Martin Majewski

A proof of the Baues-Lemaire conjecture in rational homotopy theory

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1993. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 30. pp. [113]–123.

Persistent URL: http://dml.cz/dmlcz/702134

A PROOF OF THE BAUES–LEMAIRE CONJECTURE IN RATIONAL HOMOTOPY THEORY

Martin Majewski

This paper contains an announcement of a result (Theorem 1, Section 2), which settles the connection between various algebraic models for rational homotopy theory: the models of Quillen, Sullivan and Adams-Hilton-Anick.

It is shown how this result, combined with a recent result of Anick, implies a conjecture of Baues and Lemaire from 1977 [B-L; (3.5)].

We describe in some detail the construction of these models (Section 1). We present a variant of the Adams–Hilton model, which is defined in a natural way for simplicial sets.

A forthcoming paper will contain a detailed proof of Theorem 1. A generalization to *mild* homotopy theories is in preparation, where we establish a close connection between extensions of the rational theories due to Dwyer, Cenkl–Porter and Anick.

1. Background

In this section we recall the construction of the models of Quillen, Sullivan and Adams–Hilton–Anick. We will define these models for 2-reduced simplicial sets.

The relation to geometry is the following. For a pointed topological space T, the singular simplices (continuous maps $\Delta^n \to T$, $n \geq 0$), which send the 1-skeleton to the base point, form a 2-reduced simplicial set. This construction yields a functor, which settles an equivalence of the homotopy theories (integral or rational) of 1-connected CW–complexes and 2-reduces simplicial sets, cf. [May], [Qu].

We will work over the field $\mathbb{Q}$ of rational numbers. We use DG (resp. DG*) for "differential graded with differential of degree -1 (resp. $+1$)"; *graded* means graded over the positive (= non-negative) integers. (Co)algebras are always (co)associative and have a (co)unit and a (co)augmentation (a Lie algebra is not an algebra). A simplicial, or DG, or DG* object is *r–reduced* if it is trivial in degrees $< r$, it is *reduced* if it is 1-reduced. In a category consisting of DG (or DG*) objects a *weak equivalence* is a map inducing an isomorphism in homology.

Let $\mathcal{S}$ be the category of pointed simplicial sets and $\mathcal{S}_r \subset \mathcal{S}$ that of r–reduced simplicial sets, $r \geq 1$. Let $ft_{\mathbb{Q}}$-$\mathcal{S}_r \subset \mathcal{S}_r$ be the full subcategory of r–reduced simplicial sets which are of finite rational type. Here a simplicial set X is of *finite rational type* if $H_*(X;\mathbb{Q})$ is finitely generated in each degree. We may define the homotopy groups of $X \in \mathcal{S}_1$ by $\pi_*(X) = \pi_{*-1}(GX)$ where G is Kan's loop functor

This paper is in final form and no version of it will be submitted for publication elsewhere.

to simplicial groups, cf. [May]. A simplicial map $f : X \to Y \in \mathcal{S}_2$ is a *rational homototopy equivalence* if the induced map $\pi_*(f) \otimes \mathbf{Q} : \pi_*(X) \otimes \mathbf{Q} \to \pi_*(Y) \otimes \mathbf{Q}$ is an isomorphism (equivalently, $H_*(f) \otimes \mathbf{Q}$ is an isomorphism). Before commenting on the rational homotopy theory of simplicial sets we open a subsection.

Closed model categories.

For convenience we will make use of axiomatic homotopy theory. Thus we recall some features of Quillen's closed model categories. See [Qu; II.1] for all definitions and results we need here, [Qu2] for more and [Ba2], [Ma2] for weaker axioms and many details.

A *closed model category* is a category $\mathcal{C}$ with three distinguished subclasses of the morphisms of $\mathcal{C}$, such that certain axioms are satisfied. The morphisms in these subclasses are called *weak equivalences, cofibrations*, resp. *fibrations*. By the first axiom $\mathcal{C}$ has an initial object ϕ and a final object e . An object $X \in \mathcal{C}$ is called *cofibrant* if $\phi \to X$ is a cofibration and *fibrant* if $X \to e$ is a fibration.

When X is cofibrant and Y is fibrant, there is a well-defined equivalence relation $\simeq$ ("is homotopic to") on the set $\mathcal{C}(X,Y)$. Here *well-defined* means that one can use *any* cylinder object for X or *any* path object for Y to define this relation. Recall that a *cylinder object for* X is a factorization $X \sqcup X \xrightarrow{i} IX \xrightarrow{p} X$ in $\mathcal{C}$ of the morphism $(1_X, 1_X)$ such that i is a cofibration and p is a weak equivalence (the sum $X \sqcup X$ and such a factorization exist by the axioms); two maps $f, g : X \to Y$ are *homotopic* if there is $h : IX \to Y$ with $hi = (f,g)$. Path objects are defined in a dual manner. This homotopy relation is *natural*, i.e. it behaves well with respect to composition. We denote the set of equivalence classes (= homotopy classes) by $[X,Y]$ and the homotopy class of f by $[f]$.

Another important consequence of the axioms is the following: for every object Z there are weak equivalences $RLZ \xleftarrow{\sim} LZ \xrightarrow{\sim} Z$ in $\mathcal{C}$ with RLZ fibrant and cofibrant and LZ cofibrant.

Then one constructs the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ with the *localization functor* $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, which is the identity on objects, which carries weak equivalences to isomorphisms and which induces bijections $\gamma : [X,Y] = \mathrm{Ho}\mathcal{C}(X,Y)$ for cofibrant X and fibrant Y. It follows that $\mathrm{Ho}(\mathcal{C})$ is the *localization* of $\mathcal{C}$ with respect to the class of weak equivalences, i.e. γ is universal with the property of carrying weak equivalences to isomorphisms (observe that any functor with this property carries homotopic maps to just *one* morphism). Using these facts the following is immediate.

LIFTING-LEMMA. *Let $f : X \to Y$ and $p : Z \xrightarrow{\sim} Y$ a weak equivalence, where X cofibrant and Y, Z fibrant. Then there is $g : X \to Z$ with $pg \simeq f$ and g is homotopy-unique.* □

It is proved in [Qu] that $\mathcal{S}_2$ has the structure of a closed model category, where the weak equivalences are the rational homotopy equivalences. All objects are cofibrant and $X \in \mathcal{S}_2$ is fibrant (we say $\mathbf{Q}$*-fibrant* in this case) iff X satisfies Kan's extension condition and $\pi_*(X)$ is a $\mathbf{Q}$-module. In this case we denote the homotopy category by $\mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2)$. It is equivalent to its subcategory of $\mathbf{Q}$–fibrant objects where morphisms = homotopy classes of simplicial maps (see above).

Let $\mathrm{Ho}_{\mathbf{Q}}(ft_{\mathbf{Q}}\text{-}\mathcal{S}_2) \subset \mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2)$ be the full subcategory consisting of the objects in $ft_{\mathbf{Q}}\text{-}\mathcal{S}_2$. The canonical functor $ft_{\mathbf{Q}}\text{-}\mathcal{S}_2 \longrightarrow \mathrm{Ho}_{\mathbf{Q}}(ft_{\mathbf{Q}}\text{-}\mathcal{S}_2)$ is the localization with respect to the class of rational homotopy equivalences in $ft_{\mathbf{Q}}\text{-}\mathcal{S}_2$. For this one can use that $ft_{\mathbf{Q}}\text{-}\mathcal{S}_2$ has the structure of a cofibration category in the sense of [Ba2].

The Quillen model.

Let $(\mathrm{DGL})_1$ be the category of reduced DG Lie algebras (over $\mathbf{Q}$). Quillen investigated in [Qu] the following composite functor from $\mathcal{S}_2$ to $(\mathrm{DGL})_1$:

$$(1) \qquad \mathcal{S}_2 \xrightarrow{G} (\mathrm{SG}p)_1 \xrightarrow{\hat{\mathbf{Q}}} (\mathrm{SCHA})_1 \xrightarrow[P]{} (\mathrm{SLA})_1 \xrightarrow[N]{} (\mathrm{DGL})_1$$

which we denote by λG, i.e. we put $\lambda = NP\hat{\mathbf{Q}}$.

Let $\mathbf{L} \subset (\mathrm{DGL})_1$ be the full subcategory consisting of those objects, which are free (as graded Lie algebra).

For $X \in \mathcal{S}_2$ let

$$(2) \qquad \ell_X : L_X \xrightarrow{\sim} \lambda G(X) \in (\mathrm{DGL})_1$$

be a weak equivalence with $L_X \in \mathbf{L}$. We call L_X a *Quillen model* of X.

That such a weak equivalence ℓ_X exists, follows from the fact that $(\mathrm{DGL})_1$ has the structure of a closed model category, where the weak equivalences are the canonical ones, all objects are fibrant and cofibrant = free, cf. [Qu], [Ba2]. (We remark that one can choose L_X *minimal*, cf. [B–L].) It also follows that there is a well-defined natural homotopy relation for maps (with source) in $\mathbf{L}$ (this relation can be described by means of a canonical cylinder object, cf. [Ba2], [An]). We will call this relation DGL–*homotopy*. Let $\mathrm{Ho}(\mathbf{L}) = \mathbf{L}/\simeq$ be the resulting quotient category, which is the localization with respect to the weak equivalences in $\mathbf{L}$ and a full subcategory of $\mathrm{Ho}(\mathrm{DGL})_1$.

The functor λG carries rational homotopy equivalences to weak equivalences, cf. [Qu; I.2-3]. (This follows also from the natural isomorphism $\pi_*(GX) \otimes \mathbf{Q} \cong H_*(\lambda GX)$, $X \in \mathcal{S}_2$, cf. [Qu; I.5]). Hence using ℓ_X the assignment $X \rightsquigarrow L_X$ extends to morphisms yielding the functor

$$(3) \qquad L : \mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2) \to \mathrm{Ho}(\mathbf{L}).$$

More precisely, L is the unique functor for which the $[\ell_X]$, $X \in \mathcal{S}_2$, define a natural isomorphism $L \cong (\lambda G)^{\sim}$ of functors to $\mathrm{Ho}(\mathrm{DGL})_1$. Here $(\lambda G)^{\sim}$ is the unique functor satisfying $(\lambda G)^{\sim}\gamma = \gamma(\lambda G)$, where the universal property of $\mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2)$ is used.

Remark. The principal result of Quillen's theory [Qu; Th.I] states that $(\lambda G)^{\sim}$ (thus L) is an equivalence of categories. Quillen also proves that $(\lambda G)^{\sim}$ (thus L) *preserves homotopy theory*, cf. [Qu; Th.II]. (Actually Quillen states this for a certain composition of functors, which is easily seen to coincide with $(\lambda G)^{\sim}$.)

The Sullivan model.

Let ($\mathrm{DG}^*\mathrm{A}^c$) be the category of augmented, commutative DG^* algebras (over $\mathbf{Q}$) and let

$$(4) \qquad A^* : \mathcal{S} \to (\mathrm{DG}^*\mathrm{A}^c)$$

be the *Sullivan-de Rham functor* (which is contravariant), cf. [B–G].

Let $\mathbf{M} \subset (\mathrm{DG}^*\mathrm{A}^c)$ be the full subcategory consisting of those objects, which are free (as commutative graded algebra), 2-reduced and of finite type (as graded $\mathbf{Q}$-module).

For $X \in ft_{\mathbf{Q}}\text{-}\mathcal{S}_2$ there is a weak equivalence

$$(5) \qquad m_X : M_X \xrightarrow{\sim} A^*(X) \in (\mathrm{DG}^*\mathrm{A}^c)$$

with $M_X \in \mathbf{M}$ (one can choose M_X *minimal)*, cf. [B–G]. We call M_X a *Sullivan model* of X.

The category ($\mathrm{DG}^*\mathrm{A}^c$) has the structure of a closed model category, where the weak equivalences are the canonical ones, all objects are fibrant and the objects of $\mathbf{M}$ are cofibrant, cf. [B–G]. Hence there is a well-defined natural homotopy relation for maps in $\mathbf{M}$ (this relation can be described by means of a canonical path object [B–G] or a canonical cylinder object [Su]). Let $\mathrm{Ho}(\mathbf{M}) = \mathbf{M}/\simeq$ be the resulting quotient category.

Since $H^*(A^*(X)) = H^*(X;\mathbf{Q})$, the functor A^* carries rational homotopy equivalences to weak equivalences. Hence using m_X the assignment $X \rightsquigarrow M_X$ extends to morphisms yielding the contravariant functor (cp. the construction of L in (3))

$$(6) \qquad M : \mathrm{Ho}_{\mathbf{Q}}(ft_{\mathbf{Q}}\text{-}\mathcal{S}_2) \to \mathrm{Ho}(\mathbf{M}).$$

Remark. The principal result of Sullivan's theory states that this functor M is an equivalence of categories, cf. [B–G].

The Adams-Hilton-Anick model.

First we comment on the homotopy theory of DGA's (= DG augmented algebras over $\mathbf{Q}$). These form the category (DGA), which has the structure of a closed model category, where the weak equivalences are the canonical ones, all objects are fibrant and free DGA's are cofibrant, cf. [Mu]. (This is true over any principal ideal domain, see [Un].) Here a DGA A is *free* if $A \approx \mathrm{T}(V)$ as graded algebra, where $\mathrm{T}(V)$ denotes the tensor algebra on the graded (free) module V. Hence there is a well-defined natural homotopy relation on the set of DGA maps $A \to B$, when A is free. It coincides with the derivation homotopy relation: two maps $f, g : A \to B$ are *derivation homotopic*, or DGA*-homotopic*, if there is a linear map $h : A \to B$ of degree $+1$ satisfying $dh + hd = g - f$ and $h\mu_A = \mu_B(f \otimes h + h \otimes g)$, where $\mu_A : A \otimes A \to A$, $\mu_B : B \otimes B \to B$ are the multiplication maps, cf. [Mu], [Ba2], [An].

Next we describe a simplicial version of the classical Adams-Hilton construction [A–H], [Ad].

For a simplicial set X let $C(X)$ be the normalized chain complex of X, *with coefficients in* $\mathbf{Q}$. The Alexander-Whitney diagonal gives $C(X)$ the structure of a DG coalgebra.

For $X \in \mathcal{S}_1, C(GX)$ is a DGA (multiplication is defined using the group structure on GX and the Eilenberg–Mac Lane transformation) and there is a natural *twisting morphism* ([H–M–S; II, 1.4])

$$t : C(X) \to C(GX) \tag{7}$$

satisfying $t(x) = 1 - \tau(x)^{-1}$, $x \in X_1$, where $\tau : X \to GX$ denotes the universal twisting function [May;26.3]. This is proved by the method of acyclic models, cf. [May; 31.3], [Ma]. Such t gives (and is given by) a unique natural DGA map (note that $C(X)$ is reduced)

$$\omega : \Omega C(X) \to C(GX) \tag{8}$$

satisfying $\omega t^{\Omega} = t$, where Ω is the *cobar construction* and $t^{\Omega} : C(X) \to \Omega C(X)$ the universal twisting morphism, cf. [H–M–S; Ch.II]. One can show that the DGA-homotopy class $[\omega]$ does not depend on the choice of t, cf. [Ma].

PROPOSITION. *When $X \in \mathcal{S}_2$, then ω is a weak equivalence.*

PROOF. This follows from a standard comparison theorem for spectral sequences applied to the filtration-preserving chain map

$$\mathrm{id} \otimes \omega : C(X) \otimes_{t^{\Omega}} \Omega C(X) \to C(X) \otimes_t C(GX) \tag{9}$$

between acyclic *twisted tensor products*. The filtrations are the canonical ones yielding the obvious E^2-terms since $C(X)$ is 2-reduced. Cf. [May; 30.8, 31.7], [H–M–S; Ch.II]. □

Because of this proposition we may call the free DGA $\Omega C(X)$ the *Adams–Hilton model* of $X \in \mathcal{S}_2$.

Next we enrich the structure of $\Omega C(X)$ by defining on it a diagonal (or rather a homotopy class of diagonals).

For $A \in$(DGA) we will use the DGA maps $\pi_1, \pi_2 : A \otimes A \to A$ and $\tau : A \otimes A \to A \otimes A$ defined by $\pi_1(a \otimes b) = \varepsilon(b) \cdot a, \pi_2(a \otimes b) = \varepsilon(a) \cdot b$ and $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$, where $\varepsilon : A \to \mathbf{Q}$ is the augmentation and $|a|$ = degree of A.

Definition 1. A *reduced, free* DGH$^{\simeq}$A (called *Hopf algebra up to homotopy* in [An]) is a pair (A, ψ) where A is a reduced, free DGA and $\psi : A \to A \otimes A$ (the *diagonal*) is a DGA map satisfying

$$\begin{aligned} \pi_1\psi \simeq A \simeq \pi_2\psi\,, \\ (\psi \otimes A)\psi \simeq (A \otimes \psi)\psi\,, \\ \tau\psi \simeq \psi\,, \end{aligned} \tag{10}$$

Here and in the following '$\simeq$' means 'DGA-homotopic'. The reduced, free DGH$^{\simeq}$A's form a category $\mathbf{H}$ where a morphism, called DGH$^{\simeq}$A *map*, from (A, ψ) to (B, φ) is a DGA map $f : A \to B$ satisfying $\varphi f \simeq (f \otimes f)\psi$.

Note that a reduced, free DGH$^{\simeq}$A (A, ψ) is canonically isomorphic to (A, ψ') when $\psi \simeq \psi'$, since then 1_A is a DGH$^{\simeq}$A map.

Definition 2. The *Adams-Hilton-Anick model* of a simplicial set $X \in \mathcal{S}_2$ is the reduced, free DGH$^{\simeq}$A $\mathcal{A}(X) = (\Omega C(X), \psi_X)$, where ψ_X is well-defined up to DGA-homotopy by requiring that the following diagram in (DGA) is homotopy-commutative (see Lifting-Lemma).

(11)
$$\begin{array}{ccc} \Omega C(X) & \overset{\psi_X}{\dashrightarrow} & \Omega C(X)\otimes \Omega C(X) \\ \omega \big\downarrow \sim & (\simeq) & \sim \big\downarrow \omega\otimes\omega \\ C(GX) & \xrightarrow{\ \zeta\ } & C(GX)\otimes C(GX) \end{array}$$

Here ζ is the Alexander-Whitney diagonal, which is known to be a DGA map on $C(GX)$. (This follows from the fact that the Eilenberg–Mac Lane transformation is a DG coalgebra map, cf. [G–M; A.3].)

It is implicitly claimed in the definition that any ψ_X making (11) homotopy-commutative satisfies the relations (10). This is an easy consequence of Th.1 (Sec.2). For a direct proof, which also works over the integers, see [Ma].

Note that the homotopy class $[\psi_X]$ is *canonical*, i.e. uniquely preferred and natural (recall that $[\omega]$ is canonical).

Remark. A *natural* diagonal ψ_X will be given in [Ma]; an explicit formula for a *natural*, coassociative diagonal on $\Omega C(X)$ is given in [Ba; IV.2].

Obviously, $X \rightsquigarrow \mathcal{A}(X)$ yields a functor (well-defined up to canonical natural isomorphism)

(12)
$$\mathcal{A} : \mathcal{S}_2 \to \mathbf{H}.$$

Two maps in $\mathbf{H}$ are called *homotopic* if they are DGA–homotopic. Let Ho($\mathbf{H}$) $= \mathbf{H}/\simeq$ be the quotient category. One can show that this is the localization of $\mathbf{H}$ with respect to the class of weak equivalences. Clearly $\mathcal{A}$ carries rational homotopy equivalences to weak equivalences, thus $\mathcal{A}$ induces the functor

(13)
$$\mathcal{A} : \mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2) \longrightarrow \mathrm{Ho}(\mathbf{H}).$$

by the universal property of $\mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2)$.

Remarks. 1. The construction of the Adams–Hilton–Anick model can be performed, without change, when $\mathbf{Q}$ is replaced by any principal ideal domain (e.g. a subring of $\mathbf{Q}$).

2. From our main result it follows that $\mathcal{A}$ in (13) is an equivalence of categories (see Sec.2). It also follows that this functor preserves homotopy theory in a sense, which will not be made precise here. An analogous result is due to Anick [An2]; there however the classical (non-natural) Adams-Hilton model for CW-complexes is used.

Let (DGHA)$_1$ be the category of reduced DG coassociative, cocommutative (!) Hopf algebras. An object of this category is thus a pair (A, ψ) where A is a reduced DGA and $\psi : A \longrightarrow A \otimes A$ is a DGA map satisfying the relations (10) *strictly* (with '$\simeq$' replaced by '$=$'), similar for maps.

Note that the full subcategory of (DGHA)$_1$ consisting of DGHA's which are free (as DGA), is also a subcategory (not full) of $\mathbf{H}$.

When $A = (A, \psi)$ is a (reduced) DGHA, then $P(A) = \{a \in \bar{A} | \psi(a) = a \otimes 1 + 1 \otimes a\}$ is a (reduced) DGL, called the *primitive* DGL of A. Here $\bar{A} = \ker \varepsilon$ denotes the augmentation ideal.

When L is a (reduced) DGL, then UL, the *universal enveloping algebra* of L, is a (reduced) DGHA, where the diagonal $\Delta^U : UL \longrightarrow UL \otimes UL$ is obtained by applying U to $\Delta = (1,1) : L \longrightarrow L \times L$.

When C is a (2-reduced) *cocommutative* DG coalgebra, then the cobar construction ΩC is a (reduced) DGHA, where the diagonal $\Delta^\Omega : \Omega C \longrightarrow \Omega C \otimes \Omega C$ is the composition of DGA maps

$$\Delta^\Omega : \Omega C \xrightarrow{\Omega(\varphi)} \Omega(C \otimes C) \xrightarrow{\gamma'} \Omega C \otimes \Omega C. \tag{14}$$

Here φ is the diagonal of C, which is a DG coalgebra map since C is cocommutative. The natural DGA map γ' is defined in [H–M–S; IV, 5.3].

2. Results

In this section the Baues–Lemaire conjecture (which is contained in Th.3) is proved, using a result of Anick (Th.2) and one of myself (Th.1), of which only the idea of proof will be given here.

Throughout this section, *homotopy* ($\simeq$) means DGA–homotopy, unless otherwise specified. We will use freely the notations and definitions from Sec.1.

For $X \in \mathcal{S}_2$ choose a Quillen model L_X and, when X is of finite rational type, a Sullivan model M_X. Note that M_X can be dualized to give the cocommutative DG coalgebra $M_X^\#$. Recall that the free DGA's UL_X and $\Omega(M_X^\#)$ are (cocommutative) DGHA's in a canonical way, while on the free DGA $\Omega C(X)$, the Adams–Hilton model of X, there is a canonical homotopy class $[\psi_X]$ of diagonals.

THEOREM 1. *For $X \in \mathcal{S}_2$ there is a homotopy-natural weak equivalence $\alpha_X : UL_X \xrightarrow{\simeq} \Omega C(X) \in$ (DGA) and for $X \in ft_{\mathbf{Q}}\text{-}\mathcal{S}_2$ there is a homotopy-natural weak equivalence $\beta_X : \Omega C(X) \xrightarrow{\simeq} \Omega(M_X^\#) \in$ (DGA). These maps commute up to homotopy with the canonical diagonals on these DGA's, i.e. the following diagram in (DGA) commutes up to homotopy.*

$$\begin{array}{ccccc}
UL_X & \xrightarrow[\alpha_X]{\simeq} & \Omega C(X) & \xrightarrow[\beta_X]{\simeq} & \Omega(M_X^\#) \\
\Big\downarrow \Delta^U & \textcircled{\simeq} & \Big\downarrow \psi_X & \textcircled{\simeq} & \Big\downarrow \Delta^\Omega \\
UL_X \otimes UL_X & \xrightarrow[\alpha_X \otimes \alpha_X]{} & \Omega C(X) \otimes \Omega C(X) & \xrightarrow[\beta_X \otimes \beta_X]{} & \Omega(M_X^\#) \otimes \Omega(M_X^\#)
\end{array} \tag{1}$$

The proof of Th.1 proceeds as follows:

1. Construct a chain of weak equivalences in (DGA):

$$(2)\qquad \begin{gathered} UL_X \xrightarrow{\sim} U\lambda(GX) \xrightarrow{\sim} N\hat{\mathbf{Q}}(GX) \xleftarrow{\sim} C(GX) \xleftarrow{\sim} \Omega C(X) \\ \ldots\Omega C(X) \xrightarrow{\sim} {}^{\#}B(A^*(X)) \xrightarrow{\sim} \Omega(M_X^{\#}) \end{gathered}$$

(for the second part assume that X is of finite rational type). Here ${}^{\#}B(A^*(X))$ is the dual of the bar construction of $A^*(X)$, which is actually not a *positively graded* DGA, but we obtain a weakly equivalent sub–DGA by taking (cycles in degree 0) $\cup$ (elements of degree ≥ 1), and the maps in (2) are still defined. Note there is a canonical isomorphism ${}^{\#}B(M_X) = \Omega(M_X^{\#})$.

2. Extend diagram (2) by the canonical diagonals on $UL_X, C(GX)$ and $\Omega(M_X^{\#})$, and by the tensor-square of (2), compare (1); "fill" the resulting diagram to show it becomes commutative in Ho(DGA).

Since UL_X and $\Omega C(X)$ are free (cofibrant) DGA's, this proves the theorem (α_X is defined using the Lifting–Lemma (Sec.1)).

The first and the last map in the chain (2) are given by choice of models. All the other maps are *natural transformations* in X. There is some choice in the construction of the natural transformations with source $\Omega C(X)$, but different choices yield (naturally) homotopic transformations. Whence the following...

Addendum (to Th.1). The homotopy classes $[\alpha_X]$ and $[\beta_X]$ are canonical.

A detailed proof of Th.1 will be given in [Ma]. It requires a good understanding of Quillen's paper [Qu], but his principal result (that λG induces an equivalence of homotopy categories) is *not* used. Some extensions had to be achieved concerning the homotopy theory of simplicial group rings and of simplicial complete Hopf algebras. The existence of a DGA weak equivalence β_X is stated in [B–L; (3.6)], but their proof is incorrect, as it presumes that integration is an algebra map (which is not the case).

The part of Th.1 which is concerned with the left–hand square of (1) is sort of a *differential Milnor–Moore Theorem.* In fact, passing to homology yields the canonical isomorphism $U\pi_*(GX) \otimes \mathbf{Q} \cong H_*(GX; \mathbf{Q})$ of graded Hopf algebras. This result, the classical Milnor–Moore Theorem, is not used in the proof of Th.1.

Obviously, Th.1 tells us that an Adams-Hilton-Anick model can be constructed, *in the most natural way*, from either a Quillen model or a Sullivan model. (It had been clear that this works in *some way*.)

Less obviously, Th.1 tells us that any two of these models can be constructed, *in the most natural way*, from the third. This is true because of the following deep result of Anick.

Recall from Sec.1 the definition of DGH$^{\simeq}$A and DGHA, in particular that DGHA's are *cocommutative*.

THEOREM 2. ([An;5.6,6.3]). (a) *Let* (A,ψ) *be a reduced, free* DGH$^{\simeq}$A. *Then there is a diagonal* ψ' *on* A, $\psi' \simeq \psi$, *such that* (A,ψ') *is a* DGHA.
(b) *Let* (A,ψ) *and* (B,φ) *be reduced, free* DGHA*'s. Let* $f : A \to B$ *be a* DGH$^{\simeq}$A *map. Then there is a* DGHA *map* $f' : A \to B, f' \simeq f$. □

We apply Th.2 (a) to ψ_X in (1) and obtain $\psi'_X \simeq \psi_X$, such that $(\Omega C(X), \psi'_X)$ is a DGHA. (The left–hand square of (1) shows that $(\Omega C(X), \psi_X)$ is a DGH$^{\simeq}$A.)

Next we apply Th.2 (b) to α_X and β_X in (1) and obtain

(3) $$UL_X \xrightarrow[\alpha'_X]{\sim} (\Omega C(X), \psi'_X) \xrightarrow[\beta'_X]{\sim} \Omega(M^{\#}_X) \in (\text{DGHA})_1 .$$

By a theorem of Milnor and Moore there is an adjoint equivalence of categories

(4) $$(\text{DGHA})_1 \underset{P}{\overset{U}{\rightleftarrows}} (\text{DGL})_1$$

and both U and P preserve weak equivalences, cf. [M–M], [Qu;App.B], this is also implicit in [An]. Moreover, $L \in (\text{DGL})_1$ is free iff UL is free (as a graded algebra), more precisely: $L \approx \mathbf{L}(V)$ iff $UL \approx \mathbf{T}(V)$. For such L one has $f, g : L \to L' \in (\text{DGL})_1$ DGL–homotopic iff Uf, $Ug : UL \to UL'$ DGA–homotopic, cf. [A–L]. It follows that P has analogous properties.

Thus applying P to (3) yields...

THEOREM 3. *For* $X \in \mathcal{S}_2$ *(resp.* $X \in ft_{\mathbf{Q}}\text{-}\mathcal{S}_2$*) there are* DGL*-homotopy-natural weak equivalences*

(5) $$L_X \xrightarrow{\sim} P(\Omega C(X), \psi'_X) \xrightarrow{\sim} \mathcal{L}(M^{\#}_X) \in (\text{DGL})_1 .$$

□

Here $\mathcal{L} = P\Omega$ which is called the *$\mathcal{L}$-construction*, cf. [Qu;App.B], [B–L].

When X is of finite rational type, Th.3 yields a DGL–homotopy–natural weak equivalence

(6) $$L_X \xrightarrow{\sim} \mathcal{L}(M^{\#}_X) \in (\text{DGL})_1 ,$$

thus we have proved the *Baues–Lemaire conjecture* [B–L; (3.5)]. By Addendum to Th.1 the homotopy class of the weak equivalence (6) is canonical.

Th.3 also contains the result that $P(\Omega C(X), \psi'_X)$ is a Quillen model, which provides a method of computation. Note that, forgetting differentials, there is an isomorphism

(7) $$P(\Omega C(X), \psi'_X) \approx \mathbf{L}(s^{-1}\overline{C}(X)) ,$$

since $\Omega C(X) \approx \mathbf{T}(s^{-1}\overline{C}(X))$.

Remarks and Problems. 1. The diagonal ψ'_X on $\Omega C(X)$ was "constructed" for each X separately (using Th.2); for our purposes it was sufficient to know that ψ'_X is *homotopy–natural*, which is clear by definition. Is there a *natural* choice for ψ'_X? If this is true, can one give an explicit formula? (I believe that this is true.) Then, of course, $P(\Omega C, \psi')$ would be a *functor* to $(\text{DGL})_1$.

2. Quillen's DG coalgebra functor $\mathcal{C}\lambda G$, followed by dualization, is a solution of the *commutative cochain problem* over the rationals on $\mathcal{S}_2$, so is the Sullivan–deRham functor A^*. Since the Baues–Lemaire conjecture is true, these functors are homotopy–naturally weakly equivalent. Can one prove a general homotopy-uniqueness theorem for such functors? (I have heard of some unsuccessful attempts of proving the Baues–Lemaire conjecture this way.)

Finally, let us reformulate these results in a more categorical language. In doing so, we will make precise those assertions in Th.1 and Th.3 which are concerned with homotopy–naturality.

To begin with, Th.1 says that the two left–hand subdiagrams of (8) commute up to canonical (see Add.) natural isomorphism; the same holds, by (4), for the right–hand subdiagram (i.e. $\Omega \cong U\mathcal{L}$).

(8)
$$
\begin{array}{ccc}
\mathrm{Ho}_{\mathbf{Q}}(ft_{\mathbf{Q}}\text{-}\mathcal{S}_2) & \xrightarrow{M} & \mathrm{Ho}(\mathbf{M}) \\
\downarrow & & \downarrow{\scriptstyle \Omega(-^{\#})} \\
\mathrm{Ho}_{\mathbf{Q}}(\mathcal{S}_2) & \xrightarrow{\mathcal{A}} & \mathrm{Ho}(\mathbf{H}) \\
& \searrow{\scriptstyle L} & \uparrow{\scriptstyle U} \\
& & \mathrm{Ho}(\mathbf{L})
\end{array}
\qquad \mathcal{L}(-^{\#}) : \mathrm{Ho}(\mathbf{M}) \to \mathrm{Ho}(\mathbf{L})
$$

The functors $\mathcal{L}(-^{\#})$, $\Omega(-^{\#})$ and U, defined on $\mathbf{M}$ respectively $\mathbf{L}$, preserve weak equivalences, so they induce functors, denoted by the same symbol in (8), on the respective quotient categories.

Th.2 is the main part in Anick's proof that U in (8) is an equivalence of categories. This implies, in particular, that the outer quadrangle of (8) commutes up to canonical natural isomorphism (Baues–Lemaire conjecture).

Moreover, $\mathcal{L}(-^{\#})$ in (8) is an equivalence onto the subcategory of finite type objects, similar for $\Omega(-^{\#})$, cf. [Qu; App.B].

Therefore, by virtue of Th.1, starting with Quillen's result that L is an equivalence of categories, we can deduce the same for $\mathcal{A}$ and M, which are results (proved in a completely different way) of Anick [An2], respectively Sullivan. Similarly, Th.1 together with Anick's results yields a new proof, that L is an equivalence of categories.

REFERENCES

[Ad] Adams J.F., *On the cobar construction*, Proc. Nat. Acad. Sci. **42** (1956), 409–412.

[A–H] Adams J.F., Hilton P.J., *On the chain algebra of a loop space*, Comm. Math. Helv. **30** (1955), 305–330.

[A–L] Aubry M., Lemaire J.-M., *Homotopies d'algèbre de Lie et de leurs algèbre enveloppantes*, Algebraic Topology – Rational Homotopy, L.N.M. 1318, Springer–Verlag (1988), 26–30.

[An] Anick D.J., *Hopf algebras up to homotopy*, J. of the AMS **2** (1989), 417–453.

[An2] Anick D.J., *R-local homotopy theory*, L.N.M. 1418, Springer–Verlag (1988), 78–85.

[Ba] Baues H.J., *Geometry of loop spaces and the cobar construction*, Memoirs of the AMS 230 (1980).

[Ba2] Baues H.J., *Algebraic Homotopy*, Cambridge Stud. Adv. Math. 15, Cambridge Univ. Press (1989).

[B–G] Bousfield A.K., Gugenheim V.K.A.M., *On PL De Rham theory and rational homotopy type*, Memoirs of the AMS 179 (1976).

[B–L] Baues H.J., Lemaire J.M., *Minimal models in homotopy theory*, Math. Ann. **225** (1977), 219–242.

[G–M] Gugenheim V.K.A.M., May J.P., *On the theory and application of differential torsion products*, Memoirs of the AMS 142 (1974).

[H–M–S] Husemoller D., Moore J.C., Stasheff J., *Differential homological algebra and homogeneous spaces*, J. Pure and Appl. Algebra **5** (1974), 113–185.

[Ma] Majewski M., *The connection between the rational models of Quillen, Sullivan and Adams–Hilton–Anick*, to appear.

[Ma2] Majewski M., *Tame homotopy theory via polynomial forms (I)*, Diplomarbeit, FU Berlin (1988).

[May] May J.P., *Simplicial objects in algebraic topology*, Van Nostrand Math. Studies 11 (1967).

[M–M] Milnor J., Moore J.C., *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

[Mu] Munkholm H., DGA *algebras as a Quillen model category, relations to shm maps*, J. Pure Appl. Algebra **13** (1978), 221–332.

[Qu] Quillen D.G., *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295.

[Qu2] Quillen D.G., *Homotopical algebra*, L.N.M. 43, Springer–Verlag (1967).

[Su] Sullivan D., *Infinitesimal computations in topology*, Inst. Hautes Etude Sci. Publ. Math. **47** (1978), 269–331.

[Un] Unsöld H.M., *Differential torsion and fibrations*, Preprint, FU Berlin (1990), submitted to Manuscr. Math.

M. MAJEWSKI
MATHEMATISCHES INSTITUT
FREIE UNIVERSITÄT BERLIN
1000 BERLIN 33 (FRG)