Edward Grzegorek

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$\mathfrak{c}$-Luzin sets, Nonatomic σ-Fields and σ-Independent Sets

EDWARD GRZEGOREK

Gdańsk

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It is proved that if L is a $\mathfrak{c}$-Luzin set then assuming MA + negation of CH the σ-field $\mathscr{B}_L$ of Borel subsets of L contains a nonatomic σ-field separating points. Other properties of $\mathscr{B}_L$ are also considered.

If X is a set then $|X|$ denotes then cardinality of X, $\mathscr{P}(X)$ is the power set of X, $\mathfrak{c} = 2^{\aleph_0}$, $\mathbb{R}$ is the real line. For a cardinal κ, $[X]^{\leq\kappa} = \{Y \subseteq X : |Y| \leq \kappa\}$, analogously for $[X]^{<\kappa}$. We say that a family $\mathscr{F}$ of sets satisfies ccc (countable chain condition) if there are no uncountably many pairwise disjoint sets in $\mathscr{F}$. A σ-field of subsets of a set X will be called, shortly, a σ-field on X. CH denotes Continuum Hypothesis, MA denotes Martin's Axiom. Let $\mathscr{A}$ be a σ-field on a set T. If X is an arbitrary subset of T then $\mathscr{A}_X$ denotes the σ-field $\{A \cap X : A \in \mathscr{A}\}$ on X. $\mathscr{A}$ is called separable if it is countably generated and contains all singletons. The σ-field of Borel subsets of $\mathbb{R}$ is denoted by $\mathscr{B}$. If $\mathscr{A}$ is generated by a sequence of sets $A_1, A_2, \ldots$ then let $h : T \to \mathbb{R}$ be a function defined for every $x \in T$ by $f(x) = \sum_{i=1}^{\infty} \frac{2}{3^i} K_{A_i}(x)$ where $K_{A_i}(x) = 1$ if $x \in A_i$ or $K_{A_i}(x) = 0$ if $x \notin A_i$. For such a function called Marczewski function (e.g. in [1]), $h^{-1} : \mathscr{B}_{h(T)} \to \mathscr{A}$ is an isomorphism [9]. Here $h^{-1}(B) = \{x \in T : h(x) \in B\}$ for every $B \in \mathscr{B}_{h(T)}$.

Recall that a Luzin set is an uncountable subset L of $\mathbb{R}$ such that $|L \cap K| \leqslant \aleph_0$ for every $K \subseteq \mathbb{R}$ which is of the first category. Recall also that $\mathfrak{c}$-Luzin set is a subset of $\mathbb{R}$ such that $|L| = \mathfrak{c}$ and $|L \cap K| < \mathfrak{c}$ for every $K \subseteq \mathbb{R}$ which is of the first category. If we replace the first category sets by Lebesgue null sets in the above definitions, we obtain Sierpiński or $\mathfrak{c}$-Sierpiński sets respectively. Assuming CH both Luzin and Sierpiński sets exist. If we assume MA then again $\mathfrak{c}$-Luzin and $\mathfrak{c}$-Sierpiński sets exist (see [6] or [7] and references there). A set of reals X is

Instytut Matematyki, Uniwersytet Gdański, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

a Q-set if every subset of X is relative G_σ. Assuming MA every set of reals of cardinality less then the continuum is a Q-set (see [6] or [8]) and hence for $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{c}$ we have $\mathscr{B}_X = \mathscr{P}(X)$. A σ-field $\mathscr{A}$ on T is said nonatomic (or atomless e.g. in [1]) if it has no atoms. Recall that $A \in \mathscr{A}$ is an atom of $\mathscr{A}$ if it does not contain properly any nonempty set from $\mathscr{A}$.

Let $\mathscr{F}$ be a family of subsets of a set T. Say that $\mathscr{F}$ is σ-independent family if for any countable distinct (finite or infinite) sequence of sets $\langle F_i : i \geqslant 1\rangle$ from $\mathscr{F}$ we have $\bigcap_{i\geqslant 1} F_i^{\varepsilon_i} \neq 0$ where $\varepsilon_i = 0$ or 1 and $F_i^0 = F_i$ and $F_i^1 = T \setminus F_i$ for all i.

We say that a family $\mathscr{A}$ of subsets of a set T contains κ-many σ-independent sets if there is a σ-independent family $\mathscr{F} \subseteq \mathscr{A}$ with $|\mathscr{F}| = \kappa$.

It was observed by Marzcewski that in $\mathscr{B}_C$ where C is the Cantor set there are $\mathfrak{c}$ many σ-independent sets [5]. Hence using Marczewski function it is clear that if σ-field $\mathscr{A}$ contains infinitely many σ-independent sets then $\mathscr{A}$ contains $\mathfrak{c}$ many σ-independent sets (see [1]). Observe that if $\mathscr{F}$ is an uncountable σ-independent family then each set of the form $\bigcap_{i\geqslant 1} F_i^{\varepsilon_i}$ which appears in the definition has cardinality at least $\mathfrak{c}$. Hence if we modify each set in $\mathscr{F}$ by a set of cardinality less then $\mathfrak{c}$ then such a new family is still σ-independent if we assume $|\mathscr{F}| > \aleph_0$ (recall that the cofinality of $\mathfrak{c}$ is by König's lemma strictly bigger then $\aleph_0$). We need the following

Proposition 1. (Compare [1]). *If $\mathscr{A}$ is a separable σ-field on X which contains infinitely many σ-independent sets then $\mathscr{A}$ contains $\mathfrak{c}$-independent sets separating points. If additionally $[X]^{<\mathfrak{c}} \subseteq \mathscr{A}$ then we can find in $\mathscr{A}$ $\mathfrak{c}$ many σ-independent sets separating sets from $[X]^{<\mathfrak{c}}$.*

Proof. We prove only the second part of the proposition since the first part is similar to the second one and can be found in [1]. Observe then if $[X]^{<\mathfrak{c}} \subseteq \mathscr{A}$ then $|[X]^{<\mathfrak{c}}| \leqslant |\mathscr{A}| \leqslant \mathfrak{c}$. Let $\mathscr{F}$ be a family of σ-independent sets such that $\mathscr{F} \subseteq \mathscr{A}$ and $|\mathscr{F}| = \mathfrak{c}$. Let f be a function from $\mathscr{F}$ onto $[X]^{<\mathfrak{c}} \times [X]^{<\mathfrak{c}}$. Let $f = \langle f_1, f_2\rangle$. Define $\mathscr{G} = \{(F \cup f_1(F)) - f_2(F) : F \in \mathscr{F}\}$. Then $\mathscr{G}$ is σ-independent family as required. □

It is clear a σ-field generated by uncountable σ-independent family of sets is nonatomic. Hence Propositionn 2 holds.

Proposition 2. (Compare [1]). *If a separable σ-field $\mathscr{A}$ on X contains infinitely many σ-independent sets then $\mathscr{A}$ contains a nonatomic σ-field $\mathscr{C}$ which separates points. If additionally $[X]^{<\mathfrak{c}} \subseteq \mathscr{A}$ then we can find such $\mathscr{C}$ which separates sets from $[X]^{<\mathfrak{c}}$.*

In [1] K.P.S. Bhaskara Rao and B.V. Rao have given an example of a separable σ-field $\mathscr{A}$ on a set X of cardinality $\aleph_1$ which contains a nonatomic σ-field separating points. Then assuming $\neg$CH they obtain that $\mathscr{A}$ does not contain infinitely many σ-independent sets, because of course on any set of cardinality less then $\mathfrak{c}$ there are no infinitely many σ-independent sets. Their proof works for all uncountable X with $|X| < \mathfrak{c}$ if we assume MA + $\neg$CH using known consequences of MA. Assuming also MA + $\neg$CH for X of cardinality $\mathfrak{c}$ such a σ-field is obtained in Theorem (3) and (5) of the present note.

In the present note we prove that the sentence

($\bigstar$) *There is a $\mathfrak{c}$-Luzin set such that $\mathscr{B}_L$ does not contain a nonatomic σ-field*

is independent from ZFC + $\mathfrak{c} = \aleph_2$.

First recall that it is consistent with ZFC + $\mathfrak{c} = \aleph_2$ that there is a $\mathfrak{c}$-Luzin set L which is a Luzin set [3]. For such $L(\bigstar)$ is true. Motivated by a problem of K.P.S. Bhaskara Rao and B.V. Rao (P9 in [1]) I observed that the σ-field of Borel subsets of a Luzin set does not contain a nonatomic σ-field (see [1]). To prove this I remarked that $\mathscr{B}\setminus[L]^{\leq\aleph_0}$ satisfies ccc. A proof of this observation is very similar to the proof of Theorem (1) in the present note. Our Theorem (3) shows that MA + $\mathfrak{c} = \aleph_2$ implies that ($\bigstar$) is not true.

Theorem. *Let L be a $\mathfrak{c}$-Luzin set. Then*

(1) *$\mathscr{B}_L\setminus[L]^{<\mathfrak{c}}$ satisfies ccc;*
(2) *If* MA + $\neg$CH *then $[L]^{<\mathfrak{c}} \subseteq \mathscr{B}_L$;*
(3) *If* MA + $\neg$CH *then there is a nonatomic σ-field $\mathscr{A}$ on L such that $\mathscr{A} \subseteq \mathscr{B}_L$ and $\mathscr{A}$ separates points of L;*
(4) *If $\mathscr{C}$ is a nonatomic σ-field on L and $\mathscr{C} \subseteq \mathscr{B}_L$ then there is a nonempty $C \in \mathscr{C}$ with $|C| < \mathfrak{c}$ and hence $\mathscr{C}$ does not separate sets from $[L]^{<\mathfrak{c}}$;*
(5) *$\mathscr{B}_L$ does not contain infinitely many σ-independent sets.*

Proof of (1). Let $\mathscr{F} \subseteq \mathscr{B}_L\setminus[L]^{<\mathfrak{c}}$ be a family of pairwise disjoint sets. From the definition of L it follows that each set in $\mathscr{F}$ is of the second Baire category on $\mathbb{R}$ and hence on L. Consider L as a metric space. A set F is of the first category in a point $x \in L$ if there is an open subset G of L such that $x \in G$ and $G \cap F$ is of the first Baire category on L. For every $F \in \mathscr{F}$ let $G_F = Int(D_F)$ where D_F is the set of all points of L in which F is not of the first category. Then $\langle G_F : F \in \mathscr{F}\rangle$ is a family of pairwise disjoint [4] nonempty open subsets of L and hence $\mathscr{F}$ is countable. □

Proof of (2). First observe the following

Lemma 1. *Assume MA + $\neg$CH. Let $Y \subseteq X \subseteq \mathbb{R}$, $|Y| < \mathfrak{c}$ and $Y \in \mathscr{B}_X$. Then $\mathscr{P}(Y) \subseteq \mathscr{B}_X$.*

Indeed. We have $\mathscr{P}(Y) = \mathscr{B}_Y = (\mathscr{B}_X)_Y \subseteq \mathscr{B}_X$. Let $A \in [L]^{<\mathfrak{c}}$. By a known consequence of MA ([6] or [8]) A is the first category on $\mathbb{R}$. Let A_1 be a first category F_σ set on $\mathbb{R}$ such that $A \subseteq A_1$. We have $A_1 \cap L \in \mathscr{B}_L$ and $|A_1 \cap L| < \mathfrak{c}$. Apply Lemma for $Y = A_1 \cap L$, $X = L$. From Lemma it follows $\mathscr{P}(A_1 \cap L) \subseteq \mathscr{B}_L$. Since $A \subseteq A_1 \cap L$ it follows $A \in \mathscr{B}_L$. □

Proof of (3). Let $\langle X_\alpha\rangle_{\alpha<\mathfrak{c}}$ be a family of pairwise disjoint sets such that $L = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$ and for every $\alpha < \mathfrak{c}$, $|X_\alpha| = \aleph_1$. For every $\alpha < \mathfrak{c}$ let $\mathscr{A}_\alpha$ be a nonatomic σ-field on X_α separating points of X_α. On arbitrary uncountable set there is such a σ-field as was proved in ZFC in [1]. Let $\mathscr{A}$ be the σ-field on L generated by $\bigcup_{\alpha<\mathfrak{c}} \mathscr{A}_\alpha$. It is evident that $\mathscr{A}$ is a nonatomic σ-field on L separating points, which is contained in $\mathscr{B}_L$ because $\mathscr{A}_\alpha \subseteq \mathscr{P}(X_\alpha) \subseteq \mathscr{B}_L$. □

Proof of (4). If CH then $\mathscr{B}_L$ does not contain any nonatomic σ-field. Assume $\neg$CH. Since $\mathscr{C}$ is nonatomic there are uncountably many pairwise disjoint uncountable sets in $\mathscr{C}$ (see e.g. [1]). Assume a contrario that each nonempty set $C \in \mathscr{C}$ has cardinality $\mathfrak{c}$. Hence $\mathscr{B}_L \backslash [L]^{<\mathfrak{c}}$ does not satisfy ccc. This is a contradiction with Theorem (1). Let $C \in \mathscr{C}$ be nonempty and such that $|C| < \mathfrak{c}$. If $\mathscr{C}$ separated sets from $\mathscr{P}(C)$ then $\mathscr{C}_C$ would be equal to $\mathscr{P}(C)$. But $\mathscr{C}_C$ is nonatomic. A contradiction. □

Proof of (5). If a σ-field $\mathscr{C}$ on X contains infinitely many σ-independent sets then $\mathscr{C} \backslash [X]^{<\mathfrak{c}}$ contains $\mathfrak{c}$ many pairwise disjoint sets. In particular $\mathscr{C} \backslash [X]^{<\mathfrak{c}}$ does not satisfy ccc. □

Remark that in our Theorem instead of $\mathfrak{c}$-Luzin we can take a $\mathfrak{c}$-Sierpiński set.

In connection with Theorem (2) we have

Proposition 3. *It is consistent that* $\mathfrak{c} = \aleph_2$ *and there is a* $\mathfrak{c}$*-Luzin set* L *such that* $[L]^{\leqslant \aleph_1} \not\subseteq \mathscr{B}_L$.

In fact $[L]^{\leqslant \aleph_1} \not\subseteq \mathscr{B}_L$ for every Luzin set L.

Proof. Kunen in [3] has proved that it is consistent that $\mathfrak{c} = \aleph_2$ and there is Luzin set L with $|L| = \mathfrak{c}$. Of course such L is also a $\mathfrak{c}$-Luzin set. Since L is Luzin set $\mathscr{B}_L \backslash [L]^{\leqslant \aleph_0}$ satisfies ccc. The proof is similar to the proof of Theorem (1). Let $\mathscr{F}$ be an uncountable family of pairwise disjoint subsets of L such that each set in $\mathscr{F}$ has cardinality $\aleph_1$. Then only countably many sets from $\mathscr{F}$ can belong to $\mathscr{B}_L$. Hence $\mathscr{F} \not\subseteq \mathscr{B}_L$. □

Remark. Assume MA + $\neg$CH. Let $X \subseteq \mathbb{R}$, $|X| = \mathfrak{c}$ and suppose $\mathscr{B}_X \backslash [X]^{<\mathfrak{c}}$ satisfies ccc. It easily follows from a result of Fremlin and Jasiński (see 4C Corollary on p. 527 in [2]) that $[X]^{\leqslant \aleph_1} \subseteq \mathscr{B}_X$. Hence our Theorem (1), (3), (4) and (5) is true if we replace L by the above X. The proofs are the same as for L.

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