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A Scenario for Transferring High Scores

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1. Introduction

Let $\mathscr{X}$ be a Polish space with metric d and $f : \mathscr{X} \to \mathscr{X}$ a continuous function. We recall the definition of the attacking relation $x \curvearrowright_f y$ studied in our papers [2, 3, 4]:

$$x \curvearrowright_f y \Leftrightarrow_{\mathrm{df}} \forall \varepsilon > 0 \forall m :\in \mathbb{N}\ \exists \ell [\ell \geqslant m \,\&\, d(f^{\ell}(x), y) < \varepsilon].$$

Our notation is related to a familiar one in dynamical systems: $x \curvearrowright_f y$ iff $y \in \omega_f(x)$.

An important example is *Baire space*, the space of infinite sequences of natural numbers, often denoted by $\mathscr{N}$ or ${}^{\omega}\omega$, which for each finite such sequence r has the basic open set $\{\alpha \mid \alpha \upharpoonright \ell h(r) = r\}$. In that space the (backward) *shift functions* $\mathfrak{s}: \mathscr{N} \to \mathscr{N}$ is given by $\mathfrak{s}(\alpha)(n) = \alpha(n + 1)$.

The *score*, $\theta(a, f)$, of a point a in $\mathscr{X}$ with respect to the function f, is defined to be the least ordinal θ such that $A^{\theta}(a, f) = A^{\theta+1}(a, f)$, where we define recursively a shrinking sequence of sets by $A^0(a, f) = \omega_f(a)$, $A^{\nu+1}(a, f) = \{x \mid \exists y(y \in A^{\nu}(a, f) \,\&\, y \curvearrowright_f x)\}$ and for a limit ordinal λ, $A^{\lambda}(a, f) = \bigcap_{\nu<\lambda} A^{\nu}(a, f)$.

In [3] we showed, working always with the shift function $\mathfrak{s}$, that for Baire space ${}^{\omega}\omega$ or the Cantor space ${}^{\omega}7$ there are points of score any given countable ordinal. In [4] we constructed a point in Baire space of score the first uncountable ordinal, which by results of [3] is the maximum possible. An unpublished transfer theorem of Cristian Delhommé shows that a point of uncountable score and points of all countable scores will exist in Cantor space ${}^{\omega}2$.

The question arises, which other spaces and functions will permit the existence of a point of uncountable score?

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We describe hypytheses on a dynamical system $(\mathcal{X}, f)$ which will permit such a transfer. We then describe one setting in which one can establish all but the last hypothesis; then we show that an assumption of equicontinuity will yield that last hypothesis. However, experts on dynamical systems believe that, in the given setting, it is likely that f cannot be equicontinuous. So at present no dynamical system is known (to the author) to satisfy all the given hypotheses.

We work with *DC*, the mild form of the Axiom of Choice that implies that a relation is well-founded if and only if it admits no infinite descending sequences.

2. Hypotheses leading to high scores

2·0 Theorem *Suppose that $(\mathcal{X}, f)$ is a dynamical system such that there is a continuous surjection Ψ of $\mathcal{X}$ onto either ${}^{\omega}\omega$ or some ${}^{\omega}m$ with $m \geqslant 2$, satisfying these properties, where we say that x is at α rather that $\Psi(x) = \alpha$:*

(2·0·0) *for all $x \in \mathcal{X}$, $\Psi(f(x)) = \mathfrak{s}(\Psi(x))$*

(2·0·1) *for all x and y in $\mathcal{X}$, if $x \curvearrowright_f y$ then $\Psi(x) \curvearrowright_{\mathfrak{s}} \Psi(y)$*

(2·0·2) *if x is at α and $\alpha \curvearrowright_{\mathfrak{s}} \beta$, then there is a y at β with $x \curvearrowright_f y$.*

(2·0·3) *if $\alpha \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \gamma$, a is at α, c is at γ and $a \curvearrowright_f c$, then there is a point b at β with $a \curvearrowright_f b \curvearrowright_f c$.*

Then for every x, the f-score of x equals the $\mathfrak{s}$-score of $\Psi(x)$, so that there are points in $\mathcal{X}$ of all scores up to and including ω_1.

Proof: We recall the definition from [3], page 263, of the tree $T^x_y(f)$, where $x \curvearrowright_f y$. It is the set of finite sequences $(y_0, y_1, \ldots, y_n)$ such that $y_0 = y$, each $y_{i+1} \curvearrowright_f y_i$ and $x \curvearrowright_f y_n$.

Extend the definition of Ψ to such finite sequences in the natural way: $\Psi((y_0, \ldots, y_n)) = (\Psi(y_0) \ldots \Psi(y_n))$. With this extended definition, Ψ preserves length and end-extension.

2·1 Proposition *If $x \curvearrowright_f y$, $\Psi\left[T^x_y(f)\right] = T^{\Psi(x)}_{\Psi(y)}(\mathfrak{s})$.*

Proof: by (2·0·1), the restriction of Ψ to the first tree is into the second; by repeated use of (2·0·3), we see that it is onto. ⊣(2·1)

2·2 Lemma *If x is at α, y is at β, and $x \curvearrowright_f y$, then $T^x_y(f)$ is well-founded iff $T^{\alpha}_{\beta}(\mathfrak{s})$ is.*

Proof: by repeated use of (2·0·1) and (2·0·3) to transfer infinite descending sequences from one tree to the other. ⊣(2·2)

2·3 Definition If $a \curvearrowright_f b$, $T^a_b(f)$ is well-founded, and $s \in T^a_b(f)$, write $\varrho_{a,b,f}(s)$ for the rank of s in the tree $T^a_b(f)$, as given by the recursion

$$\varrho_{a,b,f}(s) = \sup\{\varrho_{a,b,f}(s^\frown\langle y\rangle) + 1 \mid s^\frown\langle y\rangle \in T^a_b(f)\}.$$

2·4 Proposition *If* $a \curvearrowright_f b$, $\emptyset \neq s \in T^a_b(f)$ *and* $\varrho_{a,b,f}(s) = \xi$, *then the last element* $\ell(s)$ *of the finite sequence* s *is in* $A^{\xi}(a,f)\setminus A^{\xi+1}(a,f)$.

Proof: by induction on ξ, much as in the proof of Lemma 2·1 of [3]. ⊣(2·4)

2·5 Corollary $\theta(a,f)$, *the f-score of a, equals* $\sup\{\varrho_{a,b,f}(\emptyset) \mid a \curvearrowright_f b\}$.

Thus the score of a point a is computable from the ranks of the various trees $T^a_b(f)$.

2·6 Proposition *Suppose the two trees are both well-founded. Then they have the same ranks:* $\varrho_{a,b,f}(\emptyset) = \varrho_{\alpha,\beta,\mathfrak{s}}(\emptyset)$.

2·7 Proposition *If x is at* α, *then the f-score of x equals the* $\mathfrak{s}$*-score of* α; *in particular if x is at* α *and* α *is of uncountable score, then so is x.*

For points of uncountable score we can also argue as follows. A point is of uncountable score iff its abode is not a Borel set. Further the image under Ψ of the abode of x is the abode of α, and the image under Ψ of the escape of x is the escape of α.

Hence if the abode and escape of x were Borel, then both the abode and the escape of α would be analytic, and therefore by Souslin's celebrated result, would be Borel.

2·8 Remark In this connection it might be worth looking again at the "original" construction of a point of uncountable score mentioned in the penultimate paragraph of [4], and there called c, where c "neatly" attacks α_T for every countable well-founded tree T. The nodes of α_T survive for $\zeta = \varrho(T)$ steps, and ζ can be arbitrarily large.

2·9 Proposition *Suppose that* $\alpha \curvearrowright_{\mathfrak{s}} \varrho \curvearrowright_{\mathfrak{s}} \varrho$. *Let a be at* α *and x be at* ϱ *with* $a \curvearrowright_f x$. *Then there are recurrent r and s such that* $a \curvearrowright_f s \curvearrowright_f x \curvearrowright_f r$ *and* $\varrho \curvearrowright_{\mathfrak{s}} \Psi(r) \curvearrowright_{\mathfrak{s}} \varrho$.

Proof: by (2·0·2) there is a y at ϱ such that $x \curvearrowright_f y$. Taking, by (2·0·3), an infinite backward sequence of ϱ's there are points y_i with $y_0 = y$, each $y_{i+1} \curvearrowright_f y_i$ and x attacking each y_i. By proposition 3.18 of [3] there is a recurrent point r with $x \curvearrowright_f r \curvearrowright_f y$, which by (2·0·1) gives $\varrho \curvearrowright_{\mathfrak{s}} \Psi(r) \curvearrowright_{\mathfrak{s}} \varrho$.

To find s, repeat the argument, inserting a chain of attacks $a \curvearrowright_f s_{i+1} \curvearrowright_f s_i \curvearrowright_f x$. ⊣(2·9)

2·10 Corollary *If x is at* β, *a at* α, $a \curvearrowright_f x$, *and* β *is in the abode of* α, *then x is in the abode of a.*

Proof: for some ϱ, $\alpha \curvearrowright_{\mathfrak{s}} \varrho \curvearrowright_{\mathfrak{s}} \varrho \curvearrowright \beta$; we can therefore find f-recurrent r with $a \curvearrowright_f r \curvearrowright_f x$.

Many things now fit well together. For example in [3] an operator Γ was introduced: $\Gamma_f(Z) = \{x \mid \exists y(y \in Z \;\&\; y \curvearrowright_f x)\}$.

2·11 Lemma $\Psi[\omega_f(x)] = \omega_{\mathfrak{s}}(\Psi(x))$.

2·12 Lemma $\Psi[\Gamma_f(Z)] = \Gamma_{\mathfrak{s}}(\Psi[Z])$.

2.13 Proposition *For all* v, $\Psi[A^v(a, f)] = A^v(\Psi(a), \mathfrak{s})$.

Proof: the previous two lemmata cover the case of 0 and successors. At limits, we must use the analysis of trees. ⊣(2·13)

2·14 Remark We have made little use of separability: it has been used only to show that any point that vanishes does so at a countable stage, thus giving the upper bound of ω_1 to the score. But for the sake of proving the existence of points of uncountable score, it shouldn't be necessary.

3. Horseshoes

Following a lecture of Jozef Bobok we make a definition. The setting is a compact (Polish) space $\mathscr{A}$ with metric d, a continuous function $f: \mathscr{A} \to \mathscr{A}$ and an integer $m \geqslant 2$. We write $\emptyset$ for the sequence of length 0, and $\varnothing$ for the empty set. Of course in many formal presentations of mathematics the two objects are the same.

3·0 Definition An (m,f)-*strong horseshoe* is a sequence $S_0, \dots S_{m-1}$ of pairwise disjoint non-empty closed sets with the property that for all i and j less than m,

$$S_i \subseteq f[S_j].$$

3·1 Remark That is a stronger requirement that the condition met by Smale's original horseshoe, which was that $S_i \cap f[S_j]$ is non-empty for each i and j.

3·2 Lemma $S_i \cap f^{-1}[S_j] \neq \varnothing$: *indeed* $f[S_i \cap f^{-1}[S_j]] = S_j$.

3·3 Lemma $S_i \cap f^{-1}[S_j] \cap f^{-2}[S_k] \neq \varnothing$; *indeed* $f^2[S_i \cap f^{-1}[S_j] \cap f^{-2}[S_k]] = S_k$.

We shall be able to generalise the above, but must first adopt a less cumbersome notation.

3·4 Definition Set $S^{\emptyset} =_{\text{df}} \bigcup_{i<m} S_i$, and for u a sequence of length $k+1$ of numbers less than m, set $S^u =_{\text{df}} S^{u \restriction k} \cap f^{-k}[S_{u(k)}]$. For $\alpha \in {}^{\omega}m$, set $S^\alpha = \bigcap_k S^{\alpha \restriction k}$.

3·5 Proposition $S^u \neq \varnothing$; *indeed* $f^k[S^u] = S_{u(k)}$.

3·6 Proposition *If* $u = s^\frown t$, *the concatenation of* s *and* $\ell = \ell h(s)$, *then* $S^u = S^s \cap f^{-\ell}[S^t]$ *and* $f^\ell[S^u] = S^t$.

Proof: The first assertion holds by the identity $g^{-1}(C \cap D) = g^{-1}(C) \cap g^{-1}(D)$. For the second, the inclusion from left to right is evident, using the first assertion. Suppose that x is in S^t. Let $v = u \restriction (\ell + 1)$. By Proposition 3·5, there is a y in S^v

with $f^{\prime}(y) = x$; but then $y \in S^s \cap f^{-\prime}(S^t)$, and so, by the first assertion again, $x \in f^{\prime}[S^u]$. ⊣(3·6)

3·7 PROPOSITION *Each S^u is a non-empty closed subset of $\mathscr{A}$.*

3·8 DEFINITION $\mathscr{X} =_{\text{df}} \{x \in \mathscr{A} \mid \forall k \geqslant 0, f^k(x) \in S^{\emptyset}\}$.

3·9 LEMMA *Each S^α, the intersection along the path α, is non-empty.*

Proof: by compactness. ⊣(3·9)

3·10 LEMMA $\mathscr{X} = \bigcup_\alpha S^\alpha = \bigcap_k \bigcup \{S^u \mid u \in {}^k m\}$.

3·11 PROPOSITION *$\mathscr{X}$ is a closed non-empty subset of $\mathscr{A}$, and is therefore a compact Polish space.*

3·12 LEMMA *If $x \in \mathscr{X}$, then $f(x) \in \mathscr{X}$.*

3·13 LENNA *If $x_k \to x$ as $k \to \infty$, and $x_k \in S^{\alpha \restriction n_k}$, where $n_k \to \infty$ with k, then $x \in S^\alpha$.*

Henceforth we work in the space $\mathscr{X}$.

A map to *m*-Cantor space

Let $\mathscr{C}$ be the product space ${}^\omega m$, where m is given the discrete topology. $\mathscr{C}$ is compact by Tychonoff. We shall use Greek letters $\alpha, \beta, \gamma, \varrho$ for members of $\mathscr{C}$.

On $\mathscr{C}$ we define the shift function $\mathfrak{s}$ by $\mathfrak{s}(\alpha)(n) = \alpha(n + 1)$.

3·14 DEFINITON For $x \in \mathscr{X}$, define $\Psi(x)(n)$ to be the $i < m$ such that $f^n(x)$ is in S_i.

3·15 PROPOSITION *For $x \in \mathscr{X}$, $\Psi(f(x)) = \mathfrak{s}(\Psi(x))$.*

3·16 REMARK This means that Ψ is an action map in the sense defined by Akin and Kolyada [1]

3·17 PROPOSITION *If $x \in \bigcap_{k \in \omega} S^{\alpha \restriction k}$, then $\Psi(x) = \alpha$; hence Ψ is surjective.*
[The compactness ensures that the intersection along α is non-empty].

3·18 PROPOSITION *Ψ is continuous.*
We shall say that x *is at* α if $\Psi(x) = \alpha$.

Lifting an attack

3·19 PROPOSITION *$\mathscr{X}$ is $\rightharpoonup_f$ closed in the sense that if x is in $\mathscr{X}$ and $x \rightharpoonup_f y$ then $y \in \mathscr{X}$.*

3·20 PROPOSITION *If $x \rightharpoonup_f y$ then $\Psi(x) \rightharpoonup_{\mathfrak{s}} \Psi(y)$.*

Proof: Let $\hat{\varepsilon} =_{\text{df}} \min_{i \neq j} d(S_i, S_j)$; thus $\hat{\varepsilon} > 0$ and has the property that two points in $S^{\emptyset}$ within distance $\hat{\varepsilon}$ of each other are in the same S_i.

Fix a natural number k. We seek ℓ such that $f^{\ell}(x)$ shadows y for k steps, in the sense that for $n < k$, $\Psi(f^{\ell}(x))(n) = \Psi(y)(n)$.

By the continuity of f, $f^2 \ldots$ and f^{k-1} at y, there are $\delta_i > 0$ such that for each $i < k$

$$d(a,y) < \delta_i \Rightarrow d(f^i(a), f^i(y)) < \hat{\varepsilon}$$

Take δ to be the minimum of the δ_i's. Pick ℓ exceeding k and large enough so that $d(f^{\ell}(x),y) < \delta$. ⊣(3·20)

3·21 Corollary *If r is f-recurrent then $\Psi(r)$ is $\mathfrak{s}$-recurrent.*

Proof: Let $\Psi(r) = \varrho$. $r \curvearrowright_f r$, so $\varrho \curvearrowright_{\mathfrak{s}} \varrho$. ⊣(3·21)

3·22 Proposition *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \beta$, and that x is at α. Then we can find y at β such that $x \curvearrowright_f y$.*

Proof: Let $\beta \restriction k = s^{n_k}(\alpha) \restriction k$. Then put $x_k = f^{n_k}(x)$. Each x_k is in $S^{\beta \restriction k}$; by compactness some subsequence of the x's converges, to y say. Then $y \in S_\beta$ and $x \curvearrowright_f y$. ⊣(3·22)

3·23 Corollary *If x is at α and the abode of α is empty, so is the abode of x.*

Proof: the abode of a point is empty iff it attacks no recurrent points. ⊣(3·23)

The clauses (2·0·0), (2·0·1) and (2·0·2) are established in the present setting by 3·15, 3·20 and 3·22.

4. The effect of equicontinuity

Suppose now that at every point x of $\mathscr{X}$, f is equicontinuous in the sense that

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall n\, \forall y\, [d(x,y) < \delta \Rightarrow d(f^n(x), f^n(y)) < \varepsilon].$$

We derive clause (2·0·3):

4·0 Lemma *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \gamma$ and that a is at α, c at γ and $a \curvearrowright_f c$. Then there is a point b at β with $a \curvearrowright_f b \curvearrowright_f c$.*

Proof: Given k, a positive integer, there are (large) integers n_k and m_k such that for each $i < k$,

$$\gamma(i) = \beta(n_k + i) = \alpha(m_k + n_k + i)$$

and such that $f^{m_k + n_k}(a) \to c$.

Set $b_k = f^{m_k}(a)$. Some subsequence, say for $k \in B \in [\omega]^{\omega}$, of those converges, to b, say. We know that $f^{n_k}(b_k) \to d$. We shall use the equicontinuity of f at b to show that $f^{n_k}(b) \to d$.

Given ε, we seek K such that for $k > K$, $k \in B$, $d(f^{n_k}(b),d) < \varepsilon$. We know that there is a K_0 such that for all $k \geqslant K_0$, $d(f^{n_k}(b_k),c) < \varepsilon/2$. Using the equicontinuity at b, we know that there is a δ such that if $d(z,b) < \delta$, then for all k, $d(f^{n_k}(z), f^{n_k}(b)) < \varepsilon/2$. For large enough $k \in B$, we shall indeed have $d(b_k,b) < \delta$. ⊣(4.0)

4·1 Remark Equicontinuity seems formally too strong, as for given k we are only interested in the point $z = b_k$.

Finally we mention a variant of (2·0·3), of which the proof has an interesting feature.

4·2 Lemma *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} B$ and that y is at β. Then there is an x at α with $x \curvearrowright_f y$.*

Proof: By Proposition 3·6, for given k and n_k with $\mathfrak{s}^{n_k}(\alpha) \restriction k = \beta \restriction k$, we may find $x_k \in S^{\alpha \restriction n_k}$ with $f^{n_k}(x_k) = y$ (and not just near y!). Let x be the limit of some convergent subsequence of the x_k's. x will be at α by Lemma 3·13. Then by the equicontinuity of f at x, given $\varepsilon > 0$ there is a $\delta > 0$ such that for each k and z $d(x,z) < \delta \Rightarrow d(f^{n_k}(k), f^{n_k}(z)) < \varepsilon$; taking z to be an x_k in the subsequence that is suitably near x, we see that $d(f^{n_k}(x),y) < \varepsilon$, and so $x \curvearrowright_f y$, as required. ⊣(4·2)

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