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A Discontinuous Function with a Connected Closed Graph

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An example of a discontinuous function on $\mathbb{R}^2$ that has a closed connected graph is given.

On the 31st Winter School in Abstract Analysis in Lhota nad Rohanovem, Czech Republic, the question has been asked if any real function f on $\mathbb{R}^2$ that has a closed and connected graph is continuous. We will prove, constructing a counterexample, that this is not the case. First we show some properties of functions with a closed graph. The following is evident.

Proposition 1. *A real function f on a topological space $\mathscr{T}$ has a closed graph if and only if for every $t \in \mathscr{T}$ the cluster values of f at t are $f(t)$ or $\pm\infty$. Hence if $f \geq 0$ has a closed graph then the set of discontinuity points coincides with the set of points where f has a cluster value ∞.*

Proposition 2. *If a real function f on a T_2 Baire space $\mathscr{T}$ (e.g. on a Euclidean space) has a closed graph then the set of continuity points of f is open dense in $\mathscr{T}$.*

Proof. See [2]. □

Proposition 3. *If a function $f : \mathbb{R} \to \mathbb{R}$ has a closed connected graph then it is continuous.*

Proof. If, for a point $a \in \mathbb{R}$, $\lim_{x \searrow a} |f(x)| = \infty$, the graph of f could be decomposed into two separated parts: graph $f \mid]-\infty, a]$ and graph $f \mid]a, \infty[$; so it would not

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[1] A simplification is due to J. Kolář.

be connected. Consequently by Proposition 1 f has a cluster value $f(a)$ at the point a from right and analogously from left. So f is peripherally continuous at a. This notion, introduced in [3], means: for each pair of open neighbourhoods U and V of a and $f(a)$ respectively, there exists an open set $G \subseteq U$ containing a such that f maps the boundary of G into V. By [1], Theorem 4, a peripherally continuous function $f: \mathbb{R} \to \mathbb{R}$ with a closed (not necessarily connected) graph is continuous. □

The following example of a function f on $\mathbb{R}^2$ with a connected closed graph shows that such a function need not be continuous.

The Example. Choose a decreasing sequence $\{a(n)\}_{n=1}^{\infty}$ and positive numbers $r(n) \leq 1/2$ such that

$$(1) \qquad 1 > a(n) \searrow 0 \qquad (n \to \infty)$$

and that the intervals $[a(n) - r(n), a(n) + r(n)] \subset]0, 1[$ are pairwise disjoint.

Then, for any $k_1 \in \mathbb{N}$, choose a decreasing sequence $\{a(k_1, n)\}_{n=1}^{\infty}$ and positive numbers $r(k_1, n) \leq 1/4$ such that

$$a(k_1) + r(k_1) > a(k_1, n) \searrow a(k_1) \qquad (n \to \infty)$$

and the interval

$$[a(k_1, n) - r(k_1, m), a(k_1, n) + r(k_1, n)] \subset]a(k_1), a(k_1) + r(k_1)[$$

are pairwise disjoint.

Inductively, having already $a(k_1, \ldots, k_N)$ and $r(k_1, \ldots, k_N)$ $(N, k_1, \ldots, k_N \in \mathbb{N})$, choose a decreasing sequence $\{a(k_1, \ldots, k_N, n)\}_{n=1}^{\infty}$ and positive numbers $r(k_1, \ldots, k_N, n) \leq 2^{-(N+1)}$ such that

$$(2) \quad a(k_1, \ldots, k_N) + r(k_1, \ldots, k_N) > a(k_1, \ldots, k_N, n\} \searrow a(k_1, \ldots, k_N) \qquad (n \to \infty)$$

and the intervals

$$(3) \quad [a(k_1, \ldots, k_N, n) - r(k_1, \ldots, k_N, n), a(k_1, \ldots, k_N, n) + r(k_1, \ldots, k_N, n)] \subset]a(k_1, \ldots, k_N), a(k_1, \ldots, k_N) + r(k_1, \ldots, k_N)[$$

are pairwise disjoint.

Define

$$(4) \qquad \mathscr{A} := \{a(k_1, \ldots, k_N); N, k_1, \ldots, k_N \in \mathbb{N}\}.$$

Furthermore, for $a = a(k_1, \ldots, k_N) \in \mathscr{A}$ and $r = r(k_1, \ldots, k_N)$ define subsets of $\mathbb{R}^2$

$$(5) \qquad \mathscr{U}(k_1, \ldots, k_N) :=$$
$$(]a - r, a[\times]r, 2^{-N} + r[) \cup (\{a\} \times]2^{-N}, 2^{-N} + r[) \cup (]a, a + r[\times]0, 2^{-N} + r[)$$

and

$$(6) \qquad \mathscr{V}(k_1, \ldots, k_N) := \overline{\mathscr{U}(k_1, \ldots, k_N)}^{\circ} =$$
$$(]a - r, a] \times]r, 2^{-N} + r[) \cup (]a, a + r[\times]0, 2^{-N} + r[).$$

As the assignment $(k_1, \ldots, k_N) \mapsto a(k_1, \ldots, k_N)$ $(N, k_1, \ldots, k_N \in \mathbb{N})$ is injective, we can denote $r_a := r(k_1, \ldots, k_N)$, $\mathcal{U}_a := \mathcal{U}(k_1, \ldots, k_N)$ and $\mathcal{V}_a := \mathcal{V}(k_1, \ldots, k_N)$ for $a = a(k_1, \ldots, k_N) \in \mathcal{A}$. The following claims are evident.

Claim 1. *For $N, M \in \mathbb{N}$, $N < M$, $\{k_1, \ldots, k_M\} \subset \mathbb{N}$ it is*

$$r(k_1, \ldots, k_M) < r(k_1, \ldots, k_N) \leq 2^{-N}.$$

Claim 2. *If $a, b \in \mathcal{A}$, $a < b$, then either the intervals $[a - r_a, a + r_a]$, $[b - r_b, b + r_b]$ are disjoint or $[b - r_b, b + r_b] \subset]a, a + r_a[$. The latter case holds iff $a = a(k_1, \ldots, k_N)$, $b = a(k_1, \ldots, k_M)$ for some $N, M \in \mathbb{N}$, $N < M$, $\{k_1, \ldots, k_M\} \subset \mathbb{N}$.*
Consequently, under the same conditions either the sets $\overline{\mathcal{U}_a}$ and $\overline{\mathcal{U}_b}$ are disjoint or $\overline{\mathcal{U}_b} \cap]0, 1[^2 \subset \mathcal{U}_a$.

Definition of the function f. Let us define

$$f(0, y) := \frac{1}{y} \quad \text{for} \quad y \in]0, 1]. \tag{7}$$

On the remaining part of the boundary of the set $[0, 1]^2$ let

$$f(x, y) := 1. \tag{8}$$

For

$$a = a(k_1, \ldots, k_N) \in \mathcal{A} \quad \text{and} \quad y \in]0, 2^{-N}] \quad \text{let} \quad f(a, y) := \frac{1}{y}. \tag{9}$$

For a point

$$(x, y) \in]0, 1[^2 \setminus \bigcup_{n=1}^{\infty} \mathcal{V}(n) \quad \text{let} \quad f(x, y) := \operatorname{dist}^{-1}((x, y), \partial(]0, 1[^2)). \tag{10}$$

Similarly, for $N, k_1, \ldots, k_N \in \mathbb{N}$ let us define f on the set

$$\mathcal{U}(k_1, \ldots, k_N) \setminus \bigcup_{n=1}^{\infty} \mathcal{V}(k_1, \ldots, k_N, n) \tag{11}$$

by

$$f(x, y) := \operatorname{dist}^{-1}((x, y), \partial(\mathcal{U}(k_1, \ldots, k_N))). \tag{12}$$

Thus the function f is defined on $[0, 1]^2$ (see below). Finally, let us extend f to the whole plane putting

$$f(x, y) = \begin{cases} f(-x, y), & (x, y) \in [-1, 0] \times [0, 1], \\ 1 & (x, y) \notin [-1, 1] \times [0, 1]. \end{cases} \tag{13}$$

Claim 3. *The points $(a(n), y)$ with $y \in]0, r(n)]$ belong to both domains used in (9) and (10) and the functional values by both definitions coincide. Thus the function f is defined by (9) and (10) (at least) on the set*

$$\mathscr{W} := \left]0, 1\right[^2 \setminus \bigcup_{n=1}^{\infty} \mathscr{U}(n).$$

Similarly, for $N, k_1, \ldots, k_N, n \in \mathbb{N}$ and $a = a(k_1, \ldots, k_N, n) \in \mathscr{A}$, the points (a, y) with $y \in \left]0, r_a\right]$ belong to both domains used in (9) and (11) and the functional values $f(a, y)$ by both definitions coincide. Thus the function f is defined by (9) and (12) (at least) on the set

$$\mathscr{W}(k_1, \ldots, k_N) := \mathscr{U}(k_1, \ldots, k_N) \setminus \bigcup_{n=1}^{\infty} \mathscr{U}(k_1, \ldots, k_N, n).$$

Proof. It suffices to prove the second part, the first one being similar. By Claim 2,

$$[a - r_a, a + r_a] \subset \left]a(k_1, \ldots, k_N), a(k_1, \ldots, k_N) + r(k_1, \ldots, k_N)\right[$$

and by Claim 1, $2^{-N} + r(k_1, \ldots, k_N) > 2r_a$, so $(a, 0)$ is the point of $\partial\mathscr{U}(k_1, \ldots, k_N)$ (defined by (5)) closest to (a, y). Hence (12) and (9) give the same value $f(a, y)$. □

Remark. The sets $\mathscr{W}$ and $\mathscr{W}(k_1, \ldots, k_N)$ $(N, k_1, \ldots, k_N \in \mathbb{N})$ are pairwise disjoint, connected and the function f restricted to any of these sets is evidently continuous. Hence any restriction of f to $\mathscr{W}$ or to $\mathscr{W}(k_1, \ldots, k_N)$ has a connected graph.

Claim 4.

$$\mathscr{W} \cup \bigcup_{a \in \mathscr{A}} \mathscr{W}_a = \left]0, 1\right[^2,$$

so by Claim 3 the function f is well defined on $\left]0, 1\right[^2$, hence by (7), (8) and (13) on the whole plane.

Proof by contradiction. Suppose $(x, y) \in \left]0, 1\right[^2 \setminus (\mathscr{W} \cup \bigcup_{a \in \mathscr{A}} \mathscr{W}_a)$. As the point $(x, y) \in \left]0, 1\right[^2$ does not belong to $\mathscr{W}$ (defined in Claim 3), it must belong to $\mathscr{U}(k_1)$ for some $k_1 \in \mathbb{N}$. Inductively, by the same argument we get a sequence $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ such that $(x, y) \in \mathscr{U}(k_1, \ldots, k_N)$ for every $N \in \mathbb{N}$. However by (5) and Claim 1 this cannot hold if $2 \cdot 2^{-N} < y$.

Claim 5. *The graph of f is connected.*

Proof. By the Remark the graph of $f \mid \mathscr{W}(k_1, \ldots, k_N)$ is connected. The closure of this graph, being again a connected set, contains by (2), (5) and (9) the points

$$(a(k_1, \ldots, k_N), y, 1/y) = \lim_{n \to \infty} (a(k_1, \ldots, k_N, n), y, 1/y) \qquad (y \in \left]0, 2^{-(N+1)}\right])$$

belonging to the graph of $f \mid \mathscr{W}(k_1, \ldots, k_{N-1})$. Thus the graph of

$$f \mid (\mathscr{W}(k_1, \ldots, k_N) \cup \mathscr{W}(k_1, \ldots, k_{N-1}))$$

is connected. By induction, the graph of f restricted to the set

$$\mathscr{W}(k_1, \ldots, k_N) \cup \mathscr{W}(k_1, \ldots, k_{N-1}) \cup \ldots \cup \mathscr{W}(k_1) \cup \mathscr{W} \cup \partial([0, 1]^2)$$

is connected (the last step by (1), (7) and (8)). This graph contains the graph of $f \mid \mathscr{W} \cup \partial([0, 1]^2)$ not depending on the choice of $k_1, \ldots, k_N$, so by Claim 4 the graph of $f \mid [0, 1]^2$ is connected and evidently the graph of f defined on the whole plane by (13) is connected, too. □

Thus we have constructed a discontinuous function f with a connected closed graph.

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