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Supersymmetry algebras: extensions of orthgonal Lie algebras

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SUPERSYMMETRY ALGEBRAS : EXTENSIONS OF ORTHOGONAL LIE ALGEBRAS

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I. INTRODUCTION

The object of this note is to describe the supersymmetry algebras associated canonically with orthogonal Lie algebras $\mathbf{so}(\eta)$ (η being the non-degenerate quadratic form of (p,q) signature).

The above canonical correspondence is achieved by introduction of, an abstract but natural, notion of the spinorial extension $\mathfrak{S}(\eta)$ of the orthogonal Lie algebra.

In order to elucidate the way in which $\mathfrak{S}(\eta)$ corresponds to $\mathbf{so}(\eta)$, let us consider quite similar but much more familiar and simpler object associated with $\mathbf{so}(\eta)$, namely it's vectorial extension $\mathcal{E}(\eta)$. It is defined as Z_2-graded Lie algebra, such that the even subalgebra $\mathcal{E}_{(0)}(\eta)$ does contain $\mathbf{so}(\eta)$ and the odd subspace $\mathcal{E}_{(1)}(\eta)$ is isomorphic with the module $\mathbf{E}(\eta)$, of the vector representation of $\mathbf{so}(\eta)$. Moreover we assume that $\mathcal{E}_{(0)}(\eta) = [\, \mathcal{E}_{(1)}(\eta) , \ \mathcal{E}_{(1)}(\eta) \,]$.

The above conditions are satisfied by the pair of Lie algebras; namely $\mathcal{E}(\eta)$ ((p,q)-signature of the form η) is isomorphic with either $\mathbf{so}(p+1,q)$ or $\mathbf{so}(p,q+1)$. The geometrical meaning of such a correspondence is clear (e.g. [4]). The important assumption we made in definition of $\mathcal{E}(\eta)$ was the irreducibility of $\mathcal{E}_{(1)}(\eta)$ with respect to $\mathbf{so}(\eta)$. If we violate it, we can obtain infinite sequence of Lie algebras satisfying remaining conditions.

Moreover, we could then also, permit the elements from $\mathcal{E}_{(1)}(\eta)$ to satisfy symmetric structural relations (anticommutators instead of commutators) and we would get the sequence of Lie superalgebras (e.g. from osp series) satisfying the axioms of $\mathcal{E}(\eta)$. This is the reason, why we have to remain in power the assumption of irreducibility.

The spinorial extension $\mathfrak{S}(\eta)$ is defined (see Def. 1.II) in almost the same way; with the vector module $\mathbf{E}(\eta)$ replaced by the spinorial one $\mathbf{\$}(\eta)$.

The next modification consists in (according with the spirit of supersymmetry) admitance the odd elements of $\mathfrak{S}_{(1)}(\eta)$ to satisfy symmetric structural relations. However, it has to be stressed that we do not eliminate the Lie algebras from the game.

It is achieved by assuming that the spinorial extension $\mathfrak{S}(\eta)$ is the element of the category of Z_2-graded ϵ-Lie algebras [10]. It is a good point to make a short digression in order to fix some notions concerning the above category.

DEFINITION 1.I

A Z_2-graded vector space $\mathcal{A} = \mathcal{A}_{(0)} \oplus \mathcal{A}_{(1)}$, with bilinear operation

$$\mathcal{A} \times \mathcal{A} \ni (X,Y) \mapsto \langle X,Y \rangle \in \mathcal{A}$$

preserving the Z_2-graded structure of it i.e.

$$\langle \mathcal{A}_{(i)}, \mathcal{A}_{(j)} \rangle \subset \mathcal{A}_{(i+j)(\mathrm{mod}2)}$$

is called a Z_2-graded ϵ-Lie algebra if for elements, homogeneous in sense of Z_2-grading, following axioms are satisfied

i) $\langle X,Y \rangle = -\epsilon(X,Y)\langle Y,X \rangle$

ii) $\langle X,\langle Y,Z \rangle\rangle = \langle\langle X,Y \rangle,Z \rangle + \epsilon(X,Y)\langle Y,\langle X,Z \rangle\rangle$

where either $\epsilon \equiv 1$ or $\epsilon(X,Y) = (-1)^{\deg X \deg Y}$.

The category of Z_2-graded ϵ-Lie algebras consists of two separate subcategories: the category of Z_2-graded Lie algebras, which corresponds to first choice of the function ϵ and the category of Lie superalgebras, which corresponds to second choice of commutation factor. Notice that the values of commutation factors for the subclasses in the category of Z_2-graded ϵ-Lie algebras differ in sign only for two odd elements. Further on, to shorten the notation, the commutation factor will be understood as this very sign function which describes the symmetry of the operation $\langle\, , \rangle$ for odd elements. There are two important mappings associated to each Z_2-graded ϵ-Lie algebra.

A first one is the representation of $\mathcal{A}_{(0)}$ Lie algebra on $\mathcal{A}_{(1)}$ odd subspace

$$\mathcal{A}_{(0)} \ni a \mapsto H(a) \in \mathrm{End}(\mathcal{A}_{(1)}) \quad . \tag{1.I}$$

where H(a) is given according to

$$\mathcal{A}_{(1)} \ni k \mapsto H(a)(k) := \langle a,k \rangle \in \mathcal{A}_{(1)} \quad .$$

A second mapping is generated by $\langle\, , \rangle$ on $A_{(1)} \times A_{(1)}$:

$$\otimes^2 \mathcal{A}_{(1)} \ni t \mapsto g(t) \in \mathcal{A}_{(0)} \quad , \tag{2.I}$$

where g is the universal extension of the following

$$\mathcal{A}_{(1)} \times \mathcal{A}_{(1)} \ni (k,k') \mapsto \tilde{g}(k,k') := \langle k,k' \rangle \in \mathcal{A}_{(0)} \quad .$$

It will be said the structural homomorphism of $\mathcal{A}$.

Let us return to the heart of the matter. Fixing the spinorial extension $\mathfrak{S}(\eta)$ to be an element of the above category, means that the spinorially extended orthogonal algebra is not necessarily Lie superalgebra, as it equally well might appear that it is Z_2-graded Lie algebra - the eventual choice is to be decided by the form alone.

For this last problem to be well posed we are forced to assume $\mathfrak{S}_{(1)}(\eta)$ - the odd subspace of $\mathfrak{S}(\eta)$ - to be irreducible with respect to $so(\eta)$. Otherwisely, as in the case of vectorial extension $\mathcal{E}(\eta)$, the coupling between the metric structure η and the object $\mathfrak{S}(\eta)$ describing the associated supersymmetries, becomes weak enough to admit almost every Z_2-graded ϵ-Lie algebra.

In order to avoid such a uncertainity we shall restrict our considerations to the algebras describing so called N=1 supersymmetries.

For the geometrical meaning of spinorial extensions and their connection with ordinarily used superextensions to be stated precisely, let us recall at first, some essential properties of Lorenzian and Euclidean Lie groups: $SO(p,q)$ with q=1 or 0 respectively and p>2. Every special orthogonal group $SO(p,q)$ (rigorously it's connected component of unity) defines two series of finite dimensional representations of it's own Lie algebra - due to biconnectedness of the group manifold. These are integrable and non-integrable representations. A representation τ : $so(p,q) \to End(V)$ is said to be integrable iff there exists a homomorphism T: $SO(p,q) \mapsto Aut(V)$, such that the following diagram

$$\begin{array}{ccc} so(p,q) & \xmapsto{\tau} & End(V) \\ Exp \downarrow & & \downarrow \\ SO(p,q) & \xmapsto{T} & Aut(V) \end{array} \tag{3.I}$$

does commute.

Otherwisely the representation τ is said to be non-integrable. In the last case a r.h.s. exponent of (3.I) provides the representation of $Spin(p,q)$ group (rigorously it's connected component of unity). Naturally $Spin(p,q)$ is a twofold - hence universal (q = 1,0 and p>2) covering of $SO(p,q)$.

All finite dimensional representations of $Spin(\eta)$ group are realized on subspaces of the tensor algebra:

$$\otimes \mathcal{S}(\eta) = \bigoplus_{p=0}^{\infty} (\otimes_R^p \mathcal{S}(\eta)) \tag{4.I}$$

where the tensor product is meant over reals and $\mathcal{S}(\eta)$ denotes spinor module. Obviously (4.I) contains all finite dimensional $so(\eta)$ Lie algebra representations. The (4.I) decomposes onto two subspaces carrying integrable and non-integrable representations of $so(\eta)$ Lie algebra and it is this very direct sum decomposition, which coincides with the one induced by the natural Z_2-grading of the tensor product. In light of the above, our spinorial extensions (when represented on the subspaces of (4.I)) are to be interpreted as the algebras of infinitesimal transformations, which map mutually one onto another the objects belonging to integrable and non-integrable representations of a given orthogonal Lie algebra. This mixing is achieved since the spinor module $\mathfrak{S}(\eta)$ (generating whole (4.I)) is included into the structure of extension. The Z_2-gradation of $\mathcal{S}(\eta)$ is the direct consequence of the same property of (4.I).

Within the above context there is a possibility of a purely geometrical generalization of the notion of supersymmetry. The supersymmetry might mean not the symmetry between bosons and fermions (these are rather phenomenological notions) but the symmetry between the geometric objects (fields) belonging to integrable and non-integrable representations of orthogonal Lie algebras. This point of view is universal independently of the dimension (p+q) and also makes sense for classical fields.

Within the framework of quantum field theory in Minkowski spacetime, due to generally believed spin-statistics theorem [5], the above is equivalent to usual understanding of supersymmetry.

In the case, when the conditions q=1,0 and p>2 are violated i.e. when $SO(p,q)$ has fundamental group larger than Z_2 (actually infinite) (4.I) does not contain all irreducible representations modules. It means that the algebra $\mathcal{S}(p,q)$ is able to describe infinitesimal transformations between objects belonging to integrable representations of $so(p,q)$ and objects belonging to rather poor subclass of non-integrable ones - exactly that contained in (4.I).

II. SPINORIAL EXTENSIONS OF ORTHOGONAL LIE ALGEBRAS - GENERAL PROPERTIES

Let us now formulate an exact definition of the object we have roughly described in introduction. All the properties of the spinorial extension we have mentioned and used above are collected within following

DEFINITION 1.II.

A Z_2-graded ε-Lie algebra (Def. 1.I) $\mathcal{S}(\eta) = \mathcal{S}_{(0)}(\eta) \oplus \mathcal{S}_{(1)}(\eta)$ is called a spinorial extension of $so(\eta)$ orthogonal Lie algebra iff the following axioms are satisfied:

i) There exists a $\mathbb{R}$-linear bijection

$$\mathfrak{S}(\eta) \ni \psi \mapsto Q(\psi) \in \mathcal{S}_{(1)}(\eta)$$

and non-zero homomorphism

$$so(\eta) \ni \Sigma \mapsto \ell(\Sigma) \in \mathcal{S}_{(0)}(\eta)$$

such that the following diagram, with τ being the spinorial representation and H that of (1.I)

$$\begin{tikzcd}
 & \tau_\Sigma : \; \mathcal{S}(\eta) \arrow[r, mapsto] \arrow[dd, "Q"'] & \mathcal{S}(\eta) \arrow[dd, "Q"] \\
\mathfrak{so}(\eta) \ni \Sigma \arrow[ur] \arrow[dr] & & \\
 & (H\circ\ell)(\Sigma) : \; S_{(1)}(\eta) \arrow[r, mapsto] & S_{(1)}(\eta)
\end{tikzcd}$$

does commute

ii) $$S_{(0)}(\eta) = \langle S_{(1)}(\eta), S_{(1)}(\eta) \rangle \quad .$$

Axiom i) of the above definition, forcing the existence of immersions Q and ℓ of $\mathcal{S}(\eta)$ spinor space and $\mathfrak{so}(\eta)$ orthogonal Lie algebra into $S(\eta)$, establishes the equivalence of spinorial representation τ of $\mathfrak{so}(\eta)$ Lie algebra with that induced by homomorphism H. In other words the condition, the above axiom imposes on $S(\eta)$, means that the structure of spinorial module is built in the structure of extension; consequently the extension is essentially spinorial.
It should be stressed that we do not assume $S_{(0)}(\eta)$ to be isomorphic with $\mathfrak{so}(\eta)$, as it might appear to be inconsistent with the axiom ii). This last axiom excludes, in turn, a trivial extensions with $\langle S_{(1)}(\eta), S_{(1)}(\eta) \rangle = 0$ and $\mathfrak{so}(\eta) \subset S_{(0)}(\eta)$. Moreover it denotes minimality of $S(\eta)$.
For the form η being chosen, each of the objects satisfying the conditions of definition 1.II, have odd subspaces isomorphic with spinorial module $\mathcal{S}(\eta)$, therefore they can differ only in the commutation factors and $S_{(0)}(\eta)$ even subalgebras.
We are going to show firstly that all $S(\eta)$ ϵ-Lie algebras are "almost" simple.
For that purpose we introduce a universal homomorphism, transporting faithfully the representations H (1.I) of $S_{(0)}(\eta)$ on $S_{(1)}(\eta)$ onto $\mathcal{S}(\eta)$ spinor module:

$$S_{(0)}(\eta) \ni m \mapsto L(m) \in \mathrm{End}_{\mathbb{R}}(\mathcal{S}(\eta)) \quad , \qquad (1.\mathrm{II})$$

where the transformation $L(m)$ is defined according to

$$\mathcal{S}(\eta) \ni \psi \mapsto L(m)(\psi) := (Q^{-1} \circ H(m) \circ Q)(\psi) \in \mathcal{S}(\eta)$$

Almost immediately from the homomorphism property of L and ii) Def.1.II. one can show the following

LEMMA 1.II.

Ker L is central abelian ideal in $S(\eta)$. ∎

It is then natural to identify all $S(\eta)$ ϵ-Lie algebras which differ by abelian ideal, expecially in light of the following

LEMMA 2.II.

The quotient algebra $S(\eta)/\ker L$ is simple

PROOF:

The representation H of $S_{(0)}(\eta)/\ker L$ on $S_{(1)}(\eta)$ module is faithful, which combined with it's irreducibility (forced by the same property of $so(\eta)$) and condition ii) Def. 1.II., ensures ([6]) simplicity of quotient structure. ∎

The above mentioned identification is realized by the following equivalence relation in the set $\hat{\$}(\eta)$ of all spinorial extensions (for η being chosen);

$$S(\eta) \approx S'(\eta) \qquad \text{iff} \qquad S(\eta)/\ker L \sim S'(\eta)/\ker L$$

where the r.h.s. isomorphism is understood in stronger sense than general morphism of the category of Z_2-graded ϵ-Lie algebras; namely it is not allowed to identify two objects with different commutation factors.

The class of simple spinorial extensions is obviously identical to quotient set $\$(\eta) := \hat{\$}(\eta)/\approx$.

Arbitrary spinorial extension may be obtained from the simple one by it's central extension, whose form is described by the structure of the simple object and classified by respective cohomology group.

It is also clear, that the restriction of our considerations to simple algebras has not effect on classification of the commutation factors that the metric η admits.

A necessary condition for spinorial extension with a given commutation factor ϵ to exist is contained in the following

LEMMA 3. II.

If there exists the extension $S(\eta)$ with the commutation factor ϵ, then on the spinor module $S(\eta)$, there exists a non-degenerate, $so(\eta)$-invariant and ϵ-symmetric bilinear form.

PROOF:

Assume firstly $S(\eta)$ to be Lie algebra i.e. $\epsilon = 1$. Being simple (Lemma 2.II), it provides non-degenerate, invariant and symmetric Killing functional K . The form $\xi := K\circ(Q\times Q)$ on $S(\eta)\times S(\eta)$ has then all desired properties.

In the case when $S(\eta)$ is Lie superalgebra i.e. $\epsilon = -1$, the simplicity does not quarantee existence of non-degenerate Killing functional, as it happens the Killing form does vanish. To eliminate this possibility for $S(\eta)$ we are forced to use the classification of simple Lie superalgebras given in [6] and [9]. Le us note that each $S(\eta)$ is classical in sense of [6], and moreover the representation of $S_{(0)}(\eta)$ on $S_{(1)}(\eta)$ is irreducible. The only superalgebras of this class with vanishing Killing form are $D(2,1;\alpha)$ and $Q(m)$[6]. However, the structure of their odd modules is inconsistent with axiom i) of definition 1.II. Consequently $\mathcal{S}(\eta)$ provides non-degenerate Killing functional, whose restriction $\xi := K\circ(Q\times Q)$ to the product of spinor spaces is antisymmetric and $so(\eta)$-invariant. ∎

In the course of further considerations we shall prove the statement converse to the above; meanwhile we can draw from it the following

LEMMA 4. II.

If the signature (p,q) of the form η does satisfy the equations

$$\left.\begin{array}{l} p + q = 2 \\ p - q = 0 \end{array}\right\} \pmod 4$$

then $\mathcal{S}(p,q)$ does not exist.

PROOF:

The form ξ constructed in the proof of the above Lemma is $so(\eta)$-invariant. For the signatures satisfying the above equations such a form on spinor module $\mathcal{S}(\eta)$ does not exist (see below-section III). ∎

III. CANONICAL EXTENSIONS

In the proceeding section we obtained general information about the structure of spinorial extensions. Before we pass to more detailed discussion and classification of $\mathcal{S}(\eta)$ algebras we shall add one more axiom to the Definition 1.II. This axiom restricts the class of admissible extensions to the subclass containing algebras which we call canonical.

The possibility of formulating such an additional condition originates in the fact that we are working in the category of real vector spaces and modules, where the Schur's lemma is not valid in general.

Despite of irreducibility of $so(\eta)$ on $\mathcal{S}(\eta)$ it might appear and in fact appears, depending on the signature of η, that the centralizer $F(\eta)$ of $so(\eta)$ in $\mathrm{End}_R(\mathcal{S}(\eta))$ is non-trivial i.e. not isomorphic with $\mathbb{R}$ (nothing similar can happen for vector module $E(\eta)$). This, in turn, means that the associative closure of $so(\eta)$ Lie algebra in $\mathrm{End}_{\mathbb{R}}(\mathcal{S}(\eta))$ is it's proper subalgebra.

Fortunately, due to Weddeburn's theorem, we know the above centralizer $F(\eta)$ is division ring and the algebra, associatively generated by the representation of $so(\eta)$ on $\mathcal{S}(\eta)$ spinor module is isomorphic with the algebra of square matrices with entries from $F(\eta)$. Moreover this algebra is identical with the even subalgebra $\mathcal{C}_{(+)}(\eta)$ (or it's simple ideal) of the Clifford algebra $\mathcal{C}(\eta)$ corresponding to the form η (e.g. [1] or [7]).

Hence the spinor module $\mathcal{S}(\eta)$ is to be identified with the module of irreducible

representation of $\mathcal{C}_{(+)}(\eta)$.
This representation is faithful iff $\mathcal{C}_{(+)}(\eta)$ algebra is simple i.e. iff the signature (p,q) of η does satisfy the condition $p-q \neq 0 \pmod 4$ [8] .
Otherwisely i.e. iff $p-q=0 \pmod 4$, the kernel of this representation is one from the ideals $\mathcal{C}_{(+)}(\eta)$ decomposes onto:

$$\mathcal{C}_{(+)}(\eta) = 1/2\,(1+J)\,\mathcal{C}_{(+)}(\eta) \oplus 1/2\,(1-J)\,\mathcal{C}_{(+)}(\eta) \tag{1.III}$$

where $J = 1/(p+q)!\,e_1 e_2 \ldots e_{p+q}$ is the central, unimodular basic pseudoscalar and $\{e_a\}_1^{p+q}$ is η-orthonormal basis of the generator of $\mathcal{C}(\eta)$.
The Lie algebra $so(\eta)$ is to be identified with the set of bivectors $\{e_a e_b \ ; \ a<b \ \ 1 \leqslant a,b \leqslant p+q\}$. The kernels of the spinorial representations of $so(\eta)$ are non-trivial only for $(p,q) = (2,2)$, $(4,0)$, and then they are spanned by either self-dual or antiself-dual bivectors according to which one of the ideals (1.III) is the kernel of the representation of $\mathcal{C}_{(+)}(\eta)$.
To simplify notation by $\mathcal{C}_{(+)}(\eta)$ we shall denote the ideal faithfully represented on $\mathcal{S}(\eta)$ i.e. whole even subalgebra or one from ideals of (1.III).
The spinor module $\mathcal{S}(\eta)$ does admit one-sided (say right-sided) linear structure over $\mathcal{F}(\eta)$.
It is then natural to introduce the following reduced Lie algebra of endomorphisms of $\mathcal{S}(\eta)$ spinor module:

$$\mathrm{End}_{\mathbb{R}}(\mathcal{S}(\eta)) \supset \mathrm{End}\,(\mathcal{S}(\eta)) \ := \ s\,\mathcal{C}_{(+)}(\eta) \oplus \mathcal{F}(\eta) \quad , \tag{2.III}$$

where $s\mathcal{C}_{(+)}(\eta) := [\mathcal{C}_{(+)}(\eta), \mathcal{C}_{(+)}(\eta)]$ is derived Lie algebra of $\mathcal{C}_{(+)}(\eta)$.
We used $s\mathcal{C}_{(+)}(\eta)$ instead of simply $\mathcal{C}_{(+)}(\eta)$ in order to avoid the doubling of the centre of $\mathcal{C}_{(+)}(\eta)$ being identical (when (2.III) is embodied in $\mathrm{End}_{\mathbb{R}}(\mathcal{S}(\eta))$) with that of $\mathcal{F}(\eta)$.
We are now in a position to formulate the following
DEFINITION 1.III.
A spinorial extension $\mathcal{S}(\eta)$ is said to be canonical
iff $L(\mathcal{S}_{(0)}(\eta)) \subset \mathrm{End}\,(\mathcal{S}(\eta))$ (is that of 1.II).

The subclass $\mathbb{S}(\eta)$ of canonical extensions contains the objects, which admitting the decomposition required in 2.III, do "remember" the linear structure of $\mathcal{S}(\eta)$ over $\mathcal{F}(\eta)$. It is natural to accompany the axioms of definition 1.II by the postulate of spinorial extension to be canonical.
However, there arises the question, whether this sharpening of definition is that essential that it reduces the admissible types of commutation factors allowed by spinorial extension? We hasten to assure, that the answer is negative (it will become obvious at the end of this section), therefore without loss of generality we

can be concerned with only these spinorial extensions, whose even subalgebras have canonical structure.

From the fact that the algebra $L(\mathcal{S}_{(0)}(\eta))$ is contained in the direct sum of two ideals, one deduces an analogous decomposition of $\mathcal{S}_{(0)}(\eta)$ Lie algebra

$$\mathcal{S}_{(0)}(\eta) = \mathcal{G}(\eta) \oplus i(\eta) \tag{3.III}$$

The ideals $\mathcal{G}(\eta)$ and $i(\eta)$ will be called geometric and respectively internal sector of $\mathcal{S}_{(0)}(\eta)$.

Let us now define the Lie algebra

$$\mathcal{L}(\eta) := \{ c \in s\mathcal{C}_{(+)}(\eta) \; ; \; \beta(c) = -c \} \tag{4.III}$$

of the elements which are antiselfconjugate with respect to the main antiautomorphism β of $\mathcal{C}_{(+)}(\eta)$. It is the unique antiautomorphism with such property, that (4.III) does contain $so(\eta)$. Since, as we assumed, $\mathcal{C}_{(+)}(\eta)$ denotes either whole even subalgebra or it's proper simple ideal if p-q=0 (mod4), we have to check whether such an $so(\eta)$- invariant antiautomorphism does exist on $s\mathcal{C}_{(+)}(\eta)$. The answer is that it exists for all (p,q) signatures, except that satisfying the equations of Lemma 4.II. [8].

We shall now proceed to show, that independently of commutation being admissible, an arbitrary Z_2-graded ϵ-Lie algebra $\mathcal{S}(\eta)$ has the same one geometric sector $\mathcal{G}(\eta)$.

THEOREM 1.III.

$$\mathcal{G}(\eta) \sim \mathcal{L}(\eta)$$

PROOF:

The proof of this theorem will be sketched here. The details are easy to be reconstructed.

The idea is to compare the structural mapping of $\mathcal{S}(\eta)$

$$\otimes^2_R \mathcal{S}'(\eta) \ni t \mapsto \Phi(t) := (L^{-1} \circ \pi \circ g \circ (Q \otimes Q))(t) \in \mathcal{G}(\eta) \tag{5.III}$$

(π: End$(\mathcal{S}'(\eta)) \mapsto s\mathcal{C}_{(+)}(\eta)$ being the natural projection) with the surjection $\Psi_\xi : \otimes^2_R \mathcal{S}(\eta) \mapsto \mathcal{L}(\eta)$, naturally associated with the form ξ described in Lemma 3.II.

Let $\{R_i\}$ be the set of endomorphisms describing the right multiplications by imaginary units of $F(\eta)$ division ring i.e. satisfying the structural relations

$$R_i R_j = - \delta_{ij} - \epsilon_{ijk} R_k \tag{6.III}$$

We can now define $F(\eta)$ - valued form

$$S(\eta)\times S(\eta)\ni(\psi,\psi')\mapsto\Theta(\psi,\psi'):=\xi(\psi,\psi')+\sum_i\xi(\psi,R_i^{-1}\psi')R_i\in F(\eta) \tag{7.III}$$

where ξ is that of Lemma 3.II.
Define the mapping:

$$\otimes^2_R S(\eta)\ni t=\sum\varphi_i\otimes\varphi_i'\mapsto\varkappa(t):=\sum\varkappa(\varphi_i,\varphi_i')\in \mathrm{End}_R(S(\eta)) \tag{8.III}$$

where an endomorphism $\varkappa(\varphi,\varphi')$ is given according to:

$$S(\eta)\ni\psi\mapsto\varkappa(\varphi,\varphi')(\psi):=\Theta(\varphi',\psi)\varphi\in S(\eta) \tag{9.III}$$

Let $T:\otimes^2_R S(\eta)\mapsto\otimes^2_R S(\eta)$ be the transposition of tensors and let $\pi_\epsilon:=1/2(id-\epsilon T)$ denotes the projection onto the subspace of ϵ -skew symmetric tensors.
Then the mapping

$$\otimes^2_R S(\eta)\ni t\mapsto\Psi_\xi(t):=((\varkappa-\mathfrak{z}\circ\varkappa)\circ\pi_\epsilon)(t)\in\mathcal{L}(\eta) \tag{10.III}$$

where $\mathfrak{z}:\mathcal{C}_{(+)}(\eta)\mapsto\mathfrak{Z}\mathcal{C}_{(+)}(\eta)$ denotes the projection of $\mathcal{C}_{(+)}(\eta)$ onto it's centre, is surjective.
The proof of this statement consists of two steps.
i) one has to check that $\varkappa(\otimes^2_R S(\eta))=\mathcal{C}_{(+)}(\eta)$.
This follows from non-degeneracy of (7.III), which in turn implies the irreducibility on $S(\eta)$ of the image of $\otimes^2_R S(\eta)$ under $\varkappa$. Direct calculation shows that the endomorphisms (8.III) commute with that of (6.III). Consequently we have required identity.
ii) One has to show that $\beta\circ\varkappa=\epsilon\varkappa\circ T$.
The r.h.s. of the above identity is the expression for the mapping (8.III) conjugated with respect ξ (hence also Θ). Since this form is $so(\eta)$-invariant, the conjugation generated by it, has to be equal to β as it is the unique one with respect to which the elements of $so(\eta)$ are skew-symmetric.
In light of the above (10.III) describes exactly the mapping $\varkappa$ composed with the projection onto $\mathcal{L}(\eta)$. Note that (5.III) as well as (8.III) are homomorphisms of respective representations of $\mathcal{G}(\eta)$ and $\mathcal{L}(\eta)$ Lie algebras; namely that differentiating the tensor product: $d_c(\psi\otimes\psi')=c\psi\otimes\psi'+\psi\otimes c\psi'$, and adjoint one. Moreover, $\mathcal{G}(\eta)\subset\mathcal{L}(\eta)$, which follows from $\mathcal{G}(\eta)$ -invariance of (7.III). The above together with $so(\eta)\subset\mathcal{G}(\eta)$ implies simplicity of $\mathcal{G}(\eta)$.
The space $\otimes^2_R S(\eta)$ can be decomposed onto the following direct sums:

$$\operatorname{coker}\Psi_\xi\oplus\ker\Psi_\xi=\otimes^2_R S(\eta)=\operatorname{coker}\Phi\oplus\ker\Phi$$

and it is obvious that coker$\Phi \cap$ coker$\Psi_\xi \neq \{0\}$. Since all subspaces of the above decompositions are $d\mathcal{G}(\eta)$-invariant, their intersections have the same property. The simplicity of $\mathcal{G}(\eta)$ yields it's transitivity on cokerΦ. This in turn gives coker $\Phi \subset$ cokerΨ_ξ. Let now Ψ_ξ denotes it's restriction to cokerΨ_ξ. Then the mapping $\Phi\circ\Psi_\xi^{-1}: \mathcal{L}(\eta) \mapsto \mathcal{G}(\eta)$ provides the surjective homomorphism of adjoint representation of $\mathcal{L}(\eta)$ Lie algebra.
The simplicity of $\mathcal{L}(\eta)$ forces it to be an isomorphism and moreover $\Phi = \alpha\Psi_\xi$ for some $\alpha \in \mathfrak{z}\mathcal{E}_{(+)}(\eta)$. ∎

We have proven more than $\mathcal{G}(\eta) \sim \mathcal{L}(\eta)$; namely we are able to identify the structural homomorphism Φ of $\mathcal{S}(\eta)$ with that given in (10.III). Note also that the coefficient $\alpha \in \mathfrak{z}\mathcal{E}_{(+)}(\eta)$, the mappings Φ and Ψ_ξ differ in, is always real if $F(\eta) \neq \mathbb{C}$. If the case of $F(\eta) = \mathbb{C}$, it can be complex iff the mappings Φ and Ψ_ξ are analytic i.e. the algebra $\mathcal{L}(\eta)$ is the complex one.
In virtue of the above theorem (possibly after appropriate rescaling) we shall consider Φ to be equal Ψ_ξ. Note that the mapping Ψ_ξ $(\equiv\Phi)$ depends on the commutation factor of $\mathcal{S}(\eta)$; namely via the symmetry of the form $\xi = \mathrm{Re}\,\Theta$.
In order to determine the structure of the whole $\mathcal{S}_{(0)}(\eta)$ Lie algebra, we have to construct the surjective mapping $\Psi_\xi \oplus \mathfrak{f}_\xi : \otimes^2_R \mathcal{S}(\eta) \mapsto \mathcal{G}(\eta) \oplus i(\eta)$.
Hence the following
LEMMA 1.III

$$\mathfrak{f}_\xi = -\Theta^{\otimes}\circ\pi_\epsilon + \mathcal{X}\circ\mathcal{K}\circ\pi_\epsilon$$

where $\Theta^{\otimes}$ denotes universal extension of Θ onto $\otimes^2_R\mathcal{S}(\eta)$, and π_ϵ is ϵ-skew-symmetrization.
PROOF:
Follows immediately from the generalized Jacobi identity (Def.1.I i)) rewritten in terms of $\Psi_\xi \oplus \mathfrak{f}_\xi$ ∎
We are now in a position to perform the classification of all canonical spinorial extensions $\mathcal{S}(\eta)$ with respect to the signature (p,q) of the form η.
We have shown, that to arbitrary ϵ-Lie algebra $\mathcal{S}(\eta)$ there corresponds $\mathcal{L}(\eta)$-invariant and $F(\eta)$-valued form (7.III) on $\mathcal{S}(\eta)$. This form is moreover ϵ-symmetric with respect to the certain antiautomorphism of $F(\eta)$. For $J: F(\eta) \mapsto F(\eta)$ defined according to $\Theta(\alpha\cdot,\cdot) = \Theta(\cdot,\cdot)J(\alpha)$, $\alpha \in F(\eta)$ one can prove $J\circ\Theta = \epsilon\Theta\circ T$ (T-transposition). It is easy to note, that the converse is also true. To arbitrary form with the above properties there corresponds, as described in the proof of Theorem 1.III, a surjective homomorphism Ψ_ξ onto $\mathcal{L}(\eta)$ Lie algebra. This very mapping after completion by $\mathfrak{f}_\xi$ (Lemma 1.III) yields the structural homomorphism of some graded ϵ-Lie algebra from canonical class $\mathcal{S}(\eta)$.
Since the structure of $\mathcal{L}(\eta)$ Lie algebras is known [8] (to use it, one has to note

that $\mathfrak{t}_{(+)}(q,p)\sim\mathfrak{t}_{(+)}(p,q)\sim\mathfrak{t}_{(+)}(p,q-1)$ and that the image of β under the last isomorphism equals to β_- of [8]), it is easy to classify all $\mathcal{L}(\eta)$-invariant forms on $\mathcal{S}(\eta)$ and their symmetries with respect to all antiautomorphisms of $F(\eta)$ division ring. It enables us to determine all admissible commutation factors and internal sectors $\dot{\iota}(\eta)$.

All $\mathcal{L}(\eta)$ Lie algebras [8] can be divided anto four basic classes.

1° Analytic series

$$\mathcal{L}(\eta)\sim sp(\cdot\,,F)\ ,\ so(\cdot\,,F)\ ;\ F\sim\mathbb{R},\mathbb{C}$$

The forms corresponding to these Lie algebras are respectively either antisymmetric or symmetric and analytic in complex case. Hence $\epsilon=-1$ for $sp(\cdot,F)$ and $\epsilon=1$ for $so(\cdot\,,F)$. Obviously $f_\xi\equiv 0$ independently of ϵ.

2° Complex unitary series

$$\mathcal{L}(\eta)\sim su(\cdot\,,\mathbb{C})$$

The form invariant with respect to this Lie algebra is either hermitean or antihermitean. In this case the following lemma is true.

LEMMA 2.III.

$$f_\xi=-(r+\epsilon)/r\ \dot{\iota}\ \mathrm{Im}\,\Theta^{\otimes}\qquad,\ \text{hence}\quad \dot{\iota}(\eta)=\dot{\iota}\mathbb{R}$$

$$r=\dim_{\mathbb{R}}\mathcal{S}(\eta)\ .$$

PROOF:

If $\varkappa\circ\varkappa\circ\pi_\epsilon\neq 0$, then there exists $\dot{\iota}\in\mathfrak{t}_{(+)}(\eta)$, such that $\beta(\dot{\iota})=-\dot{\iota}$ and $\dot{\iota}^2=-1$ (the case $\dot{\iota}^2=1$ is excluded in Lemma 4.II). This element defines complex structure in $\mathfrak{t}_{(+)}(\eta)$. Standard manipulation gives $\dot{\iota}^{-1}\varkappa\circ\varkappa\circ\pi_\epsilon=-\epsilon/r\ \mathrm{Im}\,\Theta^{\otimes}$ and $\Theta^{\otimes}\circ\pi_\epsilon=\dot{\iota}\,\mathrm{Im}\,\Theta^{\otimes}$ ∎

3° Quaternionic unitary series

$$\mathcal{L}(\eta)\sim u(\cdot\,,\mathbb{H})\sim sp(\cdot)$$

i) The form invariant with respect to the above Lie algebra is symmetric with respect to the main quaternionic conjugation "$*$" :

$(\Theta^{\otimes})^*=\Theta^{\otimes}\circ T$, which corresponds to $\epsilon=1$. Then we have $f_\xi=-\Theta^{\otimes}\circ\pi_\epsilon=-\mathrm{Im}\,\Theta^{\otimes}$, consequently $\dot{\iota}(\eta)\sim sp(1)$.

ii) Rescaling of the above form $\tilde{\Theta}^{\otimes}=\Theta^{\otimes}\alpha$; $-\alpha=\alpha^*\in\mathbb{H}$, we obtain the form which is antisymmetric with respect to $\alpha(*)$ being the composition of $*$ with the inner

automorphism $\alpha^{-1}(\cdot)\alpha$ of $\mathbb{H}$:
$(\tilde{\Theta}^{\otimes})^{\alpha(*)} = -\tilde{\Theta}^{\otimes}\circ T$; this corresponds to $\epsilon = -1$ and $f_{\xi} = -\tilde{\Theta}^{\otimes}\circ\pi_{\epsilon} = -\alpha \mathrm{Re}\,\Theta^{\otimes}$, consequently $i(\eta) = \alpha\mathbb{R} \sim O(2)$.

4° Quaternionic antiunitary series

$$\mathcal{L}(\eta) \sim \alpha U(\cdot, \mathbb{H}) \sim so^{*}(\cdot) \quad .$$

The forms invariant with respect to this algebra are either

i) antisymmetric with respect to "$*$" :
$(\Theta^{\otimes})^{*} = -\Theta^{\otimes}\circ T$, hence $\epsilon = -1$ and $f_{\xi} = -\mathrm{Im}\,\Theta^{\otimes}$, consequently $i(\eta) \sim sp(1)$

or

ii) symmetric with respect to $\alpha(*)$ (cf.3°):
$(\tilde{\Theta}^{\otimes})^{\alpha(*)} = \tilde{\Theta}^{\otimes}\circ T$. This implies $\epsilon = 1$ and $f_{\xi} = -\alpha \mathrm{Re}\,\Theta^{\otimes}$, consequently $i(\eta) \sim O(2)$.

The above considerations lead us to the following

THEOREM 2.III

The content of $\mathcal{S}(\eta)$ canonical classes is described in Table 1.III.

PROOF:

Follows directly from classification of $\mathcal{L}(\eta)$ Lie algebras [8] and the above considerations. ∎

p−q (mod 8)

q=0	p+q (mod 8)	0	1	2	3	4	5	6	7
$so(2l+1,R)$	0 +	$so(l+1,l)$		$so(2l+1;C)$		$so^{*}(2l+2)$		$so(2l+1;C)$	
–	0 −	–		–		$\alpha uu(l\|1)$		–	
$so(2l+1,R)$	1 +		$so(l+1;l)$		$so^{*}(2l+2)$		$so^{*}(2l+2)$		$so(l+1,l)$
–	1 −		–		$\alpha uu(l\|1)$		$\alpha uu(l\|1)$		–
$su(2l+1)$	2 +	–		$su(l+1,l)$		–		$su(l+1,l)$	
$su(1\|2l)$	2 −	–		$su(l,l\|1)$		–		$su(l,l\|1)$	
$sp(l+1)$	3 +		–		$sp(l/2+1,l/2)$		$sp(l/2+1,l/2)$		–
$\alpha uu(1\|l)$	3 −		$osp(1\|2l;R)$		$\alpha uu(1\|l/2,l/2)$		$\alpha uu(1\|l/2,l/2)$		$osp(1\|2l;R)$
$sp(l+1)$	4 +	–		–		$sp(l/2+1,l/2)$		–	
$\alpha uu(1\|l)$	4 −	$osp(1\|2l;R)$		$osp(1\|2l;C)$		$\alpha uu(1\|l/2,l/2)$		$osp(1\|2l;C)$	
$sp(l+1)$	5 +		–		$sp(l/2+1,l/2)$		$sp(l/2+1,l/2)$		–
$\alpha uu(1\|l)$	5 −		$osp(1\|2l;R)$		$\alpha uu(1\|l/2,l/2)$		$\alpha uu(1\|l/2,l/2)$		$osp(1\|2l;R)$
$su(2l+1)$	6 +	–		$su(l+1;l)$		–		$su(l+1,l)$	
$su(1\|2l)$	6 −	–		$su(l,l\|1)$		–		$su(l,l\|1)$	
$so(2l+1;R)$	7 +		$so(l+1,l)$		$so^{*}(2l+2)$		$so^{*}(2l+2)$		$so(l+1,l)$
–	7 −		–		$\alpha uu(l\|1)$		$\alpha uu(l\|1)$		

TABLE 1.III. $2l = 2^{s}$; $s = [(p+q-1)/2]$

In the Table 1.III we have described all canonical spinorial extensions of $\mathbf{so}(\eta)$ Lie algebras, that signatures (p,q) of the forms η do admit. One readily sees that there exists the sequence of signatures favouring one type of commutation factor and consequently admitting as an extension either only Lie algebra or only Lie superalgebra. It always holds for $\mathbf{so}(\eta)$ Lie algebras whose spinorial modules are essentially real and for these orthogonal Lie algebras whose spinorial modules are complex but their unique $\mathbf{so}(\eta)$-invariant forms are analytic. In all other cases the form η does admit Lie algebraic and Lie superalgebraic spinorial extension. It is remarkable, that this takes place only for these $\mathbf{so}(\eta)$-orthogonal Lie algebras, whose extensions have non-zero $\mathbf{i}(\eta)$ internal sectors i.e. spinors are charged.
We are under an obligation to comment shortly the cases when the signature (p,q) does satisfy the system of equations of Lemma 4.II. i.e. does not admit any extension in sense of Definition 1.II. In order to obtain the non-trivial Z_2-graded extension in these cases one is forced to double the odd subspace $S_{(1)}(\eta)$, so that it becomes isomorphic to the direct sum of two irreducible spinor modules. The simple extending structures are then special linear Lie algebras or Lie superalgebras, both admissible by any signature under consideration.
There are two orthogonal Lie algebras distinguished by their non-simplicity i.e. their irreducible spinor modules yield non-faithful representations. Consequently the Definition 1.II describes for $\mathbf{so}(4,0)$ and $\mathbf{so}(2.2)$ the extensions of their simple ideals $\mathbf{so}(3,0)$ and $\mathbf{so}(2,1)$ respectively. The extensions containing faithful representations of these semisimple Lie algebras are to be achieved by forming formal direct sums of respective extensions of ideals. This very construction yielding semisimple Z_2-graded ϵ-Lie algebras is compatible with non-simple structure of $\mathbf{so}(4,0)$ and $\mathbf{so}(2,2)$.

IV. THE STRUCTURAL RELATIONS

Let us now describe the structural relations of canonical spinorial extensions in terms of basis elements $\{e_{ab} := 1/2\, e_a \wedge e_b,\ e_{a_1 a_2 \ldots a_k} = 1/k!\, e_{a_1} \wedge e_{a_2} \wedge \ldots e_{a_k}\ ;$ $1 \leq a,b,a_1,\ldots,a_k \leq p+q;\ 2<k=2\ (\mathrm{mod} 4)\}$ of $\mathcal{L}(\eta)$ Lie algebra and $F(\eta)$ division ring. We have to decompose the (anti)commutator $\langle Q(\psi), Q(\psi')\rangle$ of two spinors in terms of the preimages

$$\begin{aligned} \mathcal{M}_{ab} &= L^{-1}(e_{ab}) \\ T_{a_1\ldots a_k} &= L^{-1}(e_{a_1\ldots a_k}) \end{aligned} \qquad (1.\mathrm{IV})$$

of the basis elements of $\mathcal{L}(\eta)$ under homomorphism (1.II) and preimages

$$G_s = L^{-1}(R_s) \quad ; \quad s = 0 \quad \text{or } 1 \leq s \leq 3 \tag{2.IV}$$

of complex or quaternionic imaginary units.
We need the following ([3]) :
LEMMA 1.IV

$$\Psi_\xi^{a_1\dots a_k}(\psi,\psi') = -\epsilon/r \operatorname{Re}\Theta(\psi, e^{a_1\dots a_k}\psi')$$

$\psi,\psi' \in S(\eta)$, $r = \dim_R S(\eta)$ and Ψ_ξ is that of (10.III).
PROOF
The equality follows directly from $\Psi_\xi^{a_1\dots a_k} = -1/r \operatorname{tr}(e^{-1}_{a_1\dots a_k}\Psi_\xi)$. ∎

Let us choose $\{u_A\}_1^r$ - an arbitrary basis of $S(\eta)$ spinor module and take the standard notation

$$\begin{aligned}
\Gamma^{ab}_{AB} &= \operatorname{Re}\Theta(u_A, e^{ab}u_B)\\
\Gamma^{a_1\dots a_k}_{AB} &= \operatorname{Re}\Theta(u_A, e^{a_1\dots a_k}u_B)\\
\mathcal{L}^s_{AB} &= \operatorname{Re}\Theta(u_A, R_s u_B)\\
C_{AB} &= \operatorname{Re}\Theta(u_A, u_B)
\end{aligned} \tag{3.IV}$$

for the coefficients of decomposition of Ψ_ξ and f_ξ mappings in canonical bases of $\mathcal{L}(\eta)$ and $F(\eta)$ algebras.
For the basis elements $\{Q_A = Q(u_A);\ 1 \leq a \leq r\}$ of $S_{(n)}(\eta)$, corresponding to $\{u_A\}_1^r$, one can write the following structural relations

$$\langle Q_A, Q_B\rangle = -\epsilon/2r\, \Gamma^{ab}_{AB}\, \mathcal{M}_{ab} - \epsilon/r\, \Gamma^{a_1\dots a_k}_{AB}\, T_{a_1\dots a_k} + \tag{4.IV}$$

+ (internal sector).

The internal sectors are given as follows

$$\begin{array}{ll}
0 & ;\ p+q = 2,\ p+q = 0 \ (\mathrm{mod}4)\\
0 & ;\ p-q = 0,1,7 \ (\mathrm{mod}8)\ p+q \text{ arbitrary}\\
-(r+\epsilon)/r\, \mathcal{L}_{AB} G_0 & ;\ p+q = p-q = 2 \ (\mathrm{mod}4)\\
-\mathcal{L}^s_{AB} G_s & \left\{\begin{array}{l} ;\ p+q = p-q = 3,4,5 \ (\mathrm{mod}8)\ \epsilon = 1\\ ;\ p+q = 0,1,7;\ p-q = 3,4,5 \ (\mathrm{mod}8)\ \epsilon = -1\end{array}\right.
\end{array}$$

$$-C_{AB}\alpha^S G_S \quad \begin{cases} ; & p+q = p-q = 3,4,5 \ (\mathrm{mod}8) \ \epsilon = -1 \\ & p+q = 0,1,7, \ p-q = 3,4,5 (\mathrm{mod}8) \ \epsilon = 1 \end{cases}$$

The last expression of the above is written according to the convention used in considerations preluding to Theorem 2.III.

The commutators of even elements of $S(\eta)$ Lie algebras with these from $S_{(1)}(\eta)$ have the form:

$$\langle L^{-1}(c), Q_A \rangle = Q_B c^B{}_A \qquad c \in \mathcal{L}(\eta)$$
$$\langle L^{-1}(f), Q_A \rangle = Q_B f^B{}_A \qquad f \in F(\eta) \tag{5.IV}$$

and are simply given by matrix representations of respective Clifford algebras. The structural relations of basis elements (1.IV) are obviously that of $\mathcal{L}(\eta)$ and $F(\eta)$.

The considerations of this paper give the full classification of spinorial extensions of orthogonal Lie algebras and provide the possibility to describe the supersymmetry algebras used in supergravity theories. We present them elsewhere.

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