the proof of this result is Theorem 1.14.

Let Q be a subset of graph F. We denote by

$$\Gamma(Q) = \{U \subset \text{graph } F \mid Q \subset U\}$$

and suppose that $U \neq \emptyset$ if $Q = \emptyset$.

Definition 2.1. ([7, Definition 2.9]) *Let Q be a subset of* graph F, *and let Γ_0 be a subset of $\Gamma(Q)$. We say that Γ_0 is invariant with respect to (F, Q)-deformation, if the set $\eta(U, 1) \in \Gamma_0$ for every $U \in \Gamma_0$, and every continuous map η :* graph $F \times [0, 1] \to$ graph F *such that $\eta = id$ on* graph $F \times \{0\} \cup Q \times [0, 1]$ *and $\eta_2((u, b), t) \leq b$ for every $t \in [0, 1]$ and $(u, b) \in$* graph F.

Let $A, B \subset$ graph F, then in the next we use the following notations:

$$A_\delta = \{x \in \text{graph } F \mid d_g(x, A) \leq \delta\};$$
$$d(x, A) = \inf\{d_g(x, y) \mid y \in A\};$$
$$d(A, B) = \inf\{d_g(x, y) \mid x \in A, y \in B\}.$$

Definition 2.2. *Let A, Q be two subsets of* graph F, *and let Γ_0 be a nonempty subset of $\Gamma(Q)$. We say that Γ_0 intersects the set A if $U \cap A \neq \emptyset$ for every $U \in \Gamma_0$.*

The main result of this section is the following.

Theorem 2.3. *Let (X, d) be a complete metric space, and $F : X \to \bar{\mathbb{R}}$ a multivalued mapping with closed graph. We assume that, there exists a closed subset A of* graph F, *and exists $Q \subset$* graph F, *$\Gamma_0 \subset \Gamma(Q)$ a nonempty and invariant subset with respect to (F, Q)-deformation such that Γ_0 intersects A. In addition we assume that $c = \inf_{U \in \Gamma_0} \sup \mathcal{G}_F(U)$ is finite and that*

$$c_A = \inf_{U \in \Gamma_0} \sup \mathcal{G}_F(U \cap A) \geq \sup \mathcal{G}_F(Q)$$

with strict inequality if $d(A, Q) = 0$. Let $\varepsilon > 0$ be a real number such that

$$\varepsilon < \frac{d(A, Q)}{3}, \quad \text{if} \quad d(A, Q) > 0; \tag{2.3a}$$

$$\varepsilon < \frac{c_A - \sup \mathcal{G}_F(Q)}{2}, \quad \text{if} \quad d(A, Q) = 0; \tag{2.3b}$$

$$\varepsilon < \frac{c - \sup \mathcal{G}_F(Q)}{2}, \quad \text{if} \quad c > c_A; \tag{2.3c}$$

Let

$$E = \begin{cases} A_\varepsilon, & \text{if} \quad c = c_A \\ \bar{\Gamma}_0, & \text{otherwise,} \end{cases} \tag{2.3d}$$

where $\overline{\Gamma_0} = \overline{\bigcup_{U \in \Gamma_0} U}$.

Then there exists $(u, b) \in \text{graph } F$ *satisfying the following assertions*

a) $c - 2\varepsilon \le b \le c + 2\varepsilon$;

b) $d_g((u, b), E) \le 2\varepsilon$;

c) $|dF|(u, b) \le \varepsilon$.

Proof. We proceed by contradiction, i.e. we assume that

$$\forall (u, b) \in \mathcal{G}_F^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap E_{2\varepsilon} \Rightarrow |dF|(u, b) > \varepsilon .$$

The proof is divided in five steps.

Step 1. We verify that $E \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$, where $\varepsilon' = \dfrac{\varepsilon^2}{2\sqrt{1 + \varepsilon^2}}$.

If $c = c_A$ (hence $E = A_e$) let $U \in \Gamma_0$ such that $\sup \mathcal{G}_F(U) \le c + \varepsilon'$. It's enough to prove that $A \cap U \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$. It this is false, we have that $\sup \mathcal{G}_F(A \cap U) \le c - \varepsilon'$. From the definition of the c and from the hypothesis, we obtain that $c = c_A \le \sup \mathcal{G}_F(A \cap U)$, i.e. $c \le c - \varepsilon'$, contradiction. From this, it is clear that $A_\varepsilon \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$.

If $c > c_A$ (hence $E = \overline{\Gamma_0}$) we prove that $\overline{\Gamma_0} \cap \mathcal{G}_F^{c+\varepsilon'} \cap (\mathcal{G}_F)_{c-\varepsilon'} \neq \emptyset$. Let $U \in \Gamma_0$ as above, i.e. $\sup \mathcal{G}_F(U) \le c + \varepsilon'$. Let us suppose, that $U \cap (\mathcal{G}_F)_{c-\varepsilon'} = \emptyset$. From this, we obtain $\sup \mathcal{G}_F(U) \le c - \varepsilon'$. From the definition of the number c we get that $c \le \sup \mathcal{G}_F(U)$, i.e. $c \le c - \varepsilon'$, contradiction.

Step 2. For $\lambda := \varepsilon$ we use Theorem 1.14 and we get a continuous function $\eta = (\eta_1, \eta_2) : \text{graph } F \times [0, 1] \to \text{graph } F$ such that:

$$d_g(\eta((u, b), t), (u, b)) \le \varepsilon t \tag{2.3e}$$

$$\eta_2((u, b), t) \le b \tag{2.3f}$$

$$\forall (u, b) \notin \mathcal{G}_F^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap E_{2\varepsilon} \Rightarrow \eta((u, b), t) = (u, b) \tag{2.3g}$$

$$\eta(\mathcal{G}_F^{c+\varepsilon'} \cap E, 1) \subset \mathcal{G}_F^{c-\varepsilon'} . \tag{2.3h}$$

Step 3. We prove that

$$\eta((u, b), t) = (u, b), \qquad \forall (u, b) \in Q, t \in [0, 1] . \tag{2.3i}$$

If $c = c_A$ we prove that $Q \subset C_{\text{graph } F}(\mathcal{G}_F^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap A_{3\varepsilon})$, where $C_{\text{graph } F}(.)$ is the complement in rapport of graph F. We assume the contrary, i.e. there exists an $(u, b) \in Q$ such that $(u, b) \in \mathcal{G}_F^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap A_{3\varepsilon}$, then follows that $c - 2\varepsilon \le b \le c + 2\varepsilon$ and $d_g((u, b), A) \le 3\varepsilon$. From these we have:

1. If $d(A, Q) > 0$ using the relation (2.3a) we have

$$\varepsilon < \frac{d(A, Q)}{3} \le \frac{d(A, (u, b))}{3} \le \varepsilon ,$$

which is a contradiction.

2. If $d(A, Q) = 0$, from the relation (2.3b) we have

$$\varepsilon < \frac{c_A - \sup \mathcal{G}_F(Q)}{2} = \frac{c - \sup \mathcal{G}_F(Q)}{2} \le \frac{c - b}{2} .$$

But from the relation $c - 2\varepsilon \le b$ we get a contradiction.

If $c > c_A$ we prove that $Q \subset C_{\text{graph}\, F}(\mathcal{G}_F^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap (\overline{\Gamma_0})_{2\varepsilon})$. We assume that there exists $(u, b) \in Q$ such that $c - 2\varepsilon \leq b \leq c + 2\varepsilon$. From this relation and relation (2.3c) we get

$$\varepsilon < \frac{c - \sup \mathcal{G}_F(Q)}{2} \leq \frac{c - b}{2} \leq \varepsilon,$$

which is a contradiction. Therefore from (2.3g) we get the relation (2.3i).

Step 4. It is clear, that there exists an $U_0 \in \Gamma_0$ such that

$$\sup \mathcal{G}_F(U_0) \leq c + \varepsilon' \tag{2.3j}$$

Because Γ_0 is invariant with respect to (F, Q)-deformation, from the relation $\eta((u, b), 0) = (u, b)$ for every $(u, b) \in \text{graph}\, F$ and from (2.3i), (2.3f) we have that $\eta(U_0, 1) \in \Gamma_0$.

Step 5. We have that

$$\eta(U_0, 1) \cap A \subset \eta(U_0 \cap A_\varepsilon, 1). \tag{2.3k}$$

Indeed, let $w_1 \in \eta(U_0, 1) \cap A$. Then there exists an $w_2 \in U_0$ such that $w_1 = \eta(w_2, 1)$. But $d_g(w_2, \eta(w_2, 1)) \leq \varepsilon$, therefore $d_g(w_2, A) \leq \varepsilon$, i.e. $w_2 \in A_\varepsilon \cap U_0$. In conclusion $w_1 \in \eta(A_\varepsilon \cap U_0, 1)$.

If $c = c_A$, using the fact that $\eta(U_0, 1) \in \Gamma_0$ and the relations (2.3k), (2.3h), (2.3j) we obtain

$$c \leq \sup \mathcal{G}_F(\eta(U_0, 1) \cap A) \leq \sup \mathcal{G}_F(\eta(U_0 \cap A_\varepsilon, 1)) \leq c - \varepsilon',$$

which is a contradiction.

If $c > c_A$, let us consider $U_0 \in \Gamma_0$ as in relation (2.3j). Then from (2.3h) we have $\eta(U_0, 1) \subset \mathcal{G}_F^{c-\varepsilon'}$. Since $\eta(U_0, 1) \in \Gamma_0$ we have $c = \inf_{U \in \Gamma_0} \sup \mathcal{G}_F(U) \leq \sup \mathcal{G}_F(\eta(U_0, 1)) \leq c - \varepsilon'$, which is a contradiction. The proof of theorem is complete.

In the smooth case, similar results have been obtained by Brezis and Nirenberg [1], Ghoussoub [8] and Willem [17], in non-smooth case by Fang [6], Ribarska-Tsachev-Krastanov [15], [16]. As a direct consequence of the above result is Theorem 2.12 from [7].

Corollary 2.4. *Let X be a complete metric space, and $F : X \to \bar{\mathbb{R}}$ be a multivalued mapping with closed graph. We assume that, there exists a closed subset A of* graph F, *and exists $Q \subset$* graph F, *and $\Gamma_0 \subset \Gamma(Q)$ nonempty and invariant with respect to (F, Q)-deformation such that Γ_0 intersect A. In addition we suppose that*

$$\inf_{U \ \Gamma_0} \sup \mathcal{G}_F(U \cap A) \geq \sup \mathcal{G}_F(Q)$$

with strict inequality if $d(A, Q) = 0$. Let $c = \inf_{U \in \Gamma_0} \sup \mathcal{G}_F(U)$.

If $c \in \mathbb{R}$, and F satisfies the condition $(PS)_c$, then we have

$$K_c \times \{c\} \cap \overline{\Gamma}_0 \neq \emptyset, \qquad \text{where} \quad \overline{\Gamma}_0 = \overline{\bigcup_{U \in \Gamma_0} U}.$$

Moreover, if $c = \inf_{U \in \Gamma_0} \sup \mathcal{G}_F(U \cap A)$ then we have $K_c \times \{c\} \cap A \neq \emptyset$.

Proof. Theorem 2.3 implies the existence of a sequence $\{(u_n, b_n)\} \subset \text{graph } F$ such that:
Case I. $\mathcal{G}_F(u_n, b_n) \to c,\ d(u_n, b_n), \overline{\Gamma}_0) \to 0,\ |dF|\,(u_n, b_n) \to 0.$
Case II. $\mathcal{G}_F(u_n, b_n) \to c,\ d(u_n, b_n), A) \to 0,\ |dF|\,(u_n, b_n) \to 0.$
Using the condition $(PS)_c$ and the fact that the sets $\overline{\Gamma}_0$ and A are closed we get that $K_c \times \{c\} \cap \overline{\Gamma}_0 \neq \emptyset$ and $K_c \times \{c\} \cap A \neq \emptyset$, respectively.

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