Miroslav Kureš

On the simplicial structure of some Weil bundles

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 19th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2000. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 63. pp. 131–140.

Persistent URL: http://dml.cz/dmlcz/702141

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 63 (2000) pp. 131-140

# ON THE SIMPLICIAL STRUCTURE OF SOME WEIL BUNDLES

MIROSLAV KUREŠ

ABSTRACT. The description of a wide class of bundles of higher order velocities with respect to their simplicial structure including derivations of their Weil algebras is given.

## 1. STARTING-POINTS

Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor. For two manifolds $M_1$, $M_2$ we denote the standard projection onto the $i$-th factor by $\pi_i : M_1 \times M_2 \to M_i$, where $i = 1, 2$. $F$ is called *product preserving* if the mapping

$$(F(\pi_1), F(\pi_2)) : F(M_1 \times M_2) \to F(M_1) \times F(M_2)$$

is a diffeomorphism for all manifolds $M_1$, $M_2$. For a product preserving bundle functor we shall always identify $F(M_1 \times M_2)$ with $F(M_1) \times F(M_2)$ by the diffeomorphism from the definition. Further, we obtain a product preserving functor by arbitrary (finite) iterations of product preserving functors. The functor $T_k^r$ of $k$-dimensional velocities of order $r$ is product preserving and by its generalization was introduced the concept of the Weil functor.

Take any ideal $\mathfrak{I}$ in the algebra $\mathbb{R}[x_1, \dots, x_k]$ of all polynomials with $k$ variables satisfying $\langle x_1, \dots, x_k \rangle^{r+1} \subset \mathfrak{I} \subset \langle x_1, \dots, x_k \rangle$, where $\langle x_1, \dots, x_k \rangle$ is the ideal of all polynomials without an absolute term and $\langle x_1, \dots, x_k \rangle^{r+1}$ is its $(r+1)$-th power. Such an ideal is said to be a *Weil ideal* and the factor algebra $A = \mathbb{R}[x_1, \dots, x_k]/\mathfrak{I}$ is called a *Weil algebra*. We have a canonical decomposition $A = \mathbb{R} \oplus \mathfrak{N}$, where $\mathfrak{N} = \langle x_1, \dots, x_k \rangle / \mathfrak{I}$ is the ideal of all nilpotent elements of $A$. If we take the algebra $E(k)$ of all germs of smooth functions on $\mathbb{R}^k$ at zero, then $\mathfrak{I}$ generates a ideal $\hat{\mathfrak{I}} \subset E(k)$ and $A = E(k)/\hat{\mathfrak{I}}$. Two maps $g, h : \mathbb{R}^k \to M$, $g(0) = h(0) = x$ are said to be $A$-*equivalent*, if $\phi \circ g - \phi \circ h \in \hat{\mathfrak{I}}$ for every germ $\phi$ of a smooth function on a manifold $M$ at $x$. Such an equivalence class will be denoted by $j^A g$ and called $A$-*velocity*

1991 *Mathematics Subject Classification.* 58A20, 55R10.

*Key words and phrases.* Non-holonomic jet, Weil bundle, colored simplex.

Supported by GA CR, grants Nos. 201/99/0296, 201/99/P065.
This paper is in final form and no version of it will be submitted for publication elsewhere.

on $M$. Denote by $T^AM$ the set of all $A$- velocities on $M$. $T^A$ is a bundle functor which is called the *Weil functor*, $T^AM$ is called the *Weil bundle* and it is easy to see $T^A\mathbb{R} = A$. The important role of the Weil functors in differential geometry has been clarified in recent papers [2], [4], [10], where it is proved that every product preserving bundle functor $F$ on the category of all manifolds is a Weil functor. The related Weil algebra $A = F\mathbb{R}$ is endowed with the extensions $F\alpha : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$ and $F\mu : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$ of the addition $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and the multiplication $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of reals.

For specifications of many general results dealing with Weil bundles, see e.g. [8], we often need a clear description of them in local coordinates. We have a sure hope that the using of colored simplices introducing in [9] solves this problem adequately for a wide class of Weil bundles. A *$k$-colored $(r-1)$-simplex* is the collection of all non-empty subsets of the set $\{1, \dots, r\}$, where *vertices* $1, \dots, r$ can acquire all values (*colors*) from the set $\{1, \dots, k\}$. An *$h$-face* $\gamma_h$ of the simplex, $0 \le h \le r-1$, is a subset with $h+1$ fixedly colored elements. We shall designate $\gamma_h$ as a $(k/r)$-matrix $\begin{pmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{k1} & \dots & a_{kr} \end{pmatrix}$ with $a_{jp} \in \{0, 1\}$ and $h+1$ of the elements equals to unity. This corresponds to the subset containing each $s$ with color $l$ such that $a_{ls} = 1$. The zero matrix is excluded. We write the *composition* of two disjoint faces $\gamma_h$, $\bar{\gamma}_{\bar{h}}$ as $\gamma_h + \bar{\gamma}_{\bar{h}}$.

We shall use also more general simplicial structures, but generalizations are quite natural. An *orientation* for a given colored simplex is an *ordering* for the vertices. The *standard orientation* is given by the arrangement $\{1, \dots, r\}$. Hence we regard such simplices as *(standardly) oriented.* Apart from that, we can take the simplex as the collection of non-empty subsets of an ordered set of $r$ elements. Every subset of $s$ elements, $1 \le s \le r$, disposes of the determined orientation and so we give the *posterior denotations* $\{1, \dots, s\}$ to its vertices. These simplices will be called *independently oriented.* On the other hand, if we define the simplex as the collection of non-empty subsets of an set of $r$ indistinguishable elements, it is no orientation for it and we call it *non-oriented.* When the orientation is the standard one, we shall not mention it explicitly.

Further, a *subcomplex* $K$ of a given simplex is the collection of non-empty subsets of the set $\{1, \dots, r\}$ with the property that if $\mathcal{A} \in K$ and $\emptyset \neq \bar{\mathcal{A}} \subset \mathcal{A}$, then $\bar{\mathcal{A}} \in K$. A *maximal component* of $K$ is a subset $\mathcal{A} \in K$ of the set $\{1, \dots, r\}$ satisfying the property that if $\bar{\mathcal{A}} \supset \mathcal{A}$, then $\bar{\mathcal{A}} \notin K$.

Basic ideas how to associate simplices with higher order bundles are described in the book of J.E. White, [11]. White occupies himself only with the case of iterated tangent bundles and so he does not need colors, in other words, he uses matrices with an only row. (Especially, the cases $TTM$ and $TTTM$ are well-known.) His construction of vector spaces of functions from simplices to $\mathbb{R}^m$, a defining of a left action of the $r$-th differential group on the fiber product of the frame bundle and this space, etc., is very exhaustive, but it is a rather long formal process and we do not go into details here. The generalization for the case of iterated $k$-dimensional velocities of order one is very natural and direct. Roughly speaking, it is essential that the local coordinates correspond with faces of colored simplices by the following way. Given some local coordinates $x^i$ on $M$ and $t^j$ on $\mathbb{R}^k$, the iterated differentiation of $x^i(t^1,\dots,t^k)$ determines the induced coordinates

$$x^i, y^i_{\gamma_h} = \frac{\partial}{\partial t^{j_r}}(\dots(\frac{\partial}{\partial t^{j_2}}(\frac{\partial x^i}{\partial t^{j_1}}))\dots), \quad h = 0,\dots,r-1$$

on $\underbrace{T^1_k \dots T^1_k}_{r\text{-times}} M$, $j_p \in \{0,1,\dots,k\}$, when $\frac{\partial}{\partial t^0}$ represents no differentiation. In the matrix $\gamma_h$, the columns represent the order of the differentiation and the rows correspond with coordinates $t^1,\dots,t^k$. Thus, $a_{ls} = 1$ means the differentiation $\frac{\partial}{\partial t^l}$ at the $s$-th iteration.

The primary aim of this paper is the completion of the description of (non-holonomic, semi-holonomic, holonomic) bundles of multidimensional velocities of higher order with respect to the emphasized points of view. However, we have added certain new conditions on simplicial structures in Chapter 5 and especially in Chapter 6. They yield further Weil bundles which are describable by the mentioned tools. These bundles were not studied up to now. Finally, we occupy ourselves with the affine structure in the Chapter 7.

## 2. The non-holonomic case

This case is the fundamental one. As usually, we define the *non-holonomic bundle of $k$-dimensional velocities of order $r$* $\tilde{T}^r_k M = \tilde{J}^r_0(\mathbb{R}^k, M)$, where we consider the non-holonomic jets in the sense of C. Ehresmann, [3]. There is a natural equivalence (see [6]) $\epsilon_r : \tilde{T}^r_k \to \underbrace{T^1_k \dots T^1_k}_{r\text{-times}}$ defined by the following induction. For $r = 1$, $\epsilon_1$ is the identity. Let $t_u$ denote the translation on $\mathbb{R}^k$ transforming 0 into $u$. If $\sigma : \mathbb{R}^k \to \tilde{J}^{r-1}(\mathbb{R}^k, M)$ is a section, then $\sigma(u) \circ j^{r-1}_0(t_u) \in \tilde{T}^{r-1}_k M$. Hence $\epsilon_{r-1}(\sigma(u) \circ j^{r-1}_0 t_u) \in \underbrace{T^1_k \dots T^1_k}_{r-1\text{-times}} M$ and we define $\epsilon_r(j^1_0 \sigma) = j^1_0 \epsilon_{r-1}(\sigma(u) \circ j^{r-1}_0 t_u)$.

In view of the fact that local coordinates of $\tilde{T}^r_k M$ correspond with the local coordinates of $\underbrace{T^1_k \dots T^1_k}_{r\text{-times}} M$, we use the mentioned fundamental concept of the color simplex for their denotation, too. So that we have

$$x^i, y^i_{\gamma_h} = \frac{\partial}{\partial t^{j_r}}(\dots(\frac{\partial}{\partial t^{j_2}}(\frac{\partial x^i}{\partial t^{j_1}}))\dots), \quad h = 0,\dots,r-1$$

on $\tilde{T}^r_k M$ as on $\underbrace{T^1_k \dots T^1_k}_{r\text{-times}}$.

For a local expressing of the bundle morphism $\tilde{T}^r_k f : \tilde{T}^r_k M \to \tilde{T}^r_k N$, where $f : M \to N$ is an arbitrary smooth map, we state that it corresponds, roughly speaking, with all decompositions of every $h$-face ($h = 0,\dots,r-1$) to $h_1$-, $h_2$-, ... , $h_q$-faces ($h_1 + \dots + h_q + q - 1 = h$, $q = 1,\dots,h+1$). The precise formula is in [9].

Write $\tilde{\mathbb{D}}^r_k$ for the Weil algebra of $\tilde{T}^r_k$. If $c = ab$ is a coordinate form of the multiplication $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the Weil algebra $\tilde{\mathbb{D}}^r_k$ is endowed with the extension

$\tilde{T}^r_k \mu$, the coordinate form of which is

$$c_{\gamma_h} = ab_{\gamma_h} + a_{\gamma_h} b + \sum_{\rho \in \mathbb{P}(\gamma_h, 2)} a_{\gamma_{h_1}} b_{\gamma_{h_2}} ,$$

where $\mathbb{P}(\gamma_h, q)$ is the set of all decompositions of $h$–face to 2 faces $\gamma_{h_1}, \gamma_{h_2}$, $h_1 + h_2 + 1 = h$ (i.e. we sum as to all such decompositions $\rho \in \mathbb{P}(\gamma_h, 2)$). So, the element of $\tilde{\mathbb{D}}^r_k$ has a form $a + a_{\gamma_h} t^{\gamma_h}$ (Einstein summation convention) and the product of two elements has a form

$$(a + a_{\gamma_h} t^{\gamma_h})(b + b_{\gamma_h} t^{\gamma_h}) = ab + (ab_{\gamma_h} + a_{\gamma_h} b + \sum_{\rho \in \mathbb{P}(\gamma_h, 2)} a_{\gamma_{h_1}} b_{\gamma_{h_2}}) t^{\gamma_h} .$$

By comparing of coefficients, we obtain generating equations for the ideal $\mathfrak{J}$ as

$$t^{\gamma_{h_1}} t^{\gamma_{h_2}} = t^{\gamma_{h_1} + \gamma_{h_2}} \tag{1}$$

for all disjoint faces $\gamma_{h_1}$, $\gamma_{h_2}$, other

$$t^{\gamma_{h_1}} t^{\gamma_{h_2}} = 0 . \tag{1'}$$

In other words, we can consider only the generators upper indices of them are 0–faces (the consequence of the first equation), the second powers of them vanish (the consequence of the second one). If we denote $t^l_s$ such generators, where $s$, $l$ represent the position of the unity in the matrix $\gamma_h$, we obtain

$$\tilde{\mathbb{D}}^r_k \approx \underbrace{\mathbb{D}^1_k \otimes \cdots \otimes \mathbb{D}^1_k}_{r\text{–times}} = \mathbb{R}[t^1_1, \ldots, t^k_r] / \langle (t^1_1)^2, \ldots, (t^k_r)^2 \rangle .$$

## 3. The semi-holonomic case

There is the canonical projection $\pi^r_{r-1} : T^1_k(\underbrace{T^1_k \ldots T^1_k}_{(r-1)\text{–times}} M) \to \underbrace{T^1_k \ldots T^1_k}_{(r-1)\text{–times}} M$ and we denote the related direct projection $\tilde{T}^r_k M \to \tilde{T}^{r-1}_k M$ by the same symbol. Thus, we construct naturally projections $\pi^s_{\bar{s}} = \pi^{\bar{s}+1}_{\bar{s}} \circ \cdots \circ \pi^s_{s-1}$, $1 \leq \bar{s} < s \leq r$,

An element $Z \in \tilde{T}^r_k M$ is called the *semi-holonomic $r$–jet* if for any $s$, $1 \leq s < r$,

$$\pi^r_s(Z) = j^1_{z_i}(\pi^{i-1}_{s-1}) \circ \pi^r_i(Z) \tag{*}$$

is satisfied, whenever $i = s+1, \ldots, r$ and $z_i = \pi^r_{i-1}(Z)$. We denote by $\bar{T}^r_k M$ the subset of semi-holonomic jets. Of course, $\bar{T}^r_k M$ has a bundle structure and there is a canonical inclusion ${}^1i^r_k : \bar{T}^r_k M \to \tilde{T}^r_k M$. We call it the *semi-holonomic bundle of $k$–dimensional velocities of order $r$*.

$h$–faces, $h = 0, \ldots, r-2$, vertices of them are colored in order by colors $C_1$, $\ldots$ ,$C_{h+1} \in \{1, \ldots, k\}$, identify in the semi-holonomic case. However, it is necessary to respect the order of colors. Evidently, we obtain just the independently oriented colored simplex. In local coordinates, it means that we delete all zero columns with preserving the original order of non-zero ones.

For a local expressing of the bundle morphism $\bar{T}_k^r f : \bar{T}_k^r M \to \bar{T}_k^r N$, where $f : M \to N$ is an arbitrary smooth map, we state that it corresponds with all decompositions of every $h$-face to faces without zero columns. However, let us notice, that the situation in the case of independently oriented simplices is rather different. Chiefly, there is no unique composition of faces as in the case of standardly oriented simplices, but several possibilities for it.

Write $\bar{\mathbb{D}}_k^r$ for the Weil algebra of $\bar{T}_k^r$. If we evaluate the product of two elements and if we compare coefficients, we obtain generating equations for the ideal $\mathfrak{J}$ as

$$\sum_{\gamma_h} \sum_{\rho \in \mathbb{P}(\gamma_h, 2)} t^{\gamma_{h_1}} t^{\gamma_{h_2}} = \sum_{\gamma_h} \sum_{\rho \in \mathbb{P}(\gamma_h, 2)} t^{\gamma_h} \tag{2}$$

for $h_1 + h_2 + 1 = h \leq r - 1$ (in the non-holonomic sense of decomposition after which we delete zero columns), and

$$t^{\gamma_{h_1}} t^{\gamma_{h_2}} = 0 . \tag{2'}$$

for $h_1 + h_2 + 1 > r - 1$. We obtain

$$\bar{\mathbb{D}}_k^r = \mathbb{R}[t^{\gamma_0}, \dots, t^{\gamma_{r-1}}]/\mathfrak{J} .$$

The special case of the ideal for $r = 2$ is in [7].

## 4. The holonomic case

This case is the classical one. Many papers dealing with an investigation into the *bundle of $k$-dimensional velocities of order $r$ $T_k^r M = J_0^r(\mathbb{R}^k, M)$* are motivated by mechanics, especially by Lagrangian dynamics, and the geometry of $T_k^r M$ is well-known in comparison with previous cases. Of course, there is a canonical inclusion ${}^2 i_k^r : T_k^r M \to \bar{T}_k^r M$. We introduce new local coordinates on $T_k^r M$. Given some local coordinates $x^i$ on $M$ and $t^j$ on $\mathbb{R}^k$, the expansion of $x^i(t^1, \dots, t^k)$ determines the induced coordinates

$$x^i, y^{i,\mathbf{h}} = \frac{1}{(h+1)!} \frac{\partial^{h+1} x^i}{(\partial t^1)^{u_1} \dots (\partial t^k)^{u_k}}, \quad h = 0, \dots, r-1$$

on $T_k^r M$, where by the bold letter $\mathbf{h}$ we denote the multiindex $\mathbf{h} = u_1 \dots u_k$, $|\mathbf{h}| = u_1 + \dots + u_k \leq r$ (we set $\mathbf{h}! = u_1! \dots u_k!$ and $\mathbf{h} + \bar{\mathbf{h}} = u_1 + \bar{u}_1 \dots u_k + \bar{u}_k$). Equally colored $h$-faces (without any respecting of the order of colors) identify in the holonomic case. In local coordinates,

$$y^i_{\gamma_h} \mapsto \mathbf{h}! y^{i,\mathbf{h}}, \quad u_j = \sum_{p=1}^{r} a_{jp}, \quad j = 1, \dots, k .$$

The holonomic case corresponds with a non-oriented colored simplex. It implies that decompositions of faces transform to classical decompositions of numbers. Write $\mathbb{D}_k^r$ for the Weil algebra of $T_k^r$. The Weil algebra $\mathbb{D}_k^r$ is endowed with the extension $T_k^r\mu$, the coordinate form of which is

$$c_{\mathbf{h}} = ab_{\mathbf{h}} + a_{\mathbf{h}}b + \sum_{\rho\in\mathbb{P}(\mathbf{h},2)} a_{\mathbf{h}_1}b_{\mathbf{h}_2}\,,$$

where $\mathbb{P}(\mathbf{h}, q)$ is the set of all decompositions of the multiindex $\mathbf{h}$ to 2 multiindices $\mathbf{h}_1, \mathbf{h}_2$. So, the element of $\mathbb{D}_k^r$ has a form $a + a_{\mathbf{h}}t^{\mathbf{h}}$ and if we evaluate the product of two elements and if we compare coefficients, we obtain generating equations for the ideal $\mathfrak{I}$ as

(3) $$t^{\mathbf{h}_1}t^{\mathbf{h}_2} = t^{\mathbf{h}_1+\mathbf{h}_2}$$

for $|\mathbf{h}_1| + |\mathbf{h}_2| \le r$ and

(3') $$t^{\mathbf{h}_1}t^{\mathbf{h}_2} = 0$$

for $|\mathbf{h}_1| + |\mathbf{h}_2| > r$. So, we can consider only the generators upper indices of them are multiindices with an only unity and remaining zeros. If we denote $t^l$ such generators, where $l$ represents the position of the unity in the multiindex $\mathbf{h}$, we obtain

$$\mathbb{D}_k^r = \mathbb{R}[t^1, \ldots, t^k]/\langle t^1, \ldots, t^k\rangle^{r+1}\,.$$

## 5. Subcomplexes

Let us return to the fundamental case and let us concentrate upon bundles *associated* with subcomplexes of a given simplex in the sense of the correspondence of local coordinates with faces. Such bundles have already been studied in [11] and we generalize obtained results for colored simplices. Let us consider a subcomplex $K$ with maximal components $\mathcal{A}_1, \ldots, \mathcal{A}_v$, the cardinalities of them are $a_1, \ldots, a_v$, respectively. We see every component $\mathcal{A}_i$ as a colored $(a_i - 1)$–simplex and we associate it with $\tilde{T}_k^{a_i}M$ equally to Chapter 2. It gives rise to a natural fiber bundle map $(D(\mathcal{A}_i), \mathrm{id})$:

$$\begin{array}{ccc} \tilde{T}_k^r M & \xrightarrow{D(\mathcal{A}_i)} & \tilde{T}_k^{a_i} M \\ {\scriptstyle \pi_0^r}\big\downarrow & & \big\downarrow{\scriptstyle \pi_0^{a_i}} \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

From the universal property of fiber products it follows that there is a unique fiber bundle map $(F, \mathrm{id})$

$$\begin{array}{ccc} \tilde{T}_k^r M & \xrightarrow{F} & \times_{i\,M}\, \tilde{T}_k^{a_i} M \\ {\scriptstyle \pi_0^r}\big\downarrow & & \big\downarrow{\scriptstyle \times_{i\,M}\, \pi_0^{a_i}} \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

such that $D(\mathcal{A}_i) = Q(\mathcal{A}_i) \circ F$ for all $i = 1, \ldots, v$, where $Q(\mathcal{A}_i) : \times_{i\, M} \tilde{T}_k^{a_i} M \to T_k^{a_i} M$ is the standard projection onto the $i$-th maximal component. The $\operatorname{im} F \to M$ has a natural fiber bundle structure and we denote this bundle $\tilde{T}_k^r[M, K]$ and call it the *(colored) $K$-sector bundle.* (We can identify it with the fiber product bundle only if we have a family of disjoint maximal components.)

If we denote $\bar{K}$ all subsets of the set $\{1, \ldots, r\}$ with the property that if $\mathcal{B} \in \bar{K}$ then $\mathcal{A} \not\supseteq \mathcal{B}$ for every $\mathcal{A} \in K$. An *antigenerator* of $K$ is a subset $\mathcal{B} \in \bar{K}$ of the set $\{1, \ldots, r\}$ satisfying the property that if $\bar{\mathcal{B}} \subset \mathcal{B}$, then $\bar{\mathcal{B}} \notin \bar{K}$. Let us consider a subcomplex $K$ with antigenerators $\mathcal{B}_1, \ldots, \mathcal{B}_w$, the cardinalities of them are $b_1, \ldots, b_w$, respectively. Let us denote the elements (i.e. vertices) of an antigenerator $\mathcal{B}_j$ by $V_1^j, \ldots, V_{b_j}^j$. We can formulate the following assertion.

**Proposition 1.**

(i) *Let $K$ is a subcomplex of an oriented $k$-colored $(r-1)$-simplex. Then the $K$-sector bundle $\tilde{T}_k^r[M, K]$ is the fiber bundle associated with $K$ and the ideal $\mathfrak{I}$ of its Weil algebra $A = \mathbb{R}[t_1^1, \ldots, t_r^k]/\mathfrak{I}$ is generated by* (1) *and* (1') *and by the added generating equations*

$$t^l_{V_1^j} \cdots t^l_{V_{b_j}^j} = 0 \quad l = 1, \ldots, k, \quad j = 1, \ldots, w\,.$$

(ii) *Let $K$ is a subcomplex of an independently oriented $k$-colored $(r-1)$-simplex. Then the semi-holonomic bundle of $k$-dimensional velocities of higher order is the fiber bundle associated with $K$ just again.*

(iii) *Let $K$ is a subcomplex of a non-oriented $k$-colored $(r-1)$-simplex. Then the holonomic bundle of $k$-dimensional velocities of higher order is the fiber bundle associated with $K$ just again.*

*Proof.* (i) We have already demonstrated the way of the association. We discover the added generating equations by a direct evaluation. For (ii) and (iii), it is sufficient to realize the fact of an only one maximal component of $K$ in these cases. □

## 6. Restrictions on colorings

We shall restrict colorings of vertices by the following way. Let us suppose, that

— vertices $1, \ldots, r_1$ are colored by colors $C_1, \ldots, C_{k_1}$,
— vertices $r_1 + 1, \ldots, r_1 + r_2$ are colored by colors $C_1, \ldots, C_{k_2}$,
— ...
— vertices $r_{n-1} + 1, \ldots, r_{n-1} + r_n = r$ are colored by colors $C_1, \ldots, C_{k_n}$

and denote such a restriction $(RC)$. We put $r_0 = 0$. It is reasonable to suppose that $k = \max\{k_i\}$, $i = 1, \ldots, n$.

The corresponding bundle $\underbrace{T^1_{k_n} \ldots T^1_{k_n}}_{r_n\text{-times}} \ldots \underbrace{T^1_{k_1} \ldots T^1_{k_1}}_{r_1\text{-times}} M$, which we obtain in the non-holonomic case, we can identify with the bundle $\tilde{T}^{r_n}_{k_n} \ldots \tilde{T}^{r_1}_{k_1} M$ in the same way as in the Chapter 2. We must add the equations representing forbidding colorings to equations which generate the ideal of the relevant Weil algebra.

In the semi-holonomic case, as we have the inclusion

$$^{D}i : \underbrace{T^1_{k_n} \dots T^1_{k_n}}_{r_n\text{-times}} \dots \underbrace{T^1_{k_1} \dots T^1_{k_1}}_{r_1\text{-times}} M \to \underbrace{T^1_k \dots T^1_k}_{r\text{-times}} M$$

and the map

$$\tilde{\epsilon}_r : \underbrace{T^1_k \dots T^1_k}_{r\text{-times}} M \to \tilde{T}^r_k M$$

inverse to $\epsilon_r$, we can set the semi-holonomic condition (*) on elements of the fiber bundle $\operatorname{im} S \to M$, $S = \tilde{\epsilon}_r \circ {}^{D}i$. We denote the obtained bundle $\bar{T}^{r_n \dots r_1}_{k_n \dots k_1} M$. Of course, this bundle is not equivalent with $\bar{T}^{r_n}_{k_n} \dots \bar{T}^{r_1}_{k_1} M$. Obviously, we do not consider decompositions of forbidly colored faces at deriving of its Weil algebra.

The holonomic case is the most exotic one. We need the following generalization of so-called $(R,S)$-jets, see e.g. [1]. Let us consider a gradually fibred bundle

$$Y \xrightarrow{\rho_1} W_1 \xrightarrow{\rho_2} W_2 \xrightarrow{\rho_3} \dots \xrightarrow{\rho_p} W_p = B$$

If we have two maps $f$, $g$ into another manifold, we say they determine the same $(R, S_1 \dots S_p)$-jet at $y \in Y$, $R \leq S_1 \leq \dots \leq S_p$, if

$$j^R_y f = j^R_y g, \quad j^{S_1}_y (f|Y_p) = j^{S_1}_y (g|Y_p), \quad \dots \quad j^{S_p}_y (f|Y_1) = j^{S_p}_y (g|Y_1)\,,$$

where $Y_l$ are the fibers passing through $y$ with respect to the projection $\rho_l \circ \dots \circ \rho_1$, $l = 1, \dots, p$. As

$$\mathbb{R}^k \xrightarrow{\rho_1} \mathbb{R}^{k-1} \xrightarrow{\rho_2} \dots \xrightarrow{\rho_{k-1}} \mathbb{R}$$

is such a bundle, we can construct $T^{R,S_1 \dots S_{k-1}}_k M := J^{R,S_1 \dots S_{k-1}}_0 (\mathbb{R}^k, M)$ (the space of all $(R, S_1 \dots S_{k-1})$-jets with the source 0). We consider $S_{k-1} = r$. Evidently, $T^{r,r \dots r}_k M = T^r_k M$. We take the restrictions on colorings into consideration by added equations, too. After the detailed evaluation, we can summarize.

**Proposition 2.**

(i) *Let us consider (RC) on an oriented $k$-colored $(r-1)$-simplex. Then the bundle $\tilde{T}^{r_n}_{k_n} \dots \tilde{T}^{r_1}_{k_1} M$ is the fiber bundle associated with this simplex and the ideal $\mathfrak{I}$ of its Weil algebra $A = \mathbb{R}[t^1_1, \dots, t^k_r]/\mathfrak{I}$ is generated by (1) and (1') and by the added generating equations*

$$t^{l_i}_{s_i} = 0, \quad s_i = r_{i-1} + 1, \dots, r_i, \quad l_i = k_{i+1}, \dots, k, \quad i = 1, \dots, n\,.$$

(ii) *Let us consider (RC) on an independently oriented $k$-colored $(r-1)$-simplex. Then the bundle $\bar{T}^{r_n \dots r_1}_{k_n \dots k_1} M$ is the fiber bundle associated with this simplex and the ideal $\mathfrak{I}$ of its Weil algebra $A = \mathbb{R}[t^{\gamma_0}, \dots, t^{\gamma_{r-1}}]/\mathfrak{I}$ is generated by (2) and (2'), where we sum as to all faces satisfying (RC) in (2) (we write $\sum\limits_{\gamma_h|(RC)}$ in the formula).*

(iii) *Let us consider (RC) on a non-oriented k-colored $(r-1)$-simplex. Then the bundle $T_k^{R,S_1\dots S_{k-1}}M$ is the fiber bundle associated with this simplex and the ideal $\mathfrak{I}$ of its Weil algebra $\mathbb{R}[t^1,\dots,t^k]/\mathfrak{I}$ equals*

$$\bigcap_{l=1}^{k} \langle t^1, \dots t^l \rangle^{S_{k-l}+1},$$

*where we put $S_0 = R$.*

*Remark.* We can require more cunning restrictions on colorings and also restrictions on colorings of subcomplexes. We obtain a very wide class of Weil bundles by this way. We shall not describe such generalized case here, because we believe that the mentioned methods do this problem only technical and it is possible to solve it similarly in other cases as well.

## 7. On underlying bundles

The factor algebra $B = A/\mathfrak{N}^r$ is called the *underlying algebra (of order $r-1$)* and the Weil bundle $T^B M$ is said to be the *underlying bundle* of $T^A M$. We recall an important result about Weil bundles proved recently by I. Kolář in [5].

*$T^A M \to T^B M$ is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes \mathfrak{N}^r$ over $T^B M$.*

It is easy to verify that we obtain the underlying bundle by the removing maximal components with cardinalities $r$ from the complex in question. If we write $\operatorname{Under}(T^A M)$ for the underlying bundle of $T^A M$, we discover directly following examples.

**Examples**

A.

$$\operatorname{Under}(\tilde{T}_k^r M) = B_k^{r-1} M,$$

where $B_k^{r-1} M = \tilde{T}_k^r[M, K]$ and $K$ is the subcomplex of $k$-colored $(r-1)$-simplex origined by removing of the maximal component. $B_k^{r-1}M$ is the so-called *(colored) bundle of boundaries*, cf. [10].

B.

$$\operatorname{Under}(B_k^r M) = \operatorname{Under}(\tilde{T}_k^r M) = \operatorname{Under}(T_k^r M) = T_k^{r-1} M.$$

C.

$$\operatorname{Under}(\tilde{T}_{k_n}^{r_n} \dots \tilde{T}_{k_1}^{r_1} M) = B_{k_n}^{r_n-1} \tilde{T}_{k_{n-1}}^{r_{n-1}} \dots \tilde{T}_{k_1}^{r_1} M \times_{B_{k_n}^{r_n-1} \dots B_{k_1}^{r_1-1} M} \cdots \times_{B_{k_n}^{r_n-1} \dots B_{k_1}^{r_1-1} M} \tilde{T}_{k_n}^{r_n} \dots \tilde{T}_{k_2}^{r_2} B_{k_1}^{r_1-1} M.$$

**D.**

$$\operatorname{Under}(B^{r_n}_{k_n}\dots B^{r_1}_{k_1}M)=\operatorname{Under}(\bar T^{r_n\dots r_1}_{k_n\dots k_1}M)=\operatorname{Under}(T^{r_n}_{k_n}\dots T^{r_1}_{k_1}M)=$$
$$T^{r_n-1}_{k_n}T^{r_{n-1}}_{k_{n-1}}\dots T^{r_1}_{k_1}M\times_{T^{r_n-1}_{k_n}\dots T^{r_1-1}_{k_1}M}\cdots$$
$$\times_{T^{r_n-1}_{k_n}\dots T^{r_1-1}_{k_1}M}T^{r_n}_{k_n}\dots T^{r_2}_{k_2}T^{r_1-1}_{k_1}M\,.$$

**E.**

$$\operatorname{Under}(T_k^{R,S_1\dots S_{k-1}}M)=T_k^{\bar R,\bar S_1\dots\bar S_{k-1}}M\,,$$

where we substract the unit in such upper indices which are equal to $r$ (clsewhere, we do not change them).

## References


[1] Doupovec, M., Kolář, I., *On the jets of fibered manifold morphisms* (to appear).
[2] Eck, D.J., *Product-preserving functors on smooth manifolds*, J. Pure Appl. Algebra **42** (1986), 133–140.
[3] Ehresmann, C., *Extension du calcul des jets aut jets non holonomes*, CRAS Paris **239** (1954), 1762–1764.
[4] Kainz, G., Michor, P.W., *Natural transformations in differential geometry*, Czechoslovak Math. J. **37** (1987), 584–607.
[5] Kolář, I., *Affine structure on Weil bundles* (to appear).
[6] ———, *Bundle functors of the jet type* (to appear).
[7] ———, *Weil algebras* $\mathbb{D}^2_m$, $\bar{\mathbb{D}}^2_m$, $\tilde{\mathbb{D}}^2_m$, Unpublished lecture notes. (Czech)
[8] Kolář, I., Michor, P.W., Slovák, J., *Natural Operations in Differential Geometry*, Springer Verlag, 1993.
[9] Kureš, M., *Affinors and connections in higher order geometry*, Proc. 7th Int. Conf. Diff. Geometry and Appl., Brno, 1998, Masaryk University in Brno (1999), 239–246.
[10] Luciano, O.O., *Categories of multiplicative functors and Weil's infinitely near points*, Nagoya Math. J. **109** (1988), 69–89.
[11] White, J.E., *The Method of Iterated Tangents with Applications in Local Riemannian Geometry*, Pitman, London, 1982.



Department of Mathematics, Technical University of Brno
Technická 2, 616 69 Brno, Czech Republic
*E-mail*: kures@mat.fme.vutbr.cz