Peter Vojtáš

Boolean games – Classifying strategies and omitting cardinality assumptions

In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [361]–368.

Persistent URL: http://dml.cz/dmlcz/702158

BOOLEAN GAMES - CLASSIFYING STRATEGIES AND OMITTING CARDINALITY ASSUMPTIONS

Peter Vojtáš

ABSTRACT. We deal with a transfinite game on Boolean algebras introduced by T. Jech. The game yields a fine method for handling κ-closed dense subsets of Boolean algebras. We prove (without set-theoretical assumptions) the existence of a γ^+-closed dense subset for a certain type of Boolean algebras determined in the game of an uncountable length γ - a generalization of some results by M. Foreman. We investigate relationship between certain cardinal characteristics of Boolean algebras, discuss the existence of positional strategies of trees, and give a couple of problems concerning the partialy ordered set of all strategies.

1. Introduction and notation. In terminology we generally follow [8],[9],[11], but some notions are introduced in this section. Let B be an atomless Boolean algebra and α an ordinal number. Consider the following transfinite game $\mathcal{G}^{I}(B,\alpha)$, introduced by T.Jech in [5], between two players White and Black. Let White and Black define a decreasing sequence

(1) $$w_0 \geq b_0 \geq w_1 \geq \dots \geq w_\xi \geq b_\xi \geq \dots$$

of nonzero elements of B of length $\leq \alpha$ by taking turns defining its entries. I.e., first White chooses a nonzero $w_0 \in B$. Then Black chooses a nonzero $b_0 \leq w_0$. Then White chooses nonzero $w_1 \leq b_0$ ... The play is won by Black if the sequence (1) has nonzero lower bound and length α; else the White wins.

The game $\mathcal{G}^{II}(B,\alpha)$ (see [4],[3]) is defined in exactly the same way as the game $\mathcal{G}^{I}(B,\alpha)$, except that the player Black moves first at limit stages, i.e. the play of $\mathcal{G}^{II}(B,\alpha)$ looks like

$$w_0, b_0, w_1, b_1, \dots, b_\omega, w_\omega, b_{\omega+1}, w_{\omega+1}, \dots, b_\xi, w_\xi, \dots$$

T. Jech in [5] proved that if the algebra B has a κ^+-closed dense subset, then the player Black has a winning strategy in the game $\mathcal{G}^{I}(B,\kappa)$, $\mathcal{G}^{II}(B,\kappa)$. He also formulated the problem whether the inverse implication holds, i.e., does the existence of a winning strategy for the Black in the game $\mathcal{G}^{I}(B,\kappa)$ ($\mathcal{G}^{II}(B,\kappa)$) imply that the algebra B has a κ^{+}-closed dense subset ? The problem for $\kappa=\omega$ was investigated in [5],[3],[8] and [11]. For $\kappa=\omega_1$, C. Gray in [4] has constructed an algebra E such that Black wins $\mathcal{G}^{II}(E,\omega_1)$ and E has no ω_2-closed dense subset (nothing similar for the game $\mathcal{G}^{I}$ is known). M. Foreman in [3] proved that if $d(B)=\lambda^{+}=ND(B)$, where $ND(B)$ denotes the nondistributivity of B, Black wins $\mathcal{G}^{I}(B,\gamma)$ and $\lambda^{<\gamma}=\lambda$, then the algebra B has a γ^{+}-closed dense subset. We show that the saturatedness of such an algebra can be either λ^{+} or λ^{++} and in the first case the same conclusion holds without the assumption about the cardinal - exponentation (Theorem 1).

We say that $D\subseteq B^{+}$ is a $\underline{\lambda\text{-closed dense subset}}$ of algebra B (we say sometimes base instead of dense subset) if $(\forall x\in B^{+})(\exists y\in D)(y\le x)$ and for every decreasing sequence $\{a_\alpha:\alpha<\tau\}\subseteq D$ of the length $\tau<\lambda$ there is a $y\in D$ such that $y\le a_\alpha$ for each $\alpha<\tau$. Define:

$d(B) = \min\{|D|: D \text{ is a dense subset of } B\}$,

$ND(B) = \min\{\delta: B \text{ is not } (\delta,\cdot,2)\text{-distributive}\}$,

$\nabla hsat(B) = \min\{\kappa: (\forall x\in B^{+})(\text{there is no partition of } B_x \text{ of size } \kappa)\}$,

$\Delta hsat(B) = \sup\{\kappa: (\forall x\in B^{+})(\text{there is a partition of } B_x \text{ of size } \kappa)\}$,

$\nabla cds(B) = \min\{\kappa: \text{there is no } \kappa\text{-closed dense subset of } B\}$,

$\Delta cds(B) = \sup\{\kappa: \text{there is a } \kappa\text{-closed dense subset of } B\}$,

$\nu_1(B) = \sup(\mathcal{H}^{I}(B)) = \sup\{\alpha: \text{Black wins } \mathcal{G}^{I}(B,\alpha)\}$,

$\eta_1(B) = \min(\mathfrak{Z}^{I}(B)) = \min\{\alpha: \text{White wins } \mathcal{G}^{I}(B,\alpha)\}$,

analogously we define ν_2, η_2, $\mathcal{H}^{II}$, $\mathfrak{Z}^{II}$ for the game $\mathcal{G}^{II}$.

It is known that $\eta_2(B)=ND(B)$ (see [3]) and that ν_1, ν_2, η_1 are regular cardinal numbers (see [11]).

2. Omitting cardinality assumptions in the game of uncountable length.

The following facts may be belong to folklore.

Proposition. For every atomless Boolean algebra B the following hold:

(i) $\Delta hsat(B)\le d(B)$ and $\nabla hsat(B)\le d(B)$ does not hold;

(ii) $ND(B)\le\nabla hsat(B)$ and $ND(B)\le\Delta hsat(B)$ does not hold;

(iii) $\Delta cds(B)\le ND(B)$ and $\nabla cds(B)\le ND(B)$ does not hold;

(iv) $\Delta cds(B)\le\nu_1(B)$ and $\nabla cds(B)\le\nu_1(B)$ does not hold;

(v) $ND(B)\le d(B)$;

(vi) $\nabla hsat(B)\le(\Delta hsat(B))^{+}$ and $\nabla cds(B)\le(\Delta cds)^{+}$.

PROOF. The negative assertions in (i) - (iv) are trivial.

(i) Follows easily, for if P is a partition of B then $|P| \leq d(B)$.

(ii) Let $\delta = \nabla hsat(B) < ND(B)$. As B is atomless, there is a matrix $\Theta = \{P_\alpha : \alpha < ND(B)\}$ consisting of maximal partitions of B such that $\alpha < \beta$ implies P_β strictly refines P_α . Then for each $x \in P_\delta$ the set $\{y_\alpha \in P_\alpha : \alpha < \delta \;\&\; x \leq y_\alpha\}$ is a strictly decreasing tower of algebra B and $\{y_{\alpha+1} - y_\alpha : \alpha < \delta\}$ is a partition of B of size δ . Contradiction.

(iii) If B has a κ-closed dense subset, then $\kappa \leq ND(B)$. (Follows also from (iv) and $\nu_1(B) \leq \eta_2(B) = ND(B)$, see [11]).

(iv) See [5].

(v) Assume $D = \{x_\alpha : \alpha < \delta\}$ is a base of B and $\delta < ND(B)$. Let P be a strict refinement of the matrix $\{\{x_\alpha ; -x_\alpha\} : \alpha < \delta\}$. For $x \in P$, take $x_\alpha \in D$ with $x_\alpha \leq x$. Contradiction.

(vi) Obvious.

The following Lemma shows that the existence of certain algebras has influence on the exponentation of cardinal numbers.

Lemma. Assume that B is a Boolean algebra such that

$$\kappa < \nabla hsat(B) \quad \text{and} \quad \gamma \in \mathcal{H}^{II}(B) .$$

Then $\kappa^\gamma \leq \Delta hsat(B)$ and $\kappa^\gamma < \nabla hsat(B)$.

The Proof is analogous as that of Corollary 1 in [11].

The next theorem generalizes some results of M. Foreman ([3]).

Theorem 1. Assume that B is anatomless Boolean algebra such that $d(B) = ND(B) = \lambda^+$ and Black wins $\mathcal{G}^I(B,\gamma)$. Then:

(1) $\Delta hsat(B) < \nabla hsat(B)$.

(2) Either $\Delta hsat(B) = \lambda^+$ and $\nabla hsat(B) = \lambda^{++}$, or $\Delta hsat(B) = \lambda$ and $\nabla hsat(B) = \lambda^+$.

(3) If $\Delta hsat(B) = \lambda$, then the algebra B has a γ^+-closed dense subset.

PROOF. (1) If $\Delta hsat(B) = \nabla hsat(B)$, then from (i) and (ii) in Proposition we have $\Delta hsat(B) = \nabla hsat(B) = \lambda^+$. But in this case $\nabla hsat(B)$ should be a weakly inaccessible cardinal number (see [7]). Contradiction.

(2) As $\nabla hsat(B) \leq (\Delta hsat(B))^+$, (2) follows from (i) and (ii) in Proposition.

(3) Aplying Lemma, $\lambda < \nabla hsat(B)$ and $\gamma \in \mathcal{H}^I(B)$ imply $\lambda^\gamma = \lambda$. Then $\lambda \leq \lambda^{<\gamma} \leq \lambda^\gamma = \lambda$ shows that the additional Foreman's set-theoretical assumption is for algebras in question granted.

Remarks. To prove a similar result for algebras having bigger density we may be tempted to use the more general construction of base matrices from Lemma 2 of [11]. But if algebra B is $(\lambda^+,\cdot,\kappa)$-nowhere distributive, $\gamma \in \mathcal{X}^I$, $d(B) = \kappa^{\gamma}$ and $\Delta hsat(B) = \lambda$, we obtain only $d(B) \leq \lambda^+$!

It might be in place to call the reader's attention to an interesting "inverse" exponentation of cardinals in the Theorem 6 of [9].

Note that for $\Delta hsat(B) = \lambda^+$ Theorem 1 implies $(\lambda^+)^{\gamma} = \lambda^+$. Then take $\rho = \min\{\min\{\tau \leq \gamma : \lambda^{\tau} = \lambda^+\}, \gamma\}$. If $\rho > \omega_0$, then the algebra B has a ρ^+-closed dense subset.

The case $\Delta hsat(B) = \lambda^+$ will be further discussed in § 3 using positional strategies.

3. Classifying strategies and problems. The importance of classifying different types of strategies was shown in [11], namely the Gray's trick for constructing determined algebras without closed dense subset does not work below ω_1.

Definition ([5]). We say that Black has a <u>positional winning strategy</u> in the game $\mathcal{G}^I(B,\kappa)$ if there is a function $\rho : B^+ \rightarrow B^+$ such that Black wins every play of length κ in which he follows ρ : $w_0, \rho(w_0), w_1, \rho(w_1), \dots, w_{\omega}, \rho(w_{\omega}), \dots, w_{\xi}, \rho(w_{\xi}), \dots$; $\xi < \kappa$. For the motivation of the following definition see [8],[9] and [11]. Moreover, we mention the following point of view. There is a lot of games which finish after reaching the winning position (e.g. chess), or at a certain point an evaluation is made to decide the game (e.g., Mycielski's game, some topological games). Jech's game has one interesting feature: the Black's victory in fact says that we can continue the play. This enables us to study a specific type of questions that are not possible for other games:

- the questions about sets $\mathcal{X}, \mathcal{Z}$ of ordinals for which Black (White) has a winning strategy (see [11]),

- the questions about relations between strategies for games of different length (e.g. does a strategy σ for the game $\mathcal{G}(B,\alpha)$ with $\alpha > \beta$ prolongate the strategy ρ for the game $\mathcal{G}(B,\beta)$?).

So our Boolean game gives us motivation for studying such aspects for other games. For instance, we can ask (perhaps an obscure question): How long, in chess, can Black or White continue the play ?

Definition ([8]). We say that the player Black has <u>a simultaneous winning strategy</u> in the game $\mathcal{G}^I$ ($\mathcal{G}^{II}$, respectively) on algebra B if there is one strategy

$$\sigma : \bigcup\{{}^{\beta}B : \beta < \aleph_1(B)\} \longrightarrow B^+$$

such that σ is winning for Black in each game $\mathcal{G}^I(B,\alpha)$ for

$\alpha < \nu_1(B)$ ($\mathcal{G}^{II}(B, \alpha)$ for $\alpha < \nu_2(B)$, respectively).

Consider the set

$$\mathcal{S}^{I}(B) = \{\rho \,;\, \rho \text{ is a winning strategy for Black in } \mathcal{G}^{I}\}$$

and a partial ordering of $\mathcal{S}^{I}(B)$:

$$\rho \leq \tau \quad \text{if} \quad \rho \supseteq \tau$$

Then $(\mathcal{S}^{I}(B), \leq)$ is a tree of length $\nu_1(B)$ (analogously for $\mathcal{G}^{II}$). Observe that Black has a simultaneous strategy in $\mathcal{G}^{I}(B)$ if and only if in the tree $(\mathcal{S}^{I}(B), \leq)$ there is a branch of the length $\nu_1(B)$.

Games played on a partialy ordered set P and on the Boolean completion RO(P) are equivalent (see [5]). We shall consider the special case when P is a tree. It concerns algebras which have a base matrix - i.e. a base which forms a tree in the natural ordering of the algebra B .

Theorem 2. Assume that T is a tree of height κ, cf $(\kappa) > \omega$ and the player Black has a positional winning strategy in the game $\mathcal{G}^{I}(T, \gamma)$ with $\gamma > \omega_0$. Then T has a γ^{+}-closed dense subset.

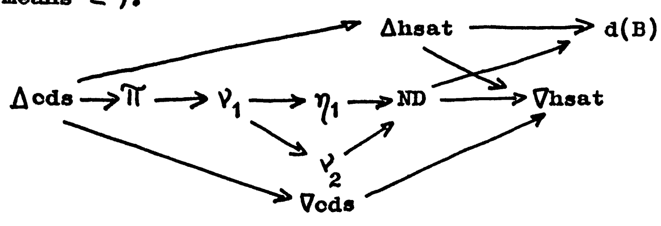
PROOF. Following the Foreman's proof (see [3]), for each $t \in T$ we will define a $t^{*} \in T$, $t^{*} \leq t$ with the property, that if $\bar{s}$ is a partial play towards t^{*} and $t' \in T$ with $\inf \bar{s} \geq t' > t^{*}$, then there is a partial play towards t^{*} extending $\bar{s}, \bar{s}'$ such that $t' \geq \inf \bar{s}' \geq t^{*}$. Using a positional strategy π for $\mathcal{G}^{I}(T, \gamma)$ define $t_0 = t$ and for $n \in \omega$ $t_{n+1} = \pi(t_n)$. The sequence $\{t_n; n \in \omega\}$ has a nonzero lower bound - take one with minimal rank in the tree T and denote it by t^{*} . Now the proof proceeds as in [3].

We remark, that Theorem 2 deals with a larger class of algebras than that treated in Theorem 1.

The following problem seems to be important.

Problem. Does the existence of a winning strategy for Black on a tree T imply the existence of a positional winning strategy for Black on T ?

Consider the following extensions of the representation problem from [11]. The results of our Proposition, [11] and further folklore results are shown below on an oriented graph (arrow $\longrightarrow$ means $\leq$).



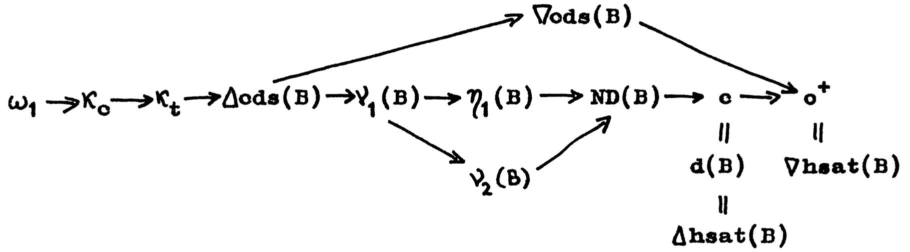
Question. If we prescribe to each vertex of our graph a cardinal number such that inequalities are fulfilled and $\pi, \nu_1, \eta_1, \nu_2$, ND, ∇hsat are regular, $\nabla \mathrm{cds} \leq (\Delta\, \mathrm{cds})^+$, $\nabla \mathrm{hsat} \leq (\Delta\, \mathrm{hsat})^+$, does then there exist a Boolean algebra B such that all its characteristics are as prescribed (here $\pi = \sup\{\alpha$: Black has a positional winning strategy in $\mathcal{G}^{I}(B, \alpha)\}$) ? Moreover, we can ask whether such an algebra exists if we prescribe the existence (or nonexistence) of the simultaneous winning strategy for $\mathcal{G}^{I}$ of the length $\nu_1(B)$ and for $\mathcal{G}^{II}$ of the length $\nu_2(B)$.

The special case of this representation problem arises if $B = \mathcal{P}(\omega)/\mathrm{fin}$ - the algebra of power set of the set of all natural numbers modulo the ideal of finite sets. We define (see also [1])

$$\kappa_c = \min\{|F| : F \subseteq \mathcal{P}(\omega)/\mathrm{fin} \text{ is centered } \& \bigwedge F = \mathbb{O}\},$$

$$\kappa_t = \min\{|T| : T \subseteq \mathcal{P}(\omega)/\mathrm{fin} \text{ is a tower and } \bigwedge T = \mathbb{O}\} .$$

In this case the graph looks like ($B = \mathcal{P}(\omega)/\mathrm{fin}$):



In [1] it is showed that ND(B) can be strictly smaller than c . In [2] Con (ZFC + $\kappa_c <$ ND) is proved and in [7] it is proved that $\kappa_c = \omega_1$ implies $\kappa_t = \omega_1$ and κ_c is a regular cardinal number. This together with Dordal's metatheorem ([2]) gives Con (ZFC+ $\kappa_t <$ ND). Is it consistent that some other inequalities are strict ? In particular, is it consistent that

$$\kappa_t < \Delta\mathrm{cds}\ (\mathcal{P}(\omega)/\mathrm{fin}) \ ?$$

At the end we mention the following problem, presented at the Logic Colloquium '82 ([10]).

Let B be a Boolean algebra. Put

$$\Delta\mathrm{IP}(B) = \sup\{\kappa : \text{there is a } \kappa^+\text{-closed dense subset of } B\},$$

$$\nabla\mathrm{IP}(B) = \min\{\kappa : \text{there is no } \kappa^+\text{-closed dense subset of } B\}.$$

The following function describes the global behaviour of our game: for a cardinal number λ define

$$b^{I}(\lambda) = \min\{\Delta\mathrm{IP}(B) : B \text{ is such that } \lambda \in \mathcal{H}^{I}(B)\}$$

(analogously b^{II})

Problem. 1. Does $(\forall\kappa)(\exists\lambda)(b^{\bullet}(\lambda) \geq \kappa)$ hold ?

2. Is there a regular cardinal number $\mathcal{S}$ such that for each $\kappa < \mathcal{S}$ there is a $\lambda < \mathcal{S}$ such that $b(\lambda) \geq \kappa$?

Note that $b(\lambda) \leq \lambda$ and the failure of the implication "the existence of a strategy for Black implies the existence of a closed dense subset" causes that the function b is regressive. This makes the questions more interesting.

REFERENCES

[1] BALCAR B., PELANT J., SIMON P. "The space of ultrafilters on N covered by nowhere dense sets", Fund. Math. 110(1980), 11-24.

[2] DORDAL P. L. "Independence results concerning some combinatorial properties of continuum", Ph. D. Thesis, Harvard Univ., Cambridge, Mass. 1982.

[3] FOREMAN M. "Games played on Boolean algebras", Manuscript.

[4] GRAY C. "Iterated forcing from the strategic point of view", Ph. D. Thesis, Berkeley, 1980.

[5] JECH T. "The game theoretic property of Boolean algebras", Logic Colloquium '77 (A. Mc Intyre et al., eds.) 135-144, NHPC Amsterdam 1978.

[6] JECH T. "Set theory", Academic Press, New York 1978.

[7] SZYMANSKI A., ZHOU HAO-XUA "The behaviour ω^{2*} under some consequences of Martin's axiom", General Topology and its Relations to Modern Analysis and Algebra V. (Proc. Fifth Prague Topological Symp. 1981) 577-584, Heldermann Verlag, Berlin, 1982.

[8] VOJTÁŠ P. "A transfinite Boolean game and a generalization of Kripke's embedding theorem", ibid, 657-662.

[9] VOJTÁŠ P. "Simultaneous strategies and Boolean games of uncountable length", (Proc. 10th Winter School on Abstract Analysis, Srní 1982) Spplemento ai Rendicanoti del Circolo Matematico di Palermo, Serie II, 2, 1982, 293-297.

[10] VOJTÁŠ P. "White and Black - a Boolean game", Abstracts of the Logic Colloquium '82, Firenze, to appear in J. Symb.Logic.

[11] VOJTÁŠ P. "Game properties of Boolean algebras", Comment. Math. Univ. Carolinae, to appear.

MATHEMATICAL INSTITUTE OF THE SLOVAK ACADEMY OF SCIENCES
KARPATSKÁ 5, 040 01 KOŠICE
CZECHOSLOVAKIA