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A few Remarks on the Set of Finite-to-One Maps of the Cantor set

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1. Introduction

Let $C(2^\omega, I^\omega)$ be the space of continuous mappings from the Cantor set $2^\omega$ to the Hilbert cube $I^\omega$, equipped with the topology of uniform convergence. A mapping $f: 2^\omega \to X$ is finite-to-one, if all fibers of $f$ are finite.

We shall consider the set

$$\mathcal{C} = \{ f \in C(2^\omega, I^\omega) : f \text{ is finite-to-one} \}.$$  \hfill (1)

One readily checks that the set $\mathcal{C}$ is coanalytic. We shall indicate a natural Lusin-Sierpiński index for $\mathcal{C}$, the transfinite order of a finite-to-one mapping on $2^\omega$ (sec. 3), and we shall verify that the transfinite order of mappings is related to the transfinite inductive dimension of compacta by a Hurewicz-type formula (sec. 4). Finally, we shall make some observations about Borel-measurable selections of finite-to-one parametrizations on $2^\omega$ for certain collections of countable-dimensional compacta (sec. 5).

These remarks are related to some open problems about the transfinite inductive dimension, discussed in [Po1] and [Po2; sec. 6].

2. Terminology and some background

Our terminology follows Kuratowski [Ku] and Nagata [Na]. We consider only separable metrizable spaces and by a compactum we mean a compact space. A set $S \subseteq T$ is residual (non-meager) in $T$, if $T \setminus S$ is of first category ($S$ is of second category) in the space $T$. The spaces of continuous functions are considered with the topology of uniform convergence.

A space is countable-dimensional (strongly countable-dimensional) if $X$ is a countable union of finite-dimensional sets (compacta)

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The transfinite inductive dimension \( \text{ind} \) is the extension, by transfinite induction, of the classical Menger-Urysohn dimension: \( \text{ind} X = -1 \) means that \( X = \emptyset \), \( \text{ind} X \leq \alpha \), if, and only if, each point in \( X \) can be separated from a closed set not containing it by a partition \( L \) with \( \text{ind} L < \alpha \); we let \( \text{ind} X \) be the smallest ordinal \( \alpha \) with \( \text{ind} X \leq \alpha \), if such an ordinal exists, and we set \( \text{ind} X = \infty \), otherwise. If \( \text{ind} X = \infty \), then \( \text{ind} X < \omega_1 \).

The following two classical results (proofs can be found in [Na; VI] and [Ku; § 45, II]) provide a link between countable-dimensionality, finite-to-one mappings on the Cantor set, and the transfinite dimension:

2.1. Theorem (Hurewicz). For a compactum \( X \) without isolated points the following conditions are equivalent:

(i) \( X \) is countable-dimensional,
(ii) \( \text{ind} K = \infty \),
(iii) there is a continuous finite-to-one mapping of \( 2^{\omega} \) onto \( X \),
(iv) the set of finite-to-one mappings is dense in the space of continuous mappings of \( 2^{\omega} \) onto \( X \).

2.2. Theorem (Kuratowski). Let \( X \) be a strongly countable-dimensional compactum without isolated points. Then the set of finite-to-one mappings is residual in the space of continuous mappings of \( 2^{\omega} \) onto \( X \).

The converse to the Kuratowski's theorem, even with "residual" weakened to "non-meager", also holds true [Po3].

3. The transfinite order of a finite-to-one mapping on \( 2^\omega \)

Here we shall adopt some general notions from descriptive set theory to the situation we are interested in, cf. Moschovakis [Mo; 2D, 2F], Kuratowski [Ku; § 39].

3.1. The transfinite length of collections of partitions of \( 2^\omega \).

Let \( \Omega \) be the countable collection of all finite partitions of the Cantor set \( 2^\omega \) into pairwise disjoint closed-and-open sets. Given \( \mathcal{U}, \mathcal{V} \in \Omega \), we write \( \mathcal{U} < \mathcal{V} \) if \( \mathcal{U} \) refines \( \mathcal{V} \) and \( \mathcal{U} \neq \mathcal{V} \).

Let \( 2^\Omega \) be the space of all subcollections of \( \Omega \) with the topology of pointwise convergence (we identify any \( \Lambda \subset \Omega \) with its characteristic function). Topologically, \( 2^\Omega \) is the Cantor set.

Let \( WF \) be the set of all collections \( \Lambda \subset \Omega \) with the property that there is no infinite descending sequence \( \mathcal{U}_1 \supset \mathcal{U}_2 \supset \ldots \) of elements of \( \Lambda \).

The set \( WF \subset 2^\Omega \) is coanalytic. For any \( \Lambda \in WF \) the rank function on \( \Lambda \) is defined inductivity as follows, cf. [Mo; pp. 83, 84]: for each \( \mathcal{U} \in \Lambda \) we set

\[ \text{rank}_\Lambda \mathcal{U} = 1 \quad \text{if there is no} \quad \mathcal{V} \in \Lambda \quad \text{with} \quad \mathcal{V} < \mathcal{U}, \]
and
\[ \text{rank}_A \mathcal{U} = \sup \{ \text{rank}_A \mathcal{V} + 1 : \mathcal{V} < \mathcal{U}, \mathcal{V} \in \Lambda \} \, . \]

The length of \( \Lambda \in WF \) is defined by the formula
\[ \text{length} \, \Lambda = \sup \{ \text{rank}_A \mathcal{U} : \mathcal{U} \in \Lambda \} \, , \]
and we set \( \text{length} \, \Lambda = \infty \) if \( \Lambda \notin WF \).

The length is a Lusin-Sierpiński index for the coanalytic set \( WF \).

3.2. The function \textit{ord}. Let \( f : 2^\omega \to I^\omega \) be a continuous mapping and let
\[ A(f) = \{ \mathcal{U} \in \Omega : \cap \{ f(F) : F \in \mathcal{U} \} = \emptyset \} \, . \]

The mapping \( f \to A(f) \) from the function space \( C(2^\omega, I^\omega) \) (see sec. 1) to the Cantor set \( 2^\Omega \) is Borel-measurable. Since, as one easily checks,
\[ A(f) \in WF \iff f \in \mathcal{C} \, , \]
where \( \mathcal{C} \) is described in (1) sec. 1, the transfinite order defined by the formula
\[ \text{ord} \, f = \text{length} \, A(f) \, , \text{ for } \, f \in C(2^\omega, I^\omega) \, , \]
is a Lusin-Sierpiński index for the coanalytic set \( \mathcal{C} \). In particular, the transfinite order is bounded on each analytic set in \( \mathcal{C} \), and each set \( \mathcal{C}_\xi = \{ f \in \mathcal{C} : \text{ord} \, f \leq \xi \} \) is Borel, see [Ku; § 39, VIII].

4. A Hurewicz-type formula for the transfinite order

The following fact is a certain substitute for a classical theorem of Hurewicz [Ku; § 45, I, Th. 2].

4.1. Proposition. Let \( f : 2^\omega \to X \) be a finite-to-one mapping of the Cantor set onto the compactum \( X \). Then
\[ \text{ind} \, X \leq \text{ord} \, f \, . \]

Proof. The proof is by induction with respect to the transfinite order of the mappings. For the mappings of finite order formula (\*) is valid by the classical result. Suppose that (\*) holds true for the mappings of order \( < \alpha, \alpha \) being a countable infinite ordinal, and let \( f : 2^\omega \to X \) be a continuous surjection with \( \text{ord} \, f = \alpha \).

Let us split \( 2^\omega \) into two nonempty closed-and-open sets \( K, L \), and let
\[ Z = f(K) \cap f(L) \, . \]

Since such sets \( Z \) separate all pairs of disjoint closed sets in \( X \), it is enough to check that
\[ \text{ind} \, Z < \text{ord} \, f \, . \]

87
We can assume that $Z$ has no isolated points, as for the set $Z'$ of points of condensation of $Z$, either $\text{ind } Z' = \text{ind } Z$, or $Z$ is countable. Let $S$ be a minimal compactum such that

\[(2) \quad S \subseteq K \quad \text{and} \quad f(S) = Z.\]

Since $S$ has no isolated points, there exists a homeomorphism $h: 2^\infty \to S$. Let

\[g = f \circ h: 2^\infty \to Z.\]

We shall check that

\[(3) \quad \text{ord } g < \text{ord } f.\]

Let $r: K \to S$ be a retraction [Ku; § 26, II, Corollary 2], and let for partition $\mathcal{U} \in \Omega$,

\[\mathcal{U}^* = \{r^{-1}(h(F)) : F \in \mathcal{U}\} \cup \{L\} \in \Omega.\]

The correspondence $\mathcal{U} \to \mathcal{U}^*$ is invertible and preserves the order $\prec$. Moreover, if $\mathcal{U} \in \Lambda(g)$, then $\mathcal{U}^* \in \Lambda(f)$. Therefore, taking into account that \(\{2^\infty\}^* = \{K, L\}\), we get (see sec. 3.1): $\text{ord } f = \text{length } \Lambda(f) = \text{rank}_{\Lambda(f)} \{2^\infty\} > \text{rank}_{\Lambda(f)} \{K, L\} \geq \text{rank}_{\Lambda(g)} \{2^\infty\} = \text{length } \Lambda(g) = \text{ord } g$, i.e. we obtain (3).

By the inductive assumption, $\text{ind } g(2^\infty) \leq \text{ord } g$, and, since $g(2^\infty) = f(S)$, (1) follows from (2) and (3).

4.2. Remark. One can define a function $\Psi: \omega_1 \to \omega_1$ such that for each compactum $X$ without isolated points, if $\text{ind } X \leq \alpha$ then there exists a finite-to-one surjection $f: 2^\infty \to X$ with $\text{ord } f \leq \Psi(\alpha)$.

To see this let us fix $\alpha < \omega_1$ and let $u: 2^\infty \to K_\alpha$ be a finite-to-one mapping onto a countable-dimensional compactum which contains topologically all compacta with $\text{ind } \leq \alpha$ (see [Po 2 sec. 3]). We let $\Psi(\alpha) = \text{ord } u$. Now, given a compactum $X$ without isolated points such that $\text{ind } X \leq \alpha$ we can assume that $X \subseteq K_\alpha$ and, for a minimal compactum $S$ in $2^\infty$ with $u(S) = X$ and for a homeomorphism $h: 2^\infty \to S$, we let $f = u \circ h: 2^\infty \to X$. Since $\text{ord } f \leq \text{ord } u$, $f$ is the required surjection.

This observation is connected to the assertion of Lemma 2.1 in [Po 1; § 3]; we do not examine, however, the relationship more closely.

4.3. Remark. The remark at the end of sec. 3.2 and Proposition 4.1 yield the following fact:

If $\mathcal{A} \subseteq \mathcal{C}$ is an analytic set of finite-to-one mappings of $2^\infty$ in $I^\infty$, then

\[\text{sup } \{\text{ind } f(2^\infty) : f \in \mathcal{A}\} < \omega_1.\]

This can be also verified directly. Let $u: \omega_1 \to \mathcal{A}$ be a continuous map of the irrationals $\omega_1$ onto $\mathcal{A}$ and let $F: \omega_1 \times 2^\infty \to \omega_1 \times I^\infty$ be defined by the formula $F(t, x) = (t, u(t)(x))$. The map $F$ is perfect and finite-to-one. Therefore, the space $E = F(\omega_1 \times 2^\infty)$ is completely metrizable and countable-dimensional and, since each $f(2^\infty), f \in \mathcal{A}$, embeds in $E$, we have $\text{sup } \{\text{ind } f(2^\infty) : f \in \mathcal{A}\} \leq \text{ind } E < \omega_1$ (cf. [Po 2; sec 6] for similar arguments).
5. Borel-measurable choice of finite-to-one parametrizations

Let $\mathcal{H}(I^\infty)$ be the hyperspace of the Hilbert cube, i.e. the space of compact subsets of $I^\infty$ with the topology induced by the Hausdorff metric.

Let
\[
\mathcal{C} = \{ K \in \mathcal{H}(I^\infty) : K \text{ is countable-dimensional} \},
\]
\[
\mathcal{C}^* = \{ K \in \mathcal{H}(I^\infty) : K \text{ is strongly countable-dimensional} \}.
\]

5.1. Proposition. For each analytic set $A \subset \mathcal{C}^*$ there exists a Borel-measurable function $\sigma$ which assigns to each compactum $K \in A$ a finite-to-one continuous mapping $\sigma(K) : 2^\infty \to K$ onto $K$.

Proof. Let $\varphi : C(2^\infty, I^\infty) \to \mathcal{H}(I^\infty)$ (see sec. 1) be defined by the formula
\[
\varphi(f) = f(2^\infty)
\]
By a result of Michael [Mi; Th. 1.1]
\[
(1) \quad \text{the mapping } \varphi \text{ is open}.
\]
Let us consider the set $\mathcal{C}$ defined in sec 1, (1). By Hurewicz’s Theorem 2.1, $\varphi(\mathcal{C}) = \mathcal{C}$ and for each $K \in \mathcal{C}$ the set $\varphi^{-1}(K) \cap \mathcal{C}$ is dense in $\varphi^{-1}(K)$. Therefore, by (1),
\[
(2) \quad \varphi | \mathcal{C} : \mathcal{C} \to \mathcal{C} \text{ is open},
\]
where $\varphi | \mathcal{C}$ is the restriction of $\varphi$ to $\mathcal{C}$. By Kuratowski’s Theorem 2.2, for each $K \in \mathcal{C}^*$ the set $\varphi^{-1}(K) \cap \mathcal{C}$ is residual in $\varphi^{-1}(K)$. Now, the set $\mathcal{C}$ being coanalytic, we can apply, by (2), to the multifunction $F(K) = \varphi^{-1}(K) \cap \mathcal{C}$ defined on $A$ a selection theorem due to Burgess [Bu; Theorem 3.1] and Cenzer and Mauldin [C-M] which provides a Borel-measurable function $\sigma : A \to \mathcal{C}$ such that $\sigma(K) \in F(K)$, i.e., $\sigma(K)(2^\infty) = K$.

5.2. Remark. By Kuratowski’s Theorem 2.2 and the remark following this theorem, $\mathcal{C}^* = \{ K \in \mathcal{H}(I^\infty) : \varphi^{-1}(K) \cap \mathcal{C} \text{ is non-meager in } \varphi^{-1}(K) \}$. Therefore, the above approach works only for analytic subsets of $\mathcal{C}^*$.

I do not know, if the assertion of Proposition 5.1 is valid for all analytic sets $A \subset \mathcal{C}$, or even for the analytic sets $\mathcal{C}_x = \{ K \in \mathcal{H}(I^\infty) : \text{ind } K \leq x \}$ (cf. the next section).

5.3. Remark. Let $\mathcal{C}_n = \{ K \in \mathcal{H}(I^\infty) : K \text{ is at most } n\text{-dimensional} \}$ and let $\mathcal{G}_n = \{ f \in \mathcal{C} : \text{the order } f \text{ is at most } n \}$. Then $\mathcal{G}_n$ and $\mathcal{C}_n$ are $G_\delta$-sets in $C(2^\infty, I^\infty)$ and $\mathcal{H}(I^\infty)$, respectively [Ku; § 45, IV, Th. 4], and, by a Kuratowski’s theorem [Ku; § 45, II, Th. 1], for each $K \in \mathcal{C}_n$, the set $\varphi^{-1}(K) \cap \mathcal{C}_n$ is dense in $\varphi^{-1}(K)$. It follows that the multifunction $K \to \varphi^{-1}(K) \cap \mathcal{C}_{n+1}$ defined on $\mathcal{C}_n$ is lower-semicontinuous.

By a selection theorem due to Kuratowski and Ryll-Nardzewski [K-RN] there exists a first Baire class function $\sigma : \mathcal{C}_n \to \mathcal{C}_{n+1}$ such that $\sigma(K)(2^\infty) = K$.

For $n = 0$ such selection $\sigma$ can be continuous, see Mägerl, Mauldin and Michael [M-M-M; Theorem 5.1(b)].
5.4. Remark. For the analytic set $C_a$, described at the end of sec. 5.2, there is an analytic set $\mathcal{A} \subset \mathcal{C}$ such that $C_a = \{ f(2^\omega) : f \in \mathcal{A} \}$. Indeed, by Remark 4.2, $C_a \subset \varphi(\mathcal{C}_\zeta)$, where $\zeta = \Psi(\alpha)$ and $\mathcal{C}_\zeta$ is defined at the end of sec. 3.2.

References

[Mi] MICHAEL E., On a map from a function space to a hyperspace, Math. Annalen 162 (1965), 87—88.