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ASYMPTOTIC FORMULAS FOR THE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1.1 In my paper I introduce some asymptotic formulas for the solutions of differential equation of the second order. For the simple formulation of results of non-oscillatory and at the same time oscillatory case it will be better to consider the differential equation in the form

$$y'' \mp q(x)\,y = 0 \quad \left({}' = \frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{1}$$

The upper sign holds for the non-oscillatory case, the lower for the oscillatory one. For the derivation of asymptotic formulas I have transformed equation (1) into

$$\ddot{Y} + Q(X)\,Y = 0 \quad \left({}^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}X}\right) \tag{2}$$

(which was investigated in detail by Prof. O. Borůvka) and have used the method of perturbation. I worked on this problem with Prof. J. Mařík, and the results will appear in the near future in the Czechoslovak Mathematical Journal.

1.2 The following notation will be used. Let m be a non-negative integer, then C_m denotes the system of all real functions with continuous derivatives of the m-th order in $J = \langle x_0, \infty)$; in the whole paper it is assumed that $q(x) \in C_0$, $Q(X) \in C_0$. Instead of $\int_a^b f(x)\,\mathrm{d}x$ I shall often write simply $\int_a^b f$, and provided no misunderstanding will appear I shall also omit x in the relations $f'(x) = g(x)$, $f(x) \geqq h(x)$ and only write $f' = g$, $f \geqq h$. If $\omega(x) \in C_0$, $\lim_{x\to\infty} \omega(x) = 1$ and $f(x) = \omega(x)\,g(x)$, we say that the functions f and g are asymptotically equal and we shall write $f \sim g$. The letter o denotes a continuous function which tends to zero as $x \to \infty$. Equation (1) is called non-oscillatory or oscillatory in J according to whether or not every non-trivial solution has finitely many or infinitely many zeros in this interval.

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2.1 Further on we assume that equations (1) and (2) are simultaneously oscillatory or non-oscillatory. The supposition $q, Q \in C_0$ guarantees the existence of the fundamental systems y_1, y_2 and Y_1, Y_2 of (1) and (2) in the whole interval J. One can easily prove the existence of functions $F \in C_2$ and $X \in C_2$ such that

$$y_j(x) = F(x)\, Y_j[X(x)]\,, \quad j = 1, 2 \tag{3}$$

in the whole interval J. The functions F and X are solutions of the nonlinear system

$$\begin{aligned} FX'' + 2F'X' &= 0\,, \\ F'' + [q - X'^2\, Q(X)]\, F &= 0 \end{aligned} \tag{4}$$

and can be expressed by using the functions y_j, Y_j in the form

$$X(x) = \int_{x_0}^{x} \frac{P^2[X(t)]}{\varrho^2(t)}\, \mathrm{d}t\,, \quad F(x) = \frac{\varrho(x)}{P[X(x)]}$$

with

$$\varrho = (Ay_1^2 + 2By_1y_2 + Cy_2^2)^{\frac{1}{2}}\,, \quad P = (AY_1^2 + 2BY_1Y_2 + CY_2^2)^{\frac{1}{2}}$$

where A, B and C denote suitable constants.

Assuming knowledge of the fundamental system Y_1, Y_2, it is easy to see that the nonlinear system (4) is equivalent to equation (1) in the sense that the solution of (4) is determined by that of (1) and vice versa. Now there is a question of the relationship between the functions in (3) and equation (1) if in (3) instead of F and X we put the functions Φ and Ξ which are in a certain sense the approximative solutions of (4). For simplicity we shall deal only with the case when the equation has the form

$$\ddot{Y} + \varepsilon Y = 0$$

with $\varepsilon = -1$ when (1) is non-oscillatory and $\varepsilon = 1$ when (1) is oscillatory. Then the following theorem holds:

2.2 *Let Φ and Ξ be positive functions in J with continuous second derivatives, $\Xi' > 0$, $\Xi(x) \to \infty$ for $x \to \infty$. If*

$$\int_{x_0}^{\infty} \left| \frac{(\Phi^2 \Xi')'}{\Phi^2 \Xi'} \right| < \infty\,,$$

$$\int_{x_0}^{\infty} |\Phi\Phi'' \mp q\Phi^2 - \varepsilon \Xi'^2 \Phi^2| < \infty\,,$$

then equation (1) has in the non-oscillatory case the fundamental system of the form

$$y_j \sim \Phi \exp\{\varepsilon_j(\Xi + o)\} \sim \Phi \exp\{\varepsilon_j \Xi\} \tag{5}$$

with $\varepsilon_1 = 1$, $\varepsilon_2 = -1$; in oscillatory case y_1 (y_2) are asymptotically equal to the real (imaginary) part of the function

$$\Phi \exp\{i(\Xi + o)\}\,. \tag{6}$$

Under further suppositions we can derive similar formulas for the derivatives of y_j.

From this theorem we can get many results by means of a suitable choice of the functions Φ and Ξ; I present the following only:

2.3 *Let $q \in C_2$, $q > 0$. Suppose that one of the four following conditions is satisfied:*

a) *There exists a $\beta \neq 3/2$ with $\int_{x_0}^{\infty} |q^{-3/2}q'' - \beta q^{-5/2}q'^2| < \infty$ and it holds that either $\beta > 3/2$ or $\beta < 1$ or $\int_{x_0}^{\infty} q^{1/2} = \infty$.*

b) *The function q^{-1} is bounded and there is a $\gamma \in (0, 1/2)$ such that $q^{-\gamma}$ is convex or concave.*

c) *The functions q, q^{-1} are bounded and there is a $\gamma \neq 0$ such that q^{γ} is convex or concave.*

d) *$\int_{x_0}^{\infty} q^{-5/2}q'^2 < \infty$ and there exists a $\gamma > 0$ such that q^{γ} is convex.*

Then $\int_{x_0}^{x} q^{1/2} = \infty$ and equation (1) has in the non-oscillatory case the fundamental system satisfying

$$y_j(x) \sim q^{-1/4}(x) \exp\left\{\varepsilon_j \int_{x_0}^{x} q^{1/2}\right\}, \quad y_j'(x) = \varepsilon_j\, q^{1/4}(x) \exp\left\{\varepsilon_j \int_{x_0}^{x} q^{1/2}\right\}$$

and in the oscillatory case

$$y_1(x) \sim q^{-1/4}(x) \sin\left(\int_{x_0}^{x} q^{1/2} + o\right), \quad y_2(x) \sim q^{-1/4}(x) \cos\left(\int_{x_0}^{x} q^{1/2} + o\right),$$

$$y_1'(x) \sim q^{1/4}(x) \cos\left(\int_{x_0}^{x} q^{1/2} + o\right), \quad y_2'(x) \sim -q^{1/4}(x) \sin\left(\int_{x_0}^{x} q^{1/2} + o\right).$$

2.4 On the basis of formulas (5) and (6) it might seem that every asymptotic formula for the solution of (1) in a non-oscillatory case has its analogy in an oscillatory case. But this assumption is not true.

Let be y_1, y_2 two independent solutions of (1). As the zeros of y_1 and y_2 separate each other we have $y_1 + iy_2 \neq 0$ in the whole interval J and we can put

$$U = \frac{y_1' + iy_2'}{y_1 + iy_2}.$$

This function satisfies the Riccati equation

$$U' + U^2 \mp q(x) = 0 \tag{7}$$

and if $U = u + iv$, then

$$u' + u^2 - v^2 \mp q = 0,$$
$$v' + 2uv = 0.$$

If $q(x) = q = \text{const.}$ then it is easy to show that the trajectories of this system $u = u(x)$, $v = v(x)$ lie in the u, v-plane on the pencil of circles which has real base-points $A[-q^{1/2}, 0]$, $B[q^{1/2}, 0]$ in the non-oscillatory case. In the oscillatory case the base-points are imaginary and A, B are limit points of the pencil. If $q(x) \sim q$, then always in the non-oscillatory case $U \sim$ const. If $U \sim$ const. in the oscillatory case, then from (7) it follows that $U \sim \pm\, q^{1/2}$. The functions u and v can be written in the form $u = \varrho'/\varrho$, $v = w/\varrho^2$ where $\varrho = (y_1^2 + y_2^2)^{1/2}$, $w = y_1 y_2' - y_1' y_2 =$ const. It is easy to see that relation $U \sim \pm\, q^{1/2} \neq 0$ is not satisfied, if ϱ is unbounded; this happens e.g. when we consider the equation $y'' + (1 + 2 \sin x/x)\, y = 0$.

2.5 In the following we shall deal only with the non-oscillatory case.

The solution y of equation (1) will be called principal, when there exists $a \geqq x_0$ such that $y(x) \neq 0$ for all $x \geqq a$ and $\int_a^\infty (1/y^2) = \infty$. Without loss of generality, it can be assumed that $x_0 = a$. It is easy to show that non-oscillatory equation (1) has at least one principal solution and that every two principal solutions are linearly dependent. If we let

$$z = \delta \frac{y'}{y} - \gamma \tag{8}$$

we get

$$\delta z' + z^2 + Pz + Q = 0\,, \tag{9}$$

where P and Q are functions of δ, γ, δ', γ'.

If y is the principal solution of (1) which has no zeros, then function (8) is the smallest of the solutions of equation (9) which are defined in J. On the basis of this dependence one can derive the series of the asymptotic estimates for the solutions of (1). I want to present some results only.

2.6 *Let $q(x) > 0$, $q \in C_2$, $(q^{-1/2})' \sim \lambda < \infty$. Further let y_1, y_2 be the fundamental system, y_2 the principal solution of (1).*

Then

$$\int_{x_0}^{\infty} q^{1/2} = \infty\,, \qquad \frac{1}{x^2\, q(x)} \sim \lambda^2\,;$$

for $\lambda = 0$,

$$\log |y_1(x)| \sim \log |y_1'(x)| \sim -\log |y_2(x)| \sim -\log |y_2'(x)| \sim \int_{x_0}^{x} q^{1/2}\,;$$

for $\lambda > 0$ one has

$$\log |y_1(x)| \sim -\log |y_2'(x)| \sim \log x^{\xi}\,,$$
$$\log |y_1'(x)| \sim -\log |y_2(x)| \sim \log x^{\xi - 1}\,,$$

with $\xi = \frac{1}{2}[1 + \sqrt{(1 + 4\lambda^{-2})}]$.

If in addition $\int_{x_0}^\infty |\mathrm{d}(q^{-1/2})'| < \infty$, then there exist constants c_j such that

$$y_j(x) \sim c_j\, q^{-1/4}(x) \exp\left\{ \varepsilon_j \int_{x_0}^{x} q^{1/2} \left(1 + \frac{q'^2}{16 q^3}\right)^{1/2} \right\},$$

$$y_j'(x) \sim \frac{1}{2} [\lambda + \varepsilon_j \sqrt{(\lambda^2 + 4)}]\, q^{1/2}(x)\, y_j(x)\,.$$

2.7 *Let p, q be continuous functions in J. Let equation* (1) *be non-oscillatory, y_2 the principal solution and y_1 a solution of* (1) *which is independent of y_2. Put $W = y_1'y_2 - y_1y_2'$, $f = y_1y_2/W$, $F = f^2(q - p)$ and assume $\inf_{x \in J} F(x) > -\frac{1}{4}$. Then the equation*

$$y'' = py \tag{10}$$

is also non-oscillatory.

Assume further that $\int_{x_0}^{\infty} |\mathrm{d}F| < \infty$ and let Y_1, Y_2 be two independent solutions and Y_2 the principal solution of (10). *Then there exist constants c_j such that*

$$Y_j(x) \sim c_j f^{1/2}(x) \exp\left\{ \varepsilon_j \int_{x_0}^{x} \frac{(1 + 4F)^{1/2}}{2f} \right\},$$

$$\frac{2f(x)\,Y_j'(x)}{Y_j(x)} - f'(x) \sim \varepsilon_j(1 + 4F)^{1/2}.$$

In concluding I would like to mention that the results of this paper are special cases of more general results.

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