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RADON-NIKODYM TYPE PROPERTIES FOR BANACH SPACES

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Let X be a real Banach space and let $\langle S,\Sigma,\lambda\rangle$ denote a finite, positive, complete measure space. In the following $\sigma(X^*,X)$ will stand for the weak* topology and K_{X^*} for the unit ball in X^*. We use the symbols $X^*\text{-}\int f d\lambda$ for the Pettis integral of a weak integrable function $f : S \to X$ and $X\text{-}\int g dm$ for the weak* integral of a weak* integrable function $g : S \to X^*$.

We write Borel $(X^*, \sigma(X^*,X))$ for the Borel σ-algebra on $\langle X^*,\sigma(X^*,X)\rangle$, i.e. the σ-algebra generated by $\sigma(X^*,X)$-open sets. By $ca(X)$ we denote the space of all λ-absolutely continuous vector measures $\mu : \Sigma \to X$ with finite variation $|\mu|$.

We shall consider the following properties of Banach spaces:

(A) For every $\langle S,\Sigma,\lambda\rangle$ and every $\mu\in ca(X)$ there exists a Pettis-integrable function $f : S \to X$ such that $\mu(A) = X^*\text{-}\int_A f d\lambda$ for every $A\in\Sigma$.

(B) For every $\langle S,\Sigma,\lambda\rangle$ and every $\mu\in ca(X)$ there exists a Pettis-integrable function $f : S \to X^{**}$, such that $\mu(A) = X^{***}\text{-}\int_A f d\lambda$ for every $A\in\Sigma$.

(C) There exists a Banach space $Z\supset X$ (isomorphically and isometrically) such that for every $\langle S,\Sigma,\lambda\rangle$ and every $\mu\in ca(X)$ there is a Pettis-integrable function $f : S\rightarrow Z$ such that $\mu(A) = Z^*-\int_A f d\lambda$ for every $A\in\Sigma$.

(D) For every $<S,\Sigma,\lambda>$ there exists a Banach space $Z\supset X$ such that for every $\mu\in ca(X)$ there is a Pettis-integrable function $f : S\rightarrow Z$ such that $\mu(A) = Z^*-\int_A f d\lambda$ for every $A\in\Sigma$.

(E) For every $<S,\Sigma,\lambda>$ and every $\mu\in ca(X)$ there exist a Banach space $Z_\mu\supset\mu(\Sigma)$ and a function $f : S\rightarrow Z_\mu$ such that $\mu(A) = Z_\mu^*-\int_A f d\lambda$ for every $A\in\Sigma$.

The property of possessing (A) was considered by Musiał [4] and it was called the Weak Radon-Nikodym Property (WRNP). Properties (B), (D), (E) was defined in the dissertation of the author. Musiał suggested consideration of (C).

It is clear that (A)$\Longrightarrow$(B)$\Longrightarrow$(C)$\Longrightarrow$(D)$\Longrightarrow$(E). Generally (A) and (B) are not equivalent. Indeed, the space B constructed by Lindenstrauss and Stegall [3] does not have property (A), since it is separable without RNP, but it satisfies (B) as B^{**} has WRNP [4]. As was proved by Drewnowski, (C), (D) and (E) are equivalent. The question whether (C) implies (B) or not remains open.

<u>Proposition.</u> c_0 does not have property (E).

Proof. Let $\{r_n(t)\}$ denote the Rademacher system on the unit interval I and consider a measure μ defined on the σ-algebra $\mathcal{L}$ of Lebesgue measurable subsets of I by $\mu(A) = l_1-\int_A\{r_n(t)\}dt$. Suppose μ has an l_∞^*-measurable de-

$g : I \to l_\infty$. Then $\{r_n(t)\}$ and $g(t)$ are l_1-equi- t an , ince l_1 is separable, these two functions are ... st everywhere. So the function $[0,1] \ni t \to \{r_n(t)\} \in l_\infty$ would have to be l_∞^* -measurable, which is impossible, n e ie function $[0,1] \ni t \to \{\frac{r_n(t)+1}{2}\}$ is not l_∞^* -measurable by a theorem of Sierpiński [7]. Thus c_0 cannot have property (D). But μ takes values in c_0 and its range generates all c_0 . Suppose there exist a Banach space $Z_\mu \supset \mu(\Sigma)$ and a function $f : S \to Z_\mu$ such that $\mu(A) = Z_\mu^* - \int_A f d\lambda$ for every $A \in \Sigma$. Then of course $\mu(A) = Z_\mu^{***} - \int_A f d\lambda$ and since l_∞ is an injective space, μ would have a Pettis derivative with values in l_∞ , which is impossible. So μ cannot satisfy the condition appearing in (E). This completes the proof.

Since (E) is hereditary, c_0 cannot be contained in a Banach space with property (E). In particular, we obtain the following Corollary which solves Problem 3 posed in [4] .

<u>Corollary 1.</u> c_0 cannot be isomorphically imbedded into any Banach space possessing WRNP.

<u>Theorem.</u> For an arbitrary Banach space X the following conditions are equivalent:

(i) $X^* \in (A)$

(ii) $X^* \in (B)$

(iii) $X^* \in (C)$

(iv) $X^* \in (D)$

(v) $X^* \in (E)$

(vi) $X \not\supset l_1$.

Proof. Of course it will be enough to prove that (v) implies (vi) and that (vi) implies (i).

(a) Suppose $X \supset l_1$ and take the measure $\mu : \mathcal{L} \to c_o$ defined in Proposition. Using Theorem 1 of [5] we can find a measure $\varkappa : \mathcal{L} \to X^*$ such that $T^*\varkappa = \mu$, where T denotes the embedding of l_1 into X. It is a standard calculation to show that $\varkappa$ does not satisfy the condition appearing in (E).

(b) Suppose $X \not\supset l_1$ and consider an arbitrary measure $\mu \in ca(X)$. We can restrict ourselves to measures which have average range $A_\mu(S) = \left\{ \frac{\mu(B)}{\lambda(B)} : B \in \Sigma, \ \lambda(B) > 0 \right\}$ contained in K_{X^*}. So $A_\mu(S)$ is relatively compact in the weak* topology. By a theorem of Rybakow [6] there exists an X-measurable function $f_o : S \to X^*$ such that $\mu(A) = X\text{-}\int_A f_o d\lambda$, $f_o(S) \subset \subset K_{X^*}$ for every $A \subset \Sigma$. Now we can use a theorem of Ionescu-Tulcea [2,p.51], and choose such a function $f : S \to X^*$ which is X-equivalent to f_o, measurable from Σ to Borel $(X^*, \sigma(X^*,X))$ and for which the measure λ_f : Borel $(X^*, \sigma(X^*,X)) \to \mathbb{R}$ defined by $\lambda_f(B) = \lambda(f^{-1}(B))$ is regular (for the generalization of this theorem see [8]).

Since λ was supposed to be complete, f is measurable from Σ to the completion of Borel $(X^*, \sigma(X^*,X))$ with respect to the measure λ_f.

By [1], Theorem 4.2, the function $x^{**} \circ f$ is measurable for every $x^{**} \in X^{**}$, which means that f is X^{**}-measurable. Now, for every $A \in \Sigma$ with $\lambda(A) > 0$ let $f|_A$ denote the restriction of f to the set A. Then $x^{**} \circ f|_A$ is measurable since $x^{**} \circ f|_A = (x^{**} \circ f)|_A$. Let x_A^* denote the barycentre of the probability measure $\frac{1}{\lambda(A)}\lambda_f^A$, where λ_f^A is defined by $\lambda_f^A(B) = \lambda(f^{-1}(B)\cap A)$ for every $B\in \mathrm{Borel}\,(X^*, \sigma(X^*,X))$. Consider the point $y_A^* = \lambda(A)x_A^*$. Then by [1] , Theorem 4.2 we can write:

$$\begin{aligned} \langle x^{**},y_A^*\rangle &= \lambda(A)\langle x^{**},x_A^*\rangle = \\ &= \lambda(A)\int_{K_{X^*}}\langle x^{**},x^*\rangle \frac{1}{\lambda(A)}\lambda_f^A(dx^*) = \\ &= \int_{K_{X^*}}\langle x^{**},x^*\rangle \lambda_f^A(dx^*) \ . \end{aligned}$$

Now, using the change-of-variables formula we have:

$$\begin{aligned} \int_{K_{X^*}}\langle x^{**},x^*\rangle \lambda_f^A(dx^*) &= \int_{f|_A^{-1}(K_{X^*})}\langle x^{**},f(s)\rangle\, \lambda(ds) = \\ &= \int_A \langle x^{**},f(s)\rangle\, \lambda\,(ds) \ . \end{aligned}$$

So $y_A^* = X^{**}\text{-}\int_A f(s)\,\lambda\,(ds)$ and $y_A^* = \mu(A)$ since X is total for X^* . This completes the proof.

<u>Corollary 2.</u> For an arbitrary Banach space X :

$$X^*\in \mathrm{WRNP} \iff X \not\supset l_1 \ .$$

The above equivalence was proved in [4] under the additional assumption that X is separably complementable. Moreover it was proved in [5] that $X\not\supset l_1$ if $X^*\in$ WRNP. Let us also remark that Theorem gives the affirmative answer to Problems 5 and 8 posed in [4] . The following Corollary solves Problem 7 from [4] .

<u>Corollary 3.</u> Let X be an arbitrary Banach space and suppose that $X^*\in$ WRNP. Then every weak* closed subspace Y of X^* possesses WRNP as well.

Proof. Every weak* closed subspace of X^* is of the form $(X/_Z)^*$ for some $Z \subset X$. So, suppose $X^* \in$ WRNP. Then $X \not\supset l_1$ and, as is easy to see, $X/_Z \not\supset l_1$ as well. By our Theorem, $(X/_Z)^* \in$ WRNP.

Let us only remark that using Theorem 2 we can also give the affirmative answer to Problem 4 from [4].

Last of all I would like to call attention to the complete analogy between characterizations of dual Banach spaces with RNP and WRNP (for references see [1]). Namely, X^* has RNP (WRNP) if and only if any of the following conditions is satisfied.

for RNP	for WRNP
1/ Every separable subspace of X has a separable dual.	1/ X contains no isomorphic copy of l_1.
2/ Every norm closed bounded convex subset of X^* is the norm closed convex hull of its extreme points.	2/ Every weak* closed bounded convex subset of X^* is the norm closed convex hull of its extreme points.
3/ The identity map from $\langle K_{X^*}, \sigma(X^*,X)\rangle$ into $\langle K_{X^*}, \\|\cdot\\| \rangle$ is universally Lusin-measurable.	3/ The identity map from $\langle K_{X^*}, \sigma(X^*,X)\rangle$ into $\langle K_{X^*}, \\|\cdot\\| \rangle$ is scalarly universally measurable.

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