Acta Universitatis Carolinae. Mathematica et Physica

Warren B. Moors; Sivajah Somasundaram

Some recent results concerning weak Asplund spaces

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 43 (2002), No. 2, 67--86

Persistent URL: http://dml.cz/dmlcz/702085

Some Recent Results Concerning Weak Asplund Spaces

WARREN B. MOORS and SIVAJAH SOMASUNDARAM

Hamilton

Received 14. March 2002

This paper provides a gentle introduction to the study of weak Asplund spaces. It begins with a brief historical review of the development of weak Asplund spaces; starting from the earliest results concerning the differentiability of convex functions on $\mathbb{R}$ through to the most recent developments concerning possible characterization of these spaces. Along the way the classes of Gâteaux differentiability spaces, Stegall spaces and fragmentable spaces are introduced and their relationship with weak Aslund spaces reviewed. Following this, we summarize some of the most recent attempts at distinguishing the classes of weak Asplund spaces; Stegall spaces and fragmentable spaces. We conclude the paper by examining a class of topological spaces, namely the class of weakly Stegall spaces, that may be useful in the problem of distinguishing the class of weak Asplund spaces from the class of Gâteaux differentiability spaces.

1. Introduction

This paper has been designed to be read by someone who is presently unfamiliar with the study of weak Asplund spaces, with the aim of leading them through the subject and bringing them to the forefront in several areas of the subject. To facilitate this the paper has been divided into three sections.

In the first section we give a brief historical account of the development of weak Asplund spaces, then in section two we present some of the most recent attempts at distinguishing the various classes of Banach spaces that have been associated with Asplund spaces. Finally, in section three we develop the theory of weakly

Department of Mathematics, The University of Waikato, Private Bag 3105, Hamilton, New Zealand
e-mail: moors@math. waikato.ac.nz; e-mail: ss15@math.waikato.ac.nz

AMS2 (2000) subject classification· 54C60, 26E25, 54C10.

Key words: weak Asplund, Stegall space, weakly Stegall space, fragmentable space.

Stegall spaces and explore how these spaces are related to the study of weak Asplund spaces.

A real-valued function φ defined on a non-empty open convex subset A of a normed linear space X is said to be *convex* if,

$$\varphi(\lambda x + (1 - \lambda) y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y) \quad \text{for all } x, y \in A \quad \text{and } \lambda \in [0, 1].$$

We say that φ is *Gâteaux differentiable* at a point $x \in A$ if there exists a continuous linear functional $x^* \in X^*$ such that

$$x^*(y) = \lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \quad \text{for all } y \in X.$$

In this case, the linear functional x^* is called the *Gâteaux derivative of* φ *at* x. If the limit above is approached uniformly with respect to all $y \in B_X$-the closed unit ball of X, then φ is said to be *Fréchet differentiable* at the point $x \in A$ and x^* is called the *Fréchet derivative of* φ *at* x. The differentiability properties of convex functions on Banach spaces have been studied for many years. The first result concerning the differentiability of convex functions is the classical result that every continuous convex function defined on $\mathbb{R}$ is differentiable everywhere except for (at most) a countable set (see [30, p. 9]). This result generalizes to $\mathbb{R}^n$ (either by using Fubini's theorem or the Kuratowski-Ulam theorem) except that one must replace "countable set" by either "measure zero set" or "first category set" (see [30, p. 11]). Indeed even the continuous convex function $(x, y) \mapsto |x|$ is non-differentiable on an uncountable set. The question of how to extend these results to infinite dimensional spaces has several difficulties. For instance, what does one mean by "except for a small set" in infinite dimensions? The two natural candidates are null sets (in a measure theoretic sense) and first category sets (in a Baire categorical sense). The first notion leads to difficulties as there is no natural notion of Haar-measure on an infinite dimensional Banach space (as it is not locally compact). However, the other notion of "smallness" readily lends itself to infinite dimensions. Indeed, using this approach S. Mazur, [24] showed in 1933 that every continuous convex function defined on a separable Banach space is Gâteaux differentiable everywhere except for (at most) a first category set. This result was revisited three decades later by J. Lindenstrauss, [23] who showed in 1963 that for reflexive separable Banach spaces the conclusions of Mazur's theorem may be strengthened to: Fréchet differentiable everywhere except for (at most) a first category set.

Five years later (i.e., in 1968) E. Asplund introduced two classes of Banach spaces defined according to the differentiability properties of the continuous convex functions defined on them (see [1]). He called the Banach spaces on which every continuous convex function defined on a non-empty open convex subset is Gâteaux differentiable everywhere except at the points of a first category set, *weak differentiability spaces* while he called the Banach spaces on which every continuous convex function defined on a non-empty open convex subset is Fréchet differentiable everywhere except on a first category set *strong differentiability spaces*. These spaces have subsequently become known as *weak Asplund spaces* (see [22]) and *Asplund spaces* respectively (see [27]). In 1968 Asplund showed that every Banach space that can be renormed to have an equivalent rotund (equivalent locally uniformly rotund) dual norm is a weak differentiability (strong differentiability) space (see [1]). This result is an improvement upon Mazur's (Lindenstrauss') result since every separable (separable reflexive) Banach space space can be renormed to have an equivalent rotund (equivalent locally uniformly rotund) dual norm.

Since the time of Asplund a considerable volume of literature has been written on Asplund spaces and it was one of the major achievements of functional analysis in the late 70's (in fact, in 1978) when the class of Asplund spaces was characterized as those Banach spaces whose dual spaces possess the Radon-Nikodým property. This was the culmination of the results from the papers [27], [33] and [34]. Subsequent to this many other characterizations as Asplund spaces have been discovered (see for example [4], [7], [10] and [30]).

By contrast, our knowledge of weak Asplund spaces is rather thin. However, several partial results are known and are summarized in [8].

In an attempt to characterize the class of weak Asplund spaces Larmen and Phelps (in 1979) considered the following class of Banach spaces (see [22]). A Banach space X is called a *Gâteaux differentiability space* (or GDS for short) if each continuous convex function defined on a non-empty open convex subset A or X is Gâteaux differentiable at the points of a dense (but not necessarily a G_δ) subset of A. Therefore, every weak Asplund space is a Gâteaux differentiability space. However, the status of the reverse implication remains unresolved. The importance of Gâteaux differentiability spaces (apart from their obvious similarily to weak Asplund spaces) stems from the fact that they admit a dual characterization analogous to that of Asplund spaces. The proof of this characterization was done in two parts. The first, in 1979 by Larmen and Phelps (see [22]) and the second by M. Fabian in 1988 (unpublished).

Theorem 1 [30, Proposition 6.5] *A Banach space X is a Gâteaux differentiability space if, and only if, every non-empty weak* compact convex subset of X^* is the weak* closed convex hull of its weak* exposed points.*

Recall that a point x^* in a weak* compact convex subset C of X^* is *weak* exposed* there exists an element $x \in X$ such that $x^*(x) > y^*(x)$ for all $y^* \in C \setminus \{x^*\}$. That is, the weak* continuous linear functional $\hat{x} : X^* \to \mathbb{R}$ defined by, $\hat{x}(y^*) := y^*(x)$ attains its maximum value on C at the single point x^*.

In another attempt to characterize the class of weak Asplund spaces C. Stegall (in 1983) introduced the following class of topological spaces, which are defined in terms of minimal uscos (see [35]). A set-valued mapping $\varphi : X \to 2^Y$ acting between topological spaces X and Y is called an *usco* mapping if for each $x \in X$, $\varphi(x)$ is a non-empty compact subset of Y and for each open set W in Y, $\{x \in X : \varphi(x) \subseteq W\}$ is open in X. An usco mapping $\varphi : X \to 2^Y$ is called a *minimal usco* if its graph does not properly contain the graph of any other usco defined on X. A topological space X is said to belong to the class of *Stegall spaces* if for every Baire space B and minimal usco $\varphi : B \to 2^X$, φ is single-valued at the points of a residual subset of B. Correspondingly, we say that a Banach space X belongs to *class*$(\tilde{\mathscr{S}})$ if $(X^*, \text{weak*})$ belongs to the class of Stegall spaces.

The relationship between Stegall spaces and weak Asplund spaces is established through the subdifferential mapping. Let $\varphi : A \to \mathbb{R}$ be a continuous convex function defined on a non-empty open convex subset A of a Banach space X. Then the *subdifferential mapping of* φ is the mapping $\partial\varphi : A \to 2^{X^*}$ defined by,

$$\partial\varphi(x) := \{x^* \in X^* : x^*(y - x) \leq \varphi(y) - \varphi(x) \quad \text{for all} \quad y \in A\}.$$

It is well-known that the subdifferential mapping is a norm-to-weak* usco (see [30, p. 19]). To establish the relationship between Stegall spaces and weak Asplund spaces we need one more fact.

Lemma 1 [5]. *Let $\varphi : X \to 2^Y$ be an usco mapping acting between topological spaces X and Y. Then there exists a minimal usco mapping $\psi : X \to 2^Y$ such that $\psi(x) \subseteq \varphi(x)$ for all $x \in X$ (i.e., every usco mapping contains a minimal usco mapping).*

Theorem 2 [35]. *Every member of class$(\tilde{\mathscr{S}})$ is weak Asplund.*

Proof. Suppose $X \in \text{class}(\tilde{\mathscr{S}})$. Let $A \to \mathbb{R}$ be a continuous convex function defined on a non-empty open convex subset A of X and let $\psi : A \to 2^{X^*}$ be a minimal usco on A such that $\psi(x) \subseteq \partial\varphi(x)$ for all $x \in A$. Since A is a Baire space and $(X^*, \text{weak*})$ belongs to the class of Stegall spaces there exists a residual subset R of A on which ψ is single-valued. Now let $\sigma : A \to (X^*, \text{weak*})$ be any selection of ψ and let $x \in R$. We claim that φ is Gâteaux differentiable at x. For any $y \in X$ and $\lambda > 0$ we have, by the definition of $\partial\varphi$, that

$$\sigma(x)(y) \leq \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \leq \sigma(x + \lambda y)(y).$$

Since σ is norm-to-weak* continuous at x we have that,

$$\sigma(x)(y) = \lim_{\lambda \to 0+} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda}.$$

This means that φ is Gâteaux diffentiable at x with derivative $\sigma(x)$; which completes the proof. □

To date, the class$(\tilde{\mathscr{S}})$ spaces are the largest known well-behaved subclass of weak Asplund spaces.

A further class of topological spaces that has played a role in the study of weak Asplund spaces is the class of fragmentable spaces. The notion of fragmentability was introduced by Jayne and Rogers in 1985 (see [11]) in regard to selection theorems. However, this class of spaces also has implications for the study of weak Asplund spaces. A topological space X is said to be *fragmentable* if there exists a metric d on X such that for every $\varepsilon > 0$ and every non-empty set A of X there exists a non-empty subset B of A that is relatively open in A and d-diam$(B) < \varepsilon$. The corresponding notion for Banach spaces is the following: A Banach space X belongs to $\tilde{\mathscr{F}}$ if (X^*, weak^*) is fragmentable.

To establish the connection between fragmentability and Stegall spaces we need the following characterization.

Lemma 2 [8]. *Let $\varphi : X \to 2^Y$ be an usco mapping acting between topological spaces X and Y. Then φ is a minimal usco if, and only if, for each pair of open subsets U of X and W and Y with $\varphi(U) \cap W \neq \emptyset$ there exists a non-empty open subset $V \subseteq U$ such that $\varphi(V) \subseteq W$.*

Theorem 3 [32]. *Every fragmentable space belongs to the class of Stegall spaces. In particular, $\tilde{\mathscr{F}} \subseteq$ class$(\tilde{\mathscr{S}})$.*

Proof. Let X be a fragmentable topological space and let d be a fragmenting metric on X. Let B be a Baire space and $\varphi : B \to 2^X$ be a minimal usco. For each $n \in \mathbb{N}$, consider the set

$$O_n := \bigcup\{\text{open sets } V : d - \text{diam}[\varphi(V)] < 1/n\}.$$

Clearly each O_n is open. We claim that each O_n is dense in B. To this end, let U be a non-empty open subset of B. Since X is fragmented there exists an open set W in X, such that $W \cap \varphi(U) \neq \emptyset$ and d-diam$[W \cap \varphi(U)] < 1/n$. From the minimality of φ there exists a non-empty open set V of U such that $\varphi(V) \subseteq W$. Hence d-diam$[\varphi(V)] < 1/n$ and so $\emptyset \neq V \subseteq O_n \cap U$. This proves that O_n is dense in B. It now follows that φ is single-valued at each point of $\bigcap_{n \in \mathbb{N}} O_n$; which completes the proof. □

In summary, the relationship between all these spaces is:

$$\tilde{\mathscr{F}} \subseteq \text{class}(\tilde{\mathscr{S}}) \subseteq \text{weak Asplund spaces} \subseteq \text{Gâteaux differentiability spaces.}$$

The question as ti whether any of these set-inclusions can be reversed is considered in the next section of this paper.

After Asplund showed that every Banach space that can be equivalently renormed to have a rotund dual norm is weak Asplund there was a considerable amount of interest in showing that every Banach space that admits an equivalent Gâteaux smooth norm is weak Asplund. The fact that this would provide a true generalization of Asplund's result follows from the fact that there are Banach spaces (e.g., $C[0, \omega_1]$) with smooth norms that cannot be equivalently renormed to have a rotund dual norm (see [38]). (Of course it is well-known that if the dual norm is rotund then the original norm is smooth (see [10, p. 107])). The first partial solution to this was obtained by Borwein and Preiss in 1987 (see [3]) when they showed that every smooth Banach space is a Gâteaux differentiability space. Finally in 1990, 22 years after Asplund's result, it was eventually shown by Preiss, Phelps and Namioka (see [31]) that every smooth Banach space is weak Asplund. Subsequently, this result has been improved (in 1992) of the following: "Every Banach space that admits a smooth Lipschitz bump function belongs to $\tilde{\mathscr{F}}$" (see [9]). A similar result may also be found in [21].

2. Distinguishing the classes of spaces

In this section of the paper we will show that under some additional set theoretic assumptions the classes: $\tilde{\mathscr{F}}$, class$(\mathscr{S})$ and weak Asplund spaces are distinct. The key to achieving this goal is the consideration of the following family of compact spaces.

2.1 Kalenda compacts

Let A be an arbitrary subset of $(0, 1)$ and let

$$K_A := [(0, 1] \times \{0\}] \cup [(\{0\} \cup A) \times \{1\}].$$

If we equip this set with the order topology generated by the lexicographical (dictionary) ordering (i.e., $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$) then with this topology K_A is a compact Hausdorff space [13, Proposition 2]. In the special case of $A = (0, 1]$, K_A reduces to the well-known "double arrow" space.

Many of the basic properties of the Kalenda compacts may be found in [13]. In particular, the following result may be found there.

Theorem 4 [13, Proposition 3]. *Let A be an arbitrary subset of $(0, 1)$. Then the following properties are equivalent:*
(i) *A is countable;*
(ii) *K_A is metrizable;*
(iii) *K_A is fragmentable.*

Hence if the set A is uncountable, then $C(K_A) \notin \tilde{\mathscr{F}}$. On the other hand, it is known that there are in fact uncountable sets A for which $C(K_A)$ is weak Asplund and so there are indeed examples of weak Asplund spaces that fail to belong to $\tilde{\mathscr{F}}$. To demonstate this we need a precise description of the duals of the $C(K_A)$ spaces. In [20], the authors give a brief description of these dual spaces, but here, for the sake of posterity, we give a more detailed description of these dual spaces.

The first step towards characterizing the dual space of $C(K_A)$ is to identify this space with something that looks more like $C[0, 1]$. (Note: Riesz's representation theorem gives a description of the dual of $C(K_A)$ in terms of regular Borel measures on K_A. However, for our purposes this representation is not so useful).

Given a subset A of $(0, 1)$ we shall denote by D_A the space of all real-valued functions on $(0, 1]$ that have (i) finite right-hand limits at the points of $[0, 1)$, (ii) are left continuous at the points of $(0, 1]$ and (iii) are continuous at the points of $(0, 1) \setminus A$. We shall consider this space endowed with the supremum norm.

Theorem 5. *For every set $A \subseteq (0, 1)$, D_A is isometrically isomorphic to $C(K_A)$.*

Proof. We define an isometry T from D_A onto $C(K_A)$ as follows: $T(f)((t, 0)) := f(t)$ for all $t \in (0, 1]$ and $T(f)((t, 1)) := \lim_{t' \to t^+} f(t')$ for $t \in \{0\} \cup A$. One can check, as in [8, p. 47] that T is in fact an isometry from D_A onto $C(K_A)$. Indeed, it is routine to verify that T is a linear isometry into $C(K_A)$, so it suffices to check that T is onto. To this end, let $g \in C(K_A)$ and define $f : (0, 1] \to \mathbb{R}$ by $f(t) := g((t, 0))$ for all $t \in (0, 1]$. Then $f \in D_A$ and $T(f) = g$. $\square$

We shall now characterize the duals of these spaces in terms of functions of bounded variation. Given bounded functions f and α defined on $(0, 1]$ and $[0, 1]$ respectively and a partition $P := \{t_k : 0 \leq k \leq n\}$ of $[0, 1]$ where

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = 1,$$

the *Riemann-Stieltjes sum of f with respect to α, determined by P*, is the real number

$$S(P, f, \alpha) := \sum_{k=1}^{n} f(t_k) \cdot [\alpha(t_k) - \alpha(t_{k-1})].$$

We say that f is *Riemann-Stieltjes integrable with respect to* α if there exists a real number I such that for every $\varepsilon > 0$ there exists a partition P_ε of $[0, 1]$ such that $|S(P, f, \alpha) - I| < \varepsilon$ for all partitions P that refine P_ε. In this case I is denoted by,

$$I := \int_{[0,1]} f(t)\, \mathrm{d}\alpha(t)$$

and is called the *Riemann-Stieltjes integral of f with respect to α.*

Remark 1. *The definition of the Riemann-Stieltjes integral given here differs from the standard definition in the sense that the assigned value of the function f over the subinterval $[t_{k-1}, t_k]$ is always taken to be the evaluation at the right hand end point. The reason for this is that with the standard definition of the Riemann-Stieltjes integral, not all the functions in $D_{(0,1)}$ are Riemann-Stieltjes integrable.*

For any subset A of $(0, 1)$ we shall denote by $BV_A[0, 1]$ the space of all real-valued functions of bounded variation on $[0, 1]$ that are right continuous at the points of $(0, 1)\backslash A$ and map 0 to 0. We will consider this space endowed with the total variation norm, i.e., for each $\alpha \in BV_A[0, 1]$,

$$\|\alpha\| := \mathrm{Var}(\alpha) = \sup\left\{\sum_{k=1}^{n} |\alpha(t_k) - \alpha(t_{k-1})| : \{t_k : 0 \le k \le n\} \text{ is a partition of } [0, 1]\right\}.$$

Lemma 3 (Uniform approximation lemma). *Let A be any dense subset of $(0, 1)$, $f \in D_A$ and $\varepsilon > 0$. Then there exists a partition $P_\varepsilon := \{t_k : 0 \le k \le n\}$ of $[0, 1]$ with $t_k \in A$ for all $1 \le k < n$ such that $\|f - f_{P_\varepsilon}\|_\infty < \varepsilon$, where $f_{P_\varepsilon} : (0, 1] \to \mathbb{R}$ is defined by*

$$f_{P_\varepsilon}(t) := \sum_{k=1}^{n} f(t_k) \cdot \chi_{(t_{k-1}, t_k]}(t).$$

One can now use this lemma to prove the following theorem.

Theorem 6 [20, Theorem 1]. *Suppose that $\alpha : [0, 1] \to \mathbb{R}$ has bounded variation and $f \in D_{(0,1)}$. Then f is Riemann-Stieltjes integrable with respect to α.*

Proof. In order to show that f is Riemann-Stieltjes integrable with respect to α we need only show that for every $\varepsilon > 0$ there exists a partition P_ε of $[0, 1]$ such that

$$|S(P_\varepsilon, f, \alpha) - S(P', f, \alpha)| < \varepsilon \quad \text{for all partitions } P' \text{ that refine } P_\varepsilon.$$

An elementary calculation shows that for any $g, g' \in D_{(0,1)}$ and partition P we have that $|S(P, g, \alpha) - S(P, g', \alpha)| \le \|g - g'\| \cdot \mathrm{Var}(\alpha)$. Therefore, if we fix $\varepsilon > 0$ and choose a partition P of $[0, 1]$ such that $\|f - f_P\| < \varepsilon/(\mathrm{Var}(\alpha) + 1)$, then

$$\begin{aligned}|S(P, f, \alpha) - S(P', f, \alpha)| &\le |S(P, f, \alpha) - S(P, f_P, \alpha)| \\ &\quad + |S(P, f_P, \alpha) - S(P', f, \alpha)| \\ &\quad + |S(P', f_P, \alpha) - S(P', f, \alpha)| \\ &< 0 + 0 + \varepsilon = \varepsilon\end{aligned}$$

for all partitions P' that refine P. □

Theorem 7 [20, Theorem 2]. *Let A be any subset of $(0, 1)$. Then the dual of D_A is isometrically isomorphic to $BV_A[0, 1]$.*

Proof. Consider the mapping $T_A : BV_A[0, 1] \to D_A^*$ defined by,

$$T_A(\alpha)(x) := \int_{[0,1]} x(t)\, \mathrm{d}\alpha(t) \quad \text{for each } x \in D_A.$$

We claim that T_A is an isometry from $BV_A[0, 1]$ onto D_A^*. First let us note that T_A really does map into D_A^*. Indeed, for any given $\alpha \in BV_A[0, 1]$ it is easy to see that the mapping $x \mapsto T_A(\alpha)(x)$ is linear and it is also reasonable routine to check that $|T_A(\alpha)(x)| \le \|\alpha\| \cdot \|x\|$ for all $x \in D_A$. Therefore, for each $\alpha \in BV_A[0, 1]$, $T_A(\alpha) \in D_A^*$.

In fact $\|T_A(\alpha)\| \leq \|\alpha\|$. It should also be clear that T_A is a bounded linear operator. We will now show that T_A is onto. To this end, let $x^* \in D_A^*$.

By the Hahn Banach extension theorem there exists a $y^* \in D_{(0,1)}^*$ such that $\|x^*\| = \|y^*\|$ and $y^*|_{D_A} = x^*$ (since D_A is a subspace of $D_{(0,1)}$). Now from Theorem 3 in [25] there exists an $\alpha \in BV_{(0,1)}[0, 1]$ such that $y^* = \int_{[0,1]} x(t)\, \mathrm{d}\alpha(t)$ for all $x \in D_{(0,1)}$ and $\|a\| = \|y^*\|$. Unfortunately, this α may not lie in $BV_A[0, 1]$. So we must consider a new function $\tilde{\alpha} : [0, 1] \to \mathbb{R}$ defined by,

$$\tilde{\alpha}(t) := \alpha(t) \text{ if } t \in \{0\} \cup A \cup \{1\} \quad \text{and} \quad \tilde{\alpha}(t) := \lim_{t' \to t+} \alpha(t') \text{ if } t \in (0, 1) \backslash A.$$

Then one can check, as in Theorem 13.2 of [2] that $\|\tilde{\alpha}\| \leq \|\alpha\|$ and

$$\int_{[0,1]} x(t)\, \mathrm{d}\tilde{\alpha}(t) = \int_{[0,1]} x(t)\, \mathrm{d}\alpha(t) = y^*(x) = x^*(x) \quad \text{for all } x \in D_A.$$

Therefore, $T_A(\tilde{\alpha}) = x^*$ and $\|x^*\| = \|T_A(\tilde{\alpha})\| \leq \|\alpha\| = \|y^*\| = \|x^*\|$. Hence T_A is an isometry onto D_A^*. □

For a non-empty subset A of $(0, 1)$ we shall denote by τ_A the topology (on $BV_A[0, 1]$) of pointwise convergence on $A \cup \{1\}$. If A is dense in $(0, 1)$, then τ_A is a Hausdorff topology. Moreover, the closed unit ball in $BV_A[0, 1]$ (with respect to the total variation norm) is τ_A-compact.

Theorem 8 [20, Corollary 1]. *For a non-empty subset A of $(0, 1)$, $(BV_A[0, 1], \tau_A)$ is homeomorphic to D_A^* endowed with the weak topology generated by the functions $\chi_{(0,a)}$ with $a \in A \cup \{1\}$. If A is dense in $(0, 1)$, then τ_A is Hausdorff and the closed unit ball $B_{BV_A[0,1]}$ in $BV_A[0, 1]$ with the τ_A-topology is homeomorphic to $(B_{D_A^*}, \text{weak}^*)$. In fact the mapping T defined in the previous theorem restricted to the ball $B_{BV_A[0,1]}$, realizes such a homeomorphism.*

Proof. The proof of the first assertion is based upon the fact that for each $\alpha \in BV_A[0, 1]$ and $t \in A \cup \{1\}$, $T(\alpha)(\chi_{(0,t]}) = \alpha(t)$. The fact that T restricted to $B_{BV_A[0,1]}$, realizes a homeomorphism onto $(B_{D_A^*}, \text{weak}^*)$, follows from the fact that on $B_{D_A^*}$ the relative weak* topology and the relative topology generated by the functions $\chi_{(0,t]}$, $t \in A \cup \{1\}$ coincide (see Lemma 3). □

2.3 Distinguishing the classes of $C(K_A)$ spaces

Equipped with the representation theorem for $C(K_A)^*$ we may now present the following theorem which will enable us to distinguish the classes: $\mathscr{F}$, class($\mathscr{S}$) and weak Asplund spaces.

Theorem 9 [14, Proposition]. *Let $\mathscr{C}$ be a class of Baire metric spaces which is closed with respect to taking open subspaces and dense Baire subsets and suppose that A is dense in $(0, 1)$. Then the following assertions are equivalent:*

(i) $(C(K_A)^*, weak^*)$ *is in Stegall's class with respect to* $\mathscr{C}$*;*
(ii) K_A *is in Stegall's class with respect to* $\mathscr{C}$*;*
(iii) *For any* $B \in \mathscr{C}$ *and any continuous function* $f : B \to A$ *the function* f *has at least one local minimum or local maximum;*
(iv) *For any* $B \in \mathscr{C}$ *and any continuous function* $f : B \to A$ *there is a non-empty open set* $U \subseteq B$ *such that* f *is constant on* U*.*

Proof. (i) $\Rightarrow$ (ii). This follows from the fact that K_A is homeomorphic to a closed subspace of $(C(K_A)^*, \text{weak}^*)$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $B \in \mathscr{C}$ and let $f : B \to A$ be a continuous function. In order to obtain a contradiction, let us assume that f has no local extrema. Then the mapping $\varphi : B \to 2^{K_A}$ defined by, $\varphi(t) := \{f(t)\} \times \{0, 1\}$ is not only an usco but in fact a minimal usco. Therefore, since K_A is in the class of Stegall spaces with respect to $\mathscr{C}$ we have our desired contradiction since φ is everywhere two-valued.

(iii) $\Rightarrow$ (iv). Suppose that (iii) holds and that there is some $B \in \mathscr{C}$ and some continuous function $f : B \to A$ that is not constant on any non-empty open subset of B. Fix a metric ρ generating the topology of B. For each $n \in \mathbb{N}$ define,

$$E_n^{\max} := \{b \in B : f(b) = \max\{f(b') : \rho(b, b') < 1/n\}\};$$
$$E_n^{\min} := \{b \in B : f(b) = \min\{f(b') : \rho(b, b') < 1/n\}\}.$$

Then clearly both of the sets $E_n^{\max}$ and $E_n^{\min}$ are closed and $E := \bigcup_{n \in \mathbb{N}} E_n^{\max} \cup E_n^{\min}$ is the set of all local extrema of f on B. If one of the sets $E_n^{\max}$ or $E_n^{\min}$ has an interior point, then f is constant on a neighbourhood of it. Indeed, if b is an interior point of $E_n^{\max}$ then $B_\rho(b; \delta) \subseteq E_n^{\max}$ for some $0 < \delta < 1/n$. Let $b' \in B(b; \delta)$. Then both $f(b) \geq f(b')$ and $f(b') \geq f(b)$ hold and so $f(b) = f(b')$; which shows that f is constant on $B(b; \delta)$. Hence both of the sets $E_n^{\max}$ and $E_n^{\min}$ are closed and nowhere dense. Therefore E is a first category set and $B' := B \setminus E$ is a dense Baire subspace of B and so it belongs to $\mathscr{C}$. Thus, by (iii), $f|_{B'}$ has a local extremum at a point b. Then, by continuity of f and density of B' in B, f has a local extremum at b, with respect to B, too. Thus $b \in E$ and hence we have a contradiction.

(iv) $\Rightarrow$ (i). This is shown in [20, Theorem 4]. □

Let $\mathscr{C}$ be a class of Baire metric spaces which is closed with respect to taking open subsets and dense Baire subspaces. We will say that a subset A of $(0, 1)$ satisfies *property* $(*)$ *with respect to* $\mathscr{C}$ if for every $B \in \mathscr{C}$ and every continuous function $f : B \to A$ there exists a non-empty open set U of B such that f is constant on U.

Corollary 1 [14, Theorem 1].

(i) *If there is an uncountable dense subset* A *of* $(0, 1)$ *that satisfies property* $(*)$ *with respect to the class of all Baire metric spaces then* $C(K_A)$ *belongs to* class$(\mathscr{S})$ *but not to* $\mathscr{F}$*;*

(ii) *If there is an uncountable dense subset A of $(0, 1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces of density at most card(A) then $C(K_A)$ is a weak Asplund space that does not belong to $\tilde{\mathscr{F}}$;*

(iii) *If there is a dense subset A of $(0, 1)$ that satisfies property $(*)$ with respect to the class of all Baire metric saces of density at most card(A), but not property $(*)$ with respect to the class of all Baire metric spaces then $C(K_A)$ is a weak Asplund space that does not belong to class$(\mathscr{S})$.*

Proof. (i) From Theorem 4 it follows that $C(K_A) \notin \tilde{\mathscr{F}}$. On the other hand, it is shown in [20, Theorem 3] that a Banach space X belongs to class$(\mathscr{S})$ if, and only if, (X^*, weak^*) is in the class of Stegall spaces with respect to the class of all Baire metric spaces. The result then follows from Theorem 9.

(ii) Again from Theorem 4 it follows that $C(K_A) \notin \tilde{\mathscr{F}}$. To show that $C(K_A)$ is weak Asplund, we need the result [8, Theorem 3.2.2] that for a Banach space X if (X^*, weak^*) is in the class of Stegall spaces with respect to the class of all Baire metric spaces with density at most equal to the density of X then X is weak Asplund. The result then follows from Theorem 9 and the fact that the density of $C(K_A)$ equals card(A).

(iii) As mentioned in part (ii) if $(C(K_A)^*, \text{weak}^*)$ is in the class of Stegall spaces with respect to the class of all Baire metric spaces with density at most equal to the density of $C(K_A)$ then $C(K_A)$ is weak Asplund. The fact that $C(K_A) \notin \text{class}(\mathscr{S})$ follows directly from Theorem 9. □

Corollary 2. *Under the assumption that there exists an uncountable dense subset A of $(0, 1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces of density at most card(A), there exists a weak Asplund space that does not admit a smooth Lipschitz bump function.*

Proof. It follows from Theorem 4 that $C(K_A) \notin \tilde{\mathscr{F}}$ and so by [9] $C(K_A)$ does not admit a smooth Lipschitz bump function. However, as in Corollary 1 part (ii) $C(K_A)$ is weak Asplund. □

Remark 2. *Till this point, we have not dwelt upon the question of whether there are in fact subsets of $(0, 1)$ that satisfy any of the hypotheses of Corollary 1 or 2. For a discussion on this see [13], [14], [15] and [28]. Let us mention here though that in all cases additional set-theoretic axioms are required.*

The question of whether the classes of Gâteaux differentiability spaces and weak Asplund spaces coincide still remains. In the next section we will discuss an attempt at distinguishing these classes.

3. The class of weakly Stegall spaces

This section is divided into three subsections. In the first subsection we will define and study a class of topological spaces (i.e., the class of weakly Stegall spaces) that may help us distinguish Gâteaux differentiability spaces from weak Asplund spaces. Then in subsection two, we shall use the class of weakly Stegall spaces to present our best known candidate for a Gâteaux differentiability space that is not a weak Asplund space. Finally in subsection three, we shall examine when Banach spaces of the form $C(K)$ belong to class$(w\mathcal{S})$ i.e., when $(C(K)^*, \text{weak}^*)$ belongs to the class of weakly Stegall spaces.

3.1 Basic properties of weakly Stegall spaces

We say that a topological space Y belongs to the class of *weakly Stegall spaces* if for every complete metric space M and minimal usco $\varphi : M \to 2^Y$, φ is single-valued at some point of M. Although the definition of being weakly Stegall is very similar to that of being Stegall, the class of weakly Stegall spaces admit an internal characterization unlike the (known) situation for the class of Stegall spaces. To obtain this characterization we need to consider the following two player topological game.

Let X be a topological space. On X we consider the $\mathcal{G}(X)$-*game* played between two players Σ and Ω. Player Σ goes first and chooses a non-empty subset A_1 of X. Player Ω must then respond by choosing a non-empty relatively open subset $B_1 \subseteq A_1$. Following this, player Σ must select another non-empty subset $A_2 \subseteq B_1 \subseteq A_1$ and in turn player Ω must again select a non-empty relatively open subset B_2 of A_2. Continuing this procedure indefinitely the players Σ and Ω produce a sequence $\{(A_n, B_n) : n \in \mathbb{N}\}$ of pairs of non-empty subsets called a *play* of the $\mathcal{G}(X)$-game. We shall declare that Ω *wins* a play $\{(A_n, B_n) : n \in \mathbb{N}\}$ if, $\bigcap_{n=1}^{\infty} B_n$ is at most one point. Otherwise the player Σ is said to have won the play. By a *strategy* for the player Σ we mean a *"rule"* that specifies each move of the player Σ in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for Σ is a sequence of set-valued mappings such that $\emptyset \neq t_{n+1}(B_1, B_2, \ldots, B_n) \subseteq B_n$ for all $n \in \mathbb{N}$. The domain of each mapping t_n is precisely the set of all finite sequences $\{B_1, B_2, \ldots, B_{n-1}\}$ of length $n - 1$ with each B_j being a non-empty relatively open subset of $t_j(B_1, B_2, \ldots, B_{j-1})$ for all $1 \leq j \leq n - 1$. (Note: the sequence of length 0 will be denoted by $\emptyset$). Such a finite sequence $\{B_1, B_2, \ldots, B_{n-1}\}$ (infinite sequence $\{B_n : n \in \mathbb{N}\}$) is called a *partial t-play* (*t-play*). A strategy $t := (t_n : n \in \mathbb{N})$ for the player Σ is called a *winning strategy* if each t-play is won by Σ. We will say that the $\mathcal{G}(X)$-game on X is Σ-*unfavourable* if the player Σ does not have a winning strategy in this game.

Our characterization of weakly Stagall spaces also relies upon the following topological notion.

We say that a subset Y of a topological space (X, τ) has *countable separation in X* if there is a countable family $\{O_n : n \in \mathbb{N}\}$ of open subsets of X such that for every pair $\{x, y\}$ with $y \in Y$ and $x \in X \setminus Y$, $\{x, y\} \cap O_n$ is a singleton for at least one $n \in \mathbb{N}$. For a completely regular topological space (X, τ) we shall simply say that X has *countable separation* if in some compactification bX, X has countable separation in bX. It is shown in [19] that if X has countable separation in one compactification then X has countable separation in every compactification and that every Čech-analytic space has countable separation.

Theorem 10 [17, Theorems 4, 5 and 6]. *For a completely regular topological space (X, τ) with countable separation the following properties are equivalent:*

(i) *(X, τ) is a weakly Stegall space;*

(ii) *the $\mathcal{G}(X)$-game on X is Σ-unfavourable;*

(iii) *every minimal usco mapping $\varphi : M \to 2^X$ acting from a complete metric space M into X is single-valued at the points of a dense Baire subspace of M.*

Remark 3 [26, Theorem 2]. *If (X, τ) is of weight $\kappa \geq \aleph_0$, then the conditions* (i) *through to* (iii) *are also equivalent to:* (iv) *every minimal usco mapping $\varphi : M \to 2^X$ acting from a complete metric space M of density at most κ into X is single-valued at some point of M.*

The previous theorem enables us to establish the connection between weakly Stegall spaces and fragmentable spaces. In [18] the authors show that a space (X, τ) is fragmentable if, and only if, the player Ω has a winning strategy in the $\mathcal{G}(X)$-game played on X. This contrasts with the situation for weakly Stegall spaces which are characterized by the lack of a winning strategy for the player Σ. Hence the distinction between being fragmentable and being weakly Stegall is equivalent to the distinction between Ω having a winning strategy and Σ not having a winning strategy.

The condition that (X, τ) has countable separation in Theorem 10 can be relaxed to "(X, τ) is game determined" (see [17] for the definition of game determined spaces).

The class of weakly Stegall spaces enjoy permanence properties similar to the class of Stegall spaces. To describe and prove these properties we will need the following definition and proposition.

For a topological space X we shall denote by $\mathcal{D}(X)$ the smallest σ-algebra of subsets on X that is stable under the Souslin operation and contains all the open subsets of X.

The following result is a consequence of Proposition 5.1 and Proposition 5.3 of [36].

Proposition 1. *Let $\varphi : B \to 2^X$ be a minimal usco acting from a Baire space B into a topological space X. If $K \in \mathcal{D}(X)$ and $\varphi^{-1}(K) := \{b \in B : \varphi(b) \cap K \neq \emptyset\}$ is second category then there exist a non-empty open subset U of B and a dense G_δ subset G of U such that $\varphi(G) \subseteq K$.*

Theorem 11. *Let (X, τ) be a topological space.*

(i) *Let $f : X \to Y$ be a perfect mapping onto Y. If X belongs to the class of weakly Stegall spaces then so does Y.*

(ii) *Let $f : X \to Y$ be a continuous one-to-one mapping. If Y belongs to the class of weakly Stegall spaces then so does X.*

(iii) *Let $\{X_n : n \in \mathbb{N}\}$ be a cover of X. If each $X_n \in \mathscr{D}(X)$ and belongs to the class of weakly Stegall spaces then X also belongs to the class of weakly Stegall spaces.*

(iv) *If X belongs to the class of weakly Stegall spaces and Y belongs to the class of Stegall spaces then the product $X \times Y$ belongs to the class of weakly Stegall spaces.*

Proof. The proofs of (i), (ii) and (iv) are similar to that given in Theorem 3.1.5 of [8]. So it remains to prove (iii). Let M be a complete metric space and let $\varphi : M \to 2^X$ be a minimal usco. For each $n \in \mathbb{N}$, let $M_n := \varphi^{-1}(X_n) := \{m \in M : \varphi(m) \cap X_n \neq \emptyset\}$. Then $M = \bigcup_{n=1}^{\infty} M_n$ and so there is some $n_0 \in \mathbb{N}$ such that M_{n_0} is second category. Therefore by Proposition 1, there exist a non-empty open subset U of M and a dense G_δ subset $G \subseteq U$ such that $\varphi(G) \subseteq X_{n_0}$. By Alexandroff's Theorem (see [16, p. 208]) G is completely metrizable and by Lemma 2 $\varphi|_G$ is a minimal usco. Therefore as X_{n_0} is a weakly Stegall space there is some point $m \in G$ such that $\varphi(m)$ is a singleton. □

Next we give some sufficiency conditions for a topological space to belong to the class of weakly Stegall spaces.

Lemma 4. *Let $\varphi : X \to 2^X$ be a minimal usco acting from a topological space X into a topological space Y. If U is a non-empty open subset of X and $\{C_k : 1 \leq k \leq n\}$ is a family of closed subsets of Y such that $\varphi(U) \subseteq \bigcup_{k=1}^{n} C_k$ then there exist a $k_0 \in \{1, 2, \ldots, n\}$ and a non-empty open subset V of U such that $\varphi(V) \subseteq C_{k_0}$.*

Proof. For each $k \in \{1, 2, \ldots, n\}$, let $U_k := \{t \in U : \varphi(t) \cap C_k \neq \emptyset\}$, then $\{U_k : 1 \leq k \leq n\}$ is a closed cover of U. Hence for some $k_0 \in \{1, 2, \ldots, n\}$, $V := \operatorname{int}(U_{k_0}) \neq \emptyset$. It now follows from Lemma 2 that $\varphi(V) \subseteq C_{k_0}$. □

The following lemma is an immediate consequence of Stone's well-known "lattice formulation" of the Stone-Weierstrass theorem (see [16, p. 244] or [37]).

Lemma 5. *Let K be a compact subset of a topological space T and let τ_p denote the topology (on $C(T)$) of pointwise convergence on T. If L is a sublattice of $C(T)$ and $f \in \bar{L}^{\tau_p}$ then for each $\varepsilon > 0$ there exist an $l_\varepsilon \in L$ and an open set U_ε containing K such that $|f(k) - l_\varepsilon(k)| < \varepsilon$ for all $k \in U_\varepsilon$.*

A topological space K is said to be *countably determined* if there exists an usco mapping ψ from a separable metric space P onto K, i.e., $K = \psi(P)$.

Theorem 12. *Let K be a countably determined topological space. Then $C_p(K)$ belongs to the class of weakly Stegall spaces.*

Proof. Let $\varphi : M \to 2^{C(K)}$ be a minimal usco acting from a complete metric space M into $C(K)$. We will show that there is a point $t_\infty \in M$ such that $\varphi(t_\infty)$ is singleton. Since K is a countably determined space there exists a separable metric space P (with countable base $\{U_n : n \in \mathbb{N}\}$) and an usco mapping $\psi : P \to 2^K$ from P onto K, i.e., $K = \psi(P)$. For each $n \in \mathbb{N}$ we shall define $p_n : C(K) \to \mathbb{R} \cup \{\infty\}$ by,

$$p_n(f) := \sup\{|f(k)| : k \in \psi(U_n)\}$$

and $B_n \subseteq C(K)$ by, $B_n := \{f \in C(K) : p_n(f) \leq 1\}$. (Note: each B_n is τ_p-closed and convex). We shall also let $\sigma : \mathbb{N} \to \mathbb{N}$ be any mapping onto $\mathbb{N}$ such that for each $n \in \mathbb{N}$ σ $^1(n)$ is cofinal in $\mathbb{N}$. We now proceed inductively.

Step 1: Choose $f_1 \in \varphi(V_0)$, with $V_0 := M$ and let $L_1 := \langle f_1 \rangle = \{f_1\}$ (i.e., the lattice generated by f_1). Then put $s_1 := \sup\{\min\{p_{\sigma(1)}(f - l) : l \in L_1\} : f \in \varphi(V_0)\}$. If $s_1 = \infty$, then choose a non-empty open subset $V_1 \subseteq \overline{V_1} \subseteq V_0$ such that

$$\operatorname{diam}(V_1) < 1 \quad \text{and} \quad \varphi(V_1) \cap [L_1 + B_{\sigma(1)}] = \emptyset.$$

Note: It is possible to do this since φ is a minimal usco, $L_1 + B_{\sigma(1)}$ is τ_p-closed and $\varphi(V_0) \not\subseteq L_1 + B_{\sigma(1)}$. Otherwise (i.e., $s_1 < \infty$ and $\varphi(V_0) \subseteq L_1 + s_1 B_{\sigma(1)}$) one of the two cases must hold: (a) $s_1 = 0$ or (b) $0 < s_1 < \infty$. In case (a) $\varphi(V_0) \subseteq L_1 + s_1 B_{\sigma(1)} = L_1$ and so there exists a non-empty open subset $V_1 \subseteq \overline{V_1} \subseteq V_0$ such that

$$\operatorname{diam}(V_1) < 1 \quad \text{and} \quad \varphi(V_1) = \{l_1\} \text{ for some } l_1 \in L_1.$$

In case (b) we may choose a non-empty open subset $W \subseteq V_0$ such that

$$\varphi(W) \cap [L_1 + (s_1/2)\, B_{\sigma(1)}] = \emptyset.$$

Note: It is possible to do this since φ is a minimal usco, $L_1 + (s_1/2)\, B_{\sigma(1)}$ is τ_p-closed and $\varphi(V_0) \not\subseteq L_1 + (s_1/2)\, B_{\sigma(1)}$. Now by Lemma 4 there exists a non-empty open subset $V_1 \subseteq \overline{V_1} \subseteq W \subseteq V_0$ such that

$$\operatorname{diam}(V_1) < 1 \quad \text{and} \quad \varphi(V_1) \subseteq l_1 + s_1 B_{\sigma(1)} \text{ for some } l_1 \in L_1.$$

In both cases (a) and (b) $\inf\{\min\{p_{\sigma(1)}(f - l) : l \in L_1\} : f \in \varphi(V_1)\} \geq s_1/2$.

In general, suppose that we have completed the first n steps of the induction. Then we will have constructed the sets: $V_n \subseteq \overline{V_n} \subseteq V_{n-1} \subseteq \ldots V_2 \subseteq \overline{V_2} \subseteq V_1$ such that $\operatorname{diam}(V_j) < 1/j$ for all $1 \leq j \leq n$ and finite lattices $L_j := \langle f_1, f_2, \ldots, f_j \rangle$ with $f_j \in \varphi(V_{j-1})$ for all $1 \leq j \leq n$ and extended real numbers $\{s_j : 1 \leq j \leq n\}$ defined by, $s_j := \sup\{\min\{p_{\sigma(j)}(f - l) : l \in L_j\} : f \in V_{j-1}\}$ such that either;

(i) $\varphi(V_j) \cap [L_j + B_{\sigma(j)}] = \emptyset$ (in the case $s_j = \infty$) or

(ii) $\inf\{\min\{p_{\sigma(j)}(f - l) : l \in L_j\} : f \in \varphi(V_j)\} \geq s_j/2$ and $\varphi(V_j) \subseteq l_j + s_j B_{\sigma(j)}$ for some $l_j \in L_j$ (in the case $0 \leq s_j < \infty$).

Step $(n + 1)$: Choose $f_{n+} \in \varphi(V_n)$ and let $L_{n+1} := \langle f_1, f_2, \ldots, f_{n+1} \rangle$ (i.e., the latice generated by $\{f_1, f_2, \ldots, f_{n+1}\}$; which is a finite set). Then put

$$s_{n+1} := \sup\{\min\{p_{\sigma(n+1)}(f - l) : l \in L_{n+1}\} : f \in \varphi(V_n)\}.$$

If $s_{n+1} = \infty$ then choose a non-empty open set $V_{n+1} \subseteq \overline{V_{n+1}} \subseteq V_n$ such that

$$\operatorname{diam}(V_{n+1}) < 1/(n + 1) \quad \text{and} \quad \varphi(V_{n+1}) \cap [L_{n+1} + B_{\sigma(n+1)}] = \emptyset.$$

Note: It is possible to do this since φ is a minimal usco, $\varphi(V_n) \nsubseteq L_{n+1} + B_{\sigma(n+1)}$ and $L_{n+1} + B_{\sigma(n+1)}$ is τ_p-closed. Otherwise, (i.e., $s_{n+1} < \infty$ and $\varphi(V_n) \subseteq L_{n+1} + s_{n+1}B_{\sigma(n+1)}$) one of the two cases must hold: (a) $s_{n+1} = 0$ or (b) $0 < s_{n+1} < \infty$. In case (a) $\varphi(V_n) \subseteq L_{n+1} + s_{n+1}B_{\sigma(n+1)} = L_{n+1}$ and so by Lemma 4 there exists a non-empty open subset $V_{n+1} \subseteq \overline{V_{n+1}} \subseteq V_n$ such that

$$\operatorname{diam}(V_{n+1}) < 1/(n+1) \quad \text{and} \quad \varphi(V_{n+1}) = \{l_{n+1}\} \text{ for some } l_{n+1} \in L_{n+1}.$$

In case (b) we may choose a non-empty open subset $W \subseteq V_n$ such that

$$\varphi(W) \cap [L_{n+1} + (s_{n+1}/2)\, B_{\sigma(n+1)}] = \emptyset.$$

Note: This is possible since φ is a minimal usco, $L_{n+1} + (s_{n+1}/2)\, B_{\sigma(n+1)}$ is τ_p-closed and $\varphi(V_n) \nsubseteq L_{n+1} + (s_{n+1}/2)\, B_{\sigma(n+1)}$. Now by Lemma 4 there exists a non-empty open subset $V_{n+1} \subseteq \overline{V_{n+1}} \subseteq W \subseteq V_0$ such that

$$\operatorname{diam}(V_{n+1}) < 1/(n+1) \quad \text{and} \quad \varphi(V_{n+1}) \subseteq l_{n+1} + s_{n+1}B_{\sigma(n+1)} \text{ for some } l_{n+1} \in L_{n+1}.$$

In both cases (a) and (b) $\inf\{\min\{p_{\sigma(n+1)}(f - l) : l \in L_{n+1}\} : f \in \varphi(V_{n+1})\} \geq s_{n+1}/2$. This completes the induction.

Let $\{t_\infty\} := \bigcap_{n=1}^{\infty} V_n$. We claim that φ is single-valued at t_∞. To justify this assertion let us consider the following argument. Let $f, g \in \varphi(t_\infty)$ and $k \in K$. We need to show that $|f(k) - g(k)| = 0$. To this end, let ε be an arbitrary positive real number less that 1. Since ψ is onto there exists a $p \in P$ such that $k \in \psi(p)$. Let $f_\infty \in \varphi(t_\infty)$ be a τ_p-cluster-point of the sequence $(f_n : n \in \mathbb{N})$. The existence of such a cluster-point follows from the fact that φ is an usco mapping. Then $f_\infty \in \overline{\{f_n : n \in \mathbb{N}\}}^{\tau_p} \subseteq \overline{L_\infty}^{\tau_p}$, where $L_\infty := \bigcup_{n \in \mathbb{N}} L_n$. Now since $\psi(p)$ is compact and L_∞ is a lattice we have by Lemma 5 that there exists an $l_\infty \in L_\infty$ and an open set U containing $\psi(p)$ such that $|f_\infty(k') - l_\infty(k')| < \varepsilon/4$ for all $k' \in U$.

Since ψ is an usco there exists an $n_0 \in \mathbb{N}$ such that $p_{n_0}(f_\infty - l_\infty) < \varepsilon/4$. Next, if we choose i sufficiently large so that $l_\infty \in L_i$ and $\sigma(i) = n_0$, then at the ith stage of the induction $s_i < \infty$ since $f_\infty \in \varphi(V_i) \cap [L_i + B_{\sigma(i)}] \neq \emptyset$ and so

$$\varepsilon/4 > p_{n_0}(f_\infty - l_\infty) \geq \inf\{\min\{p_{n_0}(f - l) : l \in L_i\} : f \in \varphi(V_i)\} \geq s_i/2.$$

Therefore $0 \leq s_i \leq \varepsilon/2$. Moreover, by the construction we have that

$$f, g \in \varphi(t_\infty) \subseteq \varphi(V_i) \subseteq l_i + s_iB_{n_0} \subseteq l_i + (\varepsilon/2)\, B_{n_0}$$

and so $|f(k) - g(k)| \leq \varepsilon$; which completes the proof. □

A subset K of a topological space T is said to be *relatively pseudo-compact in* T if every continuous real-valued mapping defined on T is bounded on K. It follows from Lemma 3.3 in [39] that the conclusion of Lemma 5 remains valid under the assumption that K is relatively pseudo-compact in T and L is countable. In this way, we can see that the conclusion of Theorem 12 (i.e., $C_p(K)$ is weakly Stegall) remains valid under the weaker assumption that K is "pseudo-countably determined", i.e., there exists a second countable space P and a set-valued mapping ψ from P onto K such that (i) for each $p \in P$, $\psi(p)$ is relatively pseudo-compact in K and (ii) for each $p \in P$ and open set W in K with $\psi(p) \subseteq W$ there exists a neighbourhood U of p such that $\psi(U) \subseteq \bar{W}$.

Let us further note that by appealing to the game characterization of fragmentability given in [19] one can modify the argument in Theorem 12 to show the following: "Suppose K is pseudo-countably determined and $B \subseteq C(K)$. If player Ω has a strategy s in the $\mathcal{G}(B)$-game played on B such that for every s-play $(A_n : n \in \mathbb{N})$ either (i) $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ or (ii) every sequence $(f_n : n \in \mathbb{N})$ with $f_n \in A_n$ for all $n \in \mathbb{N}$ has a τ_p-cluster-point in $C_p(K)$, then B is fragmented by a metric d such that the d-topology is at least as strong as the relative pointwise topology on B".

3.2 Search for a GDS that is not weak Asplund

It has been a long standing open question as to whether every Gâteaux differentiability space is a weak Asplund space. In an attempt to make some progress on this question, the authors in [26] considered the following class of Banach spaces. We say that a Banach space X belongs to *class*$(w\tilde{\mathcal{S}})$ if (X^*, weak^*) belongs to the class of weakly Stegall spaces. The relationship between class$(w\tilde{\mathcal{S}})$ spaces and both weak Asplund and Gâteaux differentiability spaces is revealed in the next theorem.

Theorem 13. *Every member of class$(w\tilde{\mathcal{S}})$ has the property that every continuous convex function defined on a non-empty open convex subset of it is Gâteaux differentiable at each point of a second category set.*

Given Theorem 13, it is natural to ask whether there is an example of a non-weak Asplund space that belongs to class$(w\tilde{\mathcal{S}})$. Unfortunately, the construction of such a Banach space remains elusive. However, below we shall present an example lf a Banach space X that belongs to class$(w\tilde{\mathcal{S}})$ but not to class$(\tilde{\mathcal{S}})$. (Disappointingly, this space is known to be weak Asplund, [14]).

Example 1. *If there exists an uncountable dense subset A of $(0, 1)$ that satisfies property $(*)$ with respect to the class of all Baire metric spaces of density at most card(A), but not property $(*)$ with respect to the class of all Baire metric spaces then there is a Banach space X that belongs to class$(w\tilde{\mathcal{S}})$ but not to class$(\tilde{\mathcal{S}})$.*

Proof. It follows from Corollary 1 that $C(K_A) \notin \text{class}(\tilde{\mathcal{S}})$. However, from Remark 3 it follows that $C(K_A) \in \text{class}(w\tilde{\mathcal{S}})$. $\square$

We note here that if a Banach space Y belongs to class$(\tilde{\mathcal{S}})$ and $A \subseteq (0, 1)$ satisfies property $(*)$ with respect to the class of Baire metric spaces of density at most card(A) then $C(K_A) \times Y$ belongs to class$(w\tilde{\mathcal{S}})$ and so is a Gâteaux differentiability space. However, it is not at all clear whether $C(K_A) \times Y$ is weak

Asplund. Certainly, the argument used in [14] to show that $C(K_A)$ is weak Asplund will fail if the density of Y is large enough (i.e., bigger than card(A)). So our best candidate for a non-weak Asplund Gâteaux differentiability space is: $C(K_A) \times Y$ for some Banach space $Y \in$ class$(\mathscr{S})$.

3.3 $C(K)$ spaces that belong to class$(w\mathscr{S})$

In the study of weak Asplund spaces, Banach spaces of the form $C(K_A)$ have received special attention. Indeed, for spaces of this form several necessary conditions (on K) are known for $C(K)$ to be weak Asplund, e.g., every closed subspace of K must contain a dense completely metrizable subspace (see [6]). Several sufficiency conditions are also known. For example it is known that $C(K)$ belongs to $\tilde{\mathscr{F}}$ if, and only if, K is fragmentable (see [32]). Unfortunately the corresponding result for the class of Stegall spaces remains unknown. However, for the class of weakly Stegall spaces we have the following result.

Let B be a Bernstein subset of $(0, 1)$, that is, B is a subset of $(0, 1)$ such that neither B nor its complement contains any perfect compact sets (see [29, p. 23]) and put

$$A := [(0, 1/2) \cap B] \cup [1/2 + (0, 1/2) \setminus B]$$

(then A does not contain any perfect compact subsets either) and then define K_A as in section two.

Example 2 [26, Example 3]. *With A as above, K_A is a weakly Stegall space but $C(K_A) \notin$ class$(w\mathscr{S})$. In fact there exists a continuous convex function defined on $C(K_A)$ whose points of Gâteaux differentiability are at most first category.*

Therefore an interesting question is to characterize those compact spaces K for which $C(K)$ belongs to class$(w\tilde{\mathscr{S}})$. In fact even those more modest question of characterizing those subsets A of $(0, 1)$ for which $C(K_A)$ belongs to class$(w\tilde{\mathscr{S}})$ is both interesting and open.

References

[1] ASPLUND E., *Fréchet differentiability of convex functions*, Acta Math. **121** (1968), 31–47.

[2] BACHMAN G. and NARICI L., *Functional Analysis*, Academic Press, New York, 1966.

[3] BORWEIN J. M. and PREISS D., *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc. **303** (1987), 515–527.

[4] BOURGIN R. D., *Geometric aspects of convex sets with the Radon-Nikodým property*, Lecture Notes in Mathematics, **993** Springer-Verlag, Berlin, 1983.

[5] CHRISTENSEN J. P. R., *Theorems of Namioka and R. E. Johnson type for upper semicontinuous and compact valued set-valued mappings*, Proc. Amer. Math. Soc. **86** (1982), 649–655.

[6] ČOBAN M. M. and KENDEROV P. S., *Generic Gâteaux differentiability of convex functionals in $C(T)$ and the topological properties of T*, Proceedings of 15th Spring Conference of the Union of Bulgarian Mathematicians, Sljančev Brjag (1986), 141–149.

[7] DIESTEL J. and UHL J. J., JR , *Vector measures*, Amer. Math. Soc. Surveys, **15** Providence, Rhode Island, 1977.

[8] FABIAN M. J., *Gâteaux differentiability of convex functions and topology: Weak Asplund spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts. Wiley-Interscience New York, 1007.

[9] FOSGERAU M., *A Banach space with Lipschitz Gâteaux smooth bump has weak* fragmentable dual*, Part of author's Thesis. University College London, 1992, 52 – 62.

[10] GILES J. R., *Convex analysis with application in differentiation of convex functions*, Research Notes in Mathematics, **58**, Pitman, Melbourne, 1982.

[11] JAYNE J. E. and ROGERS C. A., *Borel selectors for upper semi-continuous set-valued maps*, Acta Ma h. **155** (1985), 41 – 79.

[12] KALENDA O. F. K., *Weak Stegall spaces, unpublished manuscript*, Spring 1997 (3 pages).

[13] KALENDA O. F. K., *Stegall compact spaces which are not fragmentable*, Topology Appl. **96** (1999), 121 – 132.

[14] KALENDA O. F. K., *A weak Asplund space whose dual is not in Stegall's class*, Proc. Amer. Math. Soc. **130** (2002), 2139 – 2143.

[15] KALENDA O. F. K. and KUNEN K., *On subclasses of weak Asplund spaces*, Preprint No MATH-KMA-2001/62, Charles University.

[16] KELLEY J. L., *General Topology*, Graduate Texts in Mathematics, **27** Springer-Verlag, New York, 1975.

[17] KENDEROV P. S , KORTEZOV I. S. and MOORS W. B., *Continuity points of quasi-continuous mappings*, Topology Appl. **109** (2001), 321 – 346.

[18] KENDEROV P. S and MOORS W. B., *Game characterization of fragmentability of topological spaces*, Proceedings of the 25th Spring Conference of the Union of Bulgarian Mathematicians Kazanlak, Bulgaria (1996), 8 – 18.

[19] KENDEROV P. S. and MOORS W. B., *Fragmentability and sigma-fragmentability of Banach spaces*, J. London Math Soc. **60** (1999), 203 – 223.

[20] KENDEROV P. S., MOORS W. B. and SCIFFER S., *A weak Asplund space whose dual is not weak* fragmentable*, Proc. Amer. Math. Soc. **129** (2001), 3741 – 3747.

[21] KORTEZOV I. S., *Fragmentability of the dual of a Banach space with smooth bump*, Serdica Math. J. **24** (1988), 187 – 198.

[22] LARMEN D. G. and PHELPS R. R , *Gâteaux differentiability of convex functions on Banach spaces*, J. London Math. Soc. **20** (1979), 115 – 127.

[23] LINDENSTRAUSS J., *On operators which attain their norm*, Israel J. Math. **1** (1963), 139 – 148.

[24] MAZUR S., *Über konvexe Mengen in linearen normierten Raümen*, Studia Math. **4** (1933), 70 – 84.

[25] MOORS W. B. and SCIFFER S., *Sigma-fragmentable spaces that are not countable unions of fragmentable subspaces*, Topology Appl. **119** (2002), 279 – 286.

[26] MOORS W. B. and SOMASUNDARAM S., *A weakly Stegall space that is not a Stegall space*, Proc. Amer. Math. Soc. (to appear).

[27] NAMIOKA I. and PHELPS R. R., *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735 – 750.

[28] NAMIOKA I. and POL R., *Mappings of Baire spaces into function spaces and Kadeč renorming*, Israel J. Math. **78** (1992), 1 – 20.

[29] OXTOBY J. C., *Measure and Category. A survey of the analogies between topological and measure spaces*, Graduate Texts in Mathematics, **2** Springer-Verlag, New York, 1971.

[30] PH LPS R. R., *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1993.

[31] PR ISS D., PHELPS R. and NAMIOKA I., *Smooth Banach spaces, weak Asplund spaces and monotone or usco mappings*, Israel J. Math. **72** (1990), 257 – 279.

[32] RIBARSKA N. K., *Internal characterization of fragmentable spaces*, Mathematika **34** (1987), 243 – 257.

[33] STEGALL C., *The Radon-Nikodým property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213 – 223.

[34] STEGALL C., *The duality between Asplund spaces and spaces with the Radon-Nikodým property*, Israel J. Math. **29** (1978), 408 – 412.

[35] STEGALL C., *A class of topological spaces and differentiability*, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen **10** (1983), 63 – 77.

[36] STEGALL C., *The topology of certain spaces of measures*, Topology Appl. **41** (1991), 73 – 112.

[37] STONE M. H., *The generalized Weierstrass approximation theorem*, Math. Mag. **21** (1948), 167 – 184.

[38] TALAGRAND M., *Renormages des quelques $C(K)$*, Israel J. Math. **54** (1986), 327 – 334.

[39] TROALLIC J. P., *Boundedness in $C_p(X, Y)$ and equicontinuity*, Topology Appl. **108** (2000), 79 – 89.