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On invariant CCC $\sigma$-ideals


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We re-read Reclaw's proof from [6] on invariant CCC $\sigma$-ideals of subsets of reals and obtain a reasonably stronger corollary for such ideals on the Cantor space.

1. Preliminaries. In 1998 Reclaw in [6] investigated cardinal invariants of CCC $\sigma$-ideals of subsets of reals. In particular, he showed that if such a $\sigma$-ideal $I$ is invariant, then $p \leq \non(I)$, where $p$ is a pseudointersection number (cf. [8] for more details). In this paper we analyze his proof and get an apparently stronger result for $\sigma$-ideals of subsets of the Cantor space $2^\omega$.

We use standard set-theoretical notation and terminology derived from [1]. Let us remind that the cardinality of the set of all real numbers is denoted by $c$. The cardinality of a set $X$ is denoted by $|X|$. By $[\omega]^\omega$ we denote the family of all infinite subsets of $\omega$. If $\varphi : X \rightarrow Y$ is a function then $\rng(\varphi)$ denotes the range of $\varphi$.

Let $(G, +)$ be an abelian Polish (i.e. separable, completely metrizable, without isolated points) group and let $I$ be a $\sigma$ – ideal of subsets of $G$ (we assume from now on that $I$ is proper and contains all singletons). We will consider that $I$ is invariant, that is for every $A \subseteq G$ and $g \in G$ we have $A + g = \{a + g : a \in A\} \in I$ and $-A = \{-a : a \in A\} \in I$. Moreover, we will assume that the $\sigma$ – ideal $I$ has a Borel basis i.e. every set from $I$ is contained in a certain Borel set from the ideal.

We say that $I$ is CCC (countable chain condition) if the quotient Boolean algebra $\mathcal{B}(G)/I$ in CCC, where $\mathcal{B}(G)$ is the $\sigma$-algebra of all Borel subsets of $G$. 

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We define the following cardinal invariants of $\mathcal{F}$.

\[
\text{non} (\mathcal{F}) = \min \{|B| : B \subseteq G \land B \notin \mathcal{F} \},
\]
\[
\text{cov}_i (\mathcal{F}) = \min \{|T| : T \subseteq G \land (\exists A \in \mathcal{F}) A + T = G \}.
\]

We define also an operation on the $\sigma$– ideal $\mathcal{F}$ (it was introduced by Seredyński in [7], who denoted it by $\mathcal{F}^*$)

\[
s(\mathcal{F}) = \{A \subseteq G : (\forall B \in \mathcal{F}) (\exists g \in G) (A + g) \cap B = \emptyset\}.
\]

If we apply these operations to the $\sigma$– ideals of meagre sets $\mathcal{M}$ and of null sets $\mathcal{N}$ we obtain strongly null sets $s(\mathcal{M})$ and strongly meager sets $s(\mathcal{N})$. The following is well-known

\[
\text{non} (s(\mathcal{F})) = \text{cov}_i (\mathcal{F}).
\]

We define

\[
P_{\mathcal{F}} = \{f : f \text{ is a function } \land \text{ dom } (f) \in [\omega]^\omega \land \text{ rng } (f) \subseteq 2\}.
\]

If $f \in P_{\mathcal{F}}$ then we put

\[
[f] = \{x \in 2^\omega : f \leq x\}.
\]

Let $\mathfrak{S}_2$ denotes the $\sigma$-ideal of subsets of the Cantor space $2^\omega$, which is generated by the family $\{[f] : f \in P_{\mathcal{F}}\}$. It was thoroughly investigated in [2] and [4]. We recall some properties of $\mathfrak{S}_2$, which were proved in [2].

**Fact 1.1** $\mathfrak{S}_2$ is a proper, invariant $\sigma$-ideal which contains all singletons and has a Borel basis. Every $A \in \mathfrak{S}_2$ is both meager and null. Moreover, there exists a family of size $\mathfrak{c}$ of pairwise disjoint Borel subsets of $2^\omega$ that do not belong to $\mathfrak{S}_2$. Hence $\mathfrak{S}_2$ is not CCC.

Let $A$, $S$ be two infinite subsets of $\omega$. We say that $S$ splits $A$ if $|A \cap S| = |A \setminus S| = \omega$. Let us recall a cardinal number related with a notion of splitting, introduced by Malychin in [5], namely

\[
\aleph_{\omega^\omega} = \min \{|\mathcal{S} : \mathcal{S} \subseteq [\omega]^\omega \land (\forall \mathcal{A} \in [([\omega]^\omega)^\omega]) (\exists S \in \mathcal{S}) (\forall A \in \mathcal{A}) S \text{ splits } A\}.
\]

More about cardinal numbers connected with the relation of splitting can be found in [3].

2. Reclaw’s proof revisited. In [6] Reclaw proved a theorem, which can be generalized as follows.

**Theorem 2.1** Let $\mathcal{I}$ and $\mathcal{J}$ be two $\sigma$-ideals of subsets of an abelian Polish group $G$, which are invariant and have Borel bases. If $\mathcal{I}$ is CCC then

\[
\mathcal{I} \cap s(\mathcal{J}) \subseteq \mathcal{J}.
\]

**Proof.** (Reclaw) Let $X \in \mathcal{I} \cap s(\mathcal{J})$. Assume that $X \notin \mathcal{J}$. We construct a sequence $\{F_\alpha : \alpha < \omega_1\}$ of Borel sets from $\mathcal{J}$ and a sequence $\{t_\alpha : \alpha < \omega_1\}$ of elements
of $G$. Let $t_0 = 0$ and $F_0$ be any Borel set from $\mathcal{I}$ containing $X$. Suppose that we have constructed $F_\beta$ and $t_\beta$ for $\beta < \alpha$. Then from the definition of $s(\mathcal{I})$ there exists $t_\alpha \in G$ such that

$$(X + t_\alpha) \cap \bigcup_{\beta < \alpha} F_\beta = \emptyset.$$ 

As $F_\alpha$ we take any Borel set from $\mathcal{I}$ containing $\bigcup_{\beta < \alpha} F_\beta \cup (X + t_\alpha)$. Let $G_\alpha = F_\alpha \setminus \bigcup_{\beta < \alpha} F_\beta$. Thus $\{G_\alpha : \alpha < \omega_1\}$ is a family of pairwise disjoint Borel sets such that none of them belongs to $\mathcal{I}$, as $G_\alpha \ni X + t_\alpha$ and $\mathcal{I}$ is invariant. Hence $\mathcal{I}$ is not CCC, a contradiction. □

**Corollary 2.2** Let $\mathcal{I}$ and $\mathcal{J}$ be as above. If $\mathcal{I}$ is CCC then

$$\min \{\text{non}(\mathcal{I}), \text{cov}(\mathcal{I})\} \leq \text{non}(\mathcal{J}).$$

**Proof.** It is enough to observe that $\mathcal{J} \subseteq \mathcal{J}$ implies $\text{non}(\mathcal{J}) \leq \text{non}(\mathcal{J})$. □

**Corollary 2.3** Let $\mathcal{I}$ be a $\sigma$-ideal of subsets of the Cantor space $2^\omega$ (endowed with a standard group structure), which is invariant and has a Borel basis. If $\mathcal{I}$ is CCC then

$$\aleph_0^\mathcal{I} \leq \text{non}(\mathcal{I}).$$

**Proof.** In [2] it was proved that $\text{non}(\mathcal{S}_2) = \aleph_0^\mathcal{I}$ and in [4] it was proved that $\text{cov}(\mathcal{S}_2) = \mathfrak{c}$. So it is enough to apply Corollary 2.2 for $G = 2^\omega$ and $\mathcal{J} = \mathcal{S}_2$. □

**Question.** Let $\mathcal{I}$ be an invariant CCC $\sigma$-ideal of subsets of the real line $\mathbb{R}$. Is the inequality $\aleph_0^\mathcal{I} \leq \text{non}(\mathcal{I})$ still true?

**References**