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Karel Rektorys

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In: (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Prague in September 1962. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1963. pp. 137--141.

Persistent URL: http://dml.cz/dmlcz/702181

SOLUTION OF NONLINEAR PARABOLIC EQUATIONS BY FINITE DIFFERENCE METHOD FOR AN ARBITRARY TIME INTERVAL

K. REKTORYS, Praha

The so-called problem of hydratic heat in concrete massives, especially in gravitational dams, leads in a simplified form to the following problem: To solve a nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = A(x, t, u) \frac{\partial^2 u}{\partial x^2} + B(x, t, u) \left(\frac{\partial u}{\partial x}\right)^2 + C(x, t, u) \frac{\partial u}{\partial x} + D(x, t, u, z), \quad A > 0 \tag{1}$$

with the integral condition

$$z(x, t) = \int_0^t D(x, \tau, u(x, \tau), z(x, \tau)) \, d\tau \tag{2}$$

on the rectangle

$$Q(0 < x < 1, 0 < t \leqq T)$$

(T arbitrary) with discontinuous mixed boundary conditions

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial x} = k(u - g(t)) \quad \text{for} \quad x = 0, \quad u(1, t) = h(t). \tag{3}$$

We shall give an existence theorem for this problem, i.e. we will prove that under some conditions on the coefficients A, B, C, D and on the boundary functions precisely two functions $u(x, t)$, $z(x, t)$ exist – in a certain class of functions – which fulfil equation (1) and conditions (2), (3). Further, we shall suggest an approximative method for the numerical solution of this problem.

The problem is rather complicated by the integral condition (2). This condition may be very simply expressed – in the form of a sum – if for the solution of the problem finite-difference method is used. The advantage of this method lies also in the fact, that boundary conditions (3) may be very simply expressed in finite differences. Finally, this method is very attractive from the numerical point of view as a very convenient approximative method. Therefore, there are many reasons to use it. However, the well known disadvantage of this method is that existence theorems, based on standard estimates, used in this method, guarantee the existence of the solution in the case of *nonlinear* equations only in a certain time-interval, depending on the equation itself and on boundary conditions and not in an *arbitrary* time-interval as is necessary in our case. Therefore, if we are interested in using the finite-difference method, it will be necessary to work out estimates of a completely different nature.

To make the question more clear, let us first study a substantially simple problem of a quasilinear equation

(4) $$\frac{\partial u}{\partial t} = A(x, t, u)\frac{\partial^2 u}{\partial x^2} + C(x, t, u)\frac{\partial u}{\partial x} + E(x, t, u), \quad A > 0$$

on Q with sufficiently smooth Dirichlet's boundary conditions

(5) $$u(x, 0) = f(x), \quad u(0, t) = g(t), \quad u(1, t) = h(t).$$

Problems like this were solved in one or more dimensions by many authors and by different methods (e.g. by the method of successive approximations in classical form or in the modern form of the fixed-point theorem, by the generalized method of Rothe etc.). Existence theorems were proved for the classical solution as well as for solutions in different functional spaces under certain conditions posed on boundary functions and on coefficients of equation (4). But for existence theorems based on finite-difference methods, nothing was proved — as far as I know — for an arbitrary time-interval (see also [2]). What is the matter?

The basic idea of the finite-difference methods is well known. We choose two natural numbers M and N and divide the rectangle Q by a net of parallells with the t- and the x-axis

$$x = h,\ x = 2h,\ \ldots,\ x = Mh = 1,$$
$$t = l,\ t = 2l,\ \ldots,\ t = Nl = T$$

into $M \,.\, N$ rectangles. We do not choose M and N arbitrarily, but so that

(6) $$l = ch^2,$$

where c is a constant, depending on the values of the function A. Let us denote this net by S_1. Let us replace the differential equation (4) by the difference equation

(7) $$q_{ik} = A_{ik}r_{ik} + C_{ik}p_{ik} + E_{ik}$$

where u_{ik} is the value of the so-called net-function u at the point (x_i, t_k) of the net and

$$q_{ik} = \frac{u_{i,k+1} - u_{ik}}{l},$$

$$p_{ik} = \frac{\Delta u_{ik}}{h} = \frac{u_{i+1,k} - u_{ik}}{h},$$

$$r_{ik} = \frac{\Delta^2 u_{ik}}{h^2} = \frac{u_{i+1,k} - 2u_{ik} + u_{i-1,k}}{h^2},$$

$$A_{ik} = A(x_i, t_k, u_{ik})$$

etc. The values of u_{ik} on the boundary of Q follow from conditions (5), the values in the inner of Q follow successively from (7). Having u_{ik}, we may define a function $U_1(x, t)$ sectionally linear on Q and such, that at the points of the net S_1 $U_1(x, t) = u_{ik}$.

To get an existence theorem for problem (4), (5), we construct a sequence of nets S_n, choosing for each n

$$h = \frac{1}{2^{n-1}M}, \quad l = \frac{T}{2^{2n-2}N}$$

(so that condition (6) is fulfilled). For each S_n we get a sectionally linear function U_n. If it can be proved that u_{ik}, p_{ik}, q_{ik} are bounded on $\bar{Q}$ uniformly with respect to n, then the functions $U_n(x, t)$ are uniformly bounded and uniformly continuous on $\bar{Q}$ and according to the well-known Arzela's theorem, a subsequence may be found uniformly convergent on $\bar{Q}$ to a function which we denote by $U(x, t)$. If further difference-quotients of sufficiently high order are uniformly bounded, then $U(x, t)$ may be easily shown to be the desired solution.

Let us denote

$$u_{kn} = \max_i |{}^n u_{ik}|, \quad p_{kn} = \max_i |{}^n p_{ik}|$$

etc., where the index n corresponds to the net S_n. Then, if boundary conditions (5) are bounded, it may be easily shown that u_{kn} may be majorised (uniformly with respect to n) by the solution of a linear ordinary differential equation with a suitable initial condition. If equation (4) is *linear* then this holds also for p_{kn} and for higher difference-quotients under some rather natural suppositions about boundary functions. So in this case of a linear differential equation, the uniform boundedness of u_{ik} and of the desired difference-coefficients can be easily proved by this majorising method, and the existence theorem is finished. This is a well-known procedure.

If equation (7) is *nonlinear* – and that is our case – then this result holds only for u_{kn}. For p_{kn} the majorising ordinary differential equation is in general nonlinear and its solution exists only in a certain interval, depending on boundary conditions and on coefficients of the equation (4). A similar result holds for q_{kn} etc. Therefore, in this case of a nonlinear equation, this majorising method is not suitable for the proof of an existence theorem in an *arbitrary* time-interval. How may this difficulty be removed?

In [3] a very nice method is shown for estimating $\partial u/\partial x$ for a problem like problem (4), (5). Using this method, the authors show, that $\partial u/\partial x$ will be neither too positive nor too negative. The proof of the existence theorem in their work is made by the method of successive approximations, and the just mentioned proof of the boundedness of $\partial u/\partial x$ cannot be applied to finite-difference method, where the just explained majorising method is used. But it suggested to me to work out a method in finite differences – I shall call it the *balance method* – permitting the estimation od difference-quotients for an arbitrary time-interval. The idea of this balance method is very simple. Instead of majorising u_{kn}, p_{kn} etc. by the solution of an ordinary differential equation, I prove at first that if boundary functions (5) are bounded and if equation (4) is of a special type, then if u_{ik} is bounded in the k-th row of the net S_n by a sufficiently high constant, then it remains bounded in the $(k + 1)$-th row by the same constant, and this holds for all nets with sufficiently high n. On the basis of this result, I prove (roughly stated) that if boundary functions (5) are bounded and if the absolute term of the difference equation (4) is negative for all sufficiently positive u and positive for all sufficiently negative u, then u_{ik} is uniformly bounded in $\bar{Q}$ with respect to u. Having this basic result, I use a suitable substitution and I get for the net-function P_{ik}, which corresponds to the net-function p_{ik}, the parabolic equation with the just mentioned property; showing, that P_{ik} is uniformly bounded on the boundary of Q — under some suppositions about the boundary functions — I have the uniform boundedness of P_{ik} on the whole $\bar{Q}$ and in consequence of the used substitution the uniform boundedness of p_{ik} on the whole $\bar{Q}$. Having it, I show by a similar procedure the uniform boundedness of r_{ik} (and by this, of q_{ik}) etc. So that, using the balance method, I can easily prove the uniform boundedness of u_{ik} and of the desired difference-quotients for an arbitrary time-interval.

The idea of the balance method is then very simple, but the whole procedure is not so simple, for it requires the solution of a lot of small problems related to it. This is all worked out in detail in [1].

So I come to the basic theorem of the work, to the existence theorem for the classical solution of the problem (4), (5). The exact formulation of the problem can be found in [1]; roughly stated, I suppose Lipschitzian derivatives of the second order of the functions A, C, E; the function $E(x, t, u)$ is assumed to be not too much increasing (in absolute value) with $|u|$; e.g. it may be of the form $E(x, t, u) = F(x, t, u)\,u + G(x, t, u)$, where F, G are bounded functions. The proof of uniqueness is very simple.

Having this basic existence theorem, all is prepared for an existence theorem for problem (1), (2), (3). Boundary functions (3), however, being supposed to be discontinuous, I first proved an existence theorem for the problem (4), (5) and for discontinuous boundary conditions. For my purpose, it was sufficient to suppose boundary functions $f(x)$, $g(t)$ and $h(t)$ sectionally continuous. By the solution of the problem (4), (5) I mean a function $u(x, t)$ which is a classical solution of equation (4) in the inner of Q, which is bounded on Q and continuous up to the boundary at every point of continuity of the boundary functions. The method of the proof of existence theorem seems to be self evident: I express the boundary functions (5) as a limit of a sequence of sufficiently smooth boundary functions and I have to prove that the sequence of the corresponding solutions $u_n(x, t)$ converges to a function $u(x, t)$ which is the desired solution. But a question comes up here: It seemed at first that to get the desired result, it would be necessary to pose further conditions on the coefficients A, C, E. I proved that this is not the case. In addition to the approximation of the boundary functions, I also approximated the functions A, C, E. In order to get the desired result, this aproximation could not be arbitrary. I approximated the functions A, C, E by the sequence of the so-called regularized functions. In one dimension, the regularized function $f_\varrho(\xi)$ corresponding to the function $f(x)$ is a function of the form

$$f_\varrho(\xi) = C_0 \int_{\xi-\varrho}^{\xi+\varrho} f(x) \exp\left[\frac{(x-\xi)^2}{(x-\xi)^2-\varrho^2}\right] \mathrm{d}x \,,$$

where C_0 is a suitable constant and $\exp[(x - \xi^2)/((x - \zeta)^2 - \varrho^2)]$ is the regularizing kernel. It is well known that $f_\varrho(\xi)$ is infinitely differentiable. Now, it may be easily shown, that if $f(x)$ satisfies the Lipschitz condition with the constant L, that then $|df_\varrho/d\xi| \leqq L$. This result may be extended for higher dimensions and for derivatives of higher order. On base of this property, I get the convergence of the functions A_n, C_n, E_n and of their derivatives to the functions A, C, E and to the corresponding derivatives locally uniform on Q and I get an existence theorem for the problem (4), (5) and for discontinuous boundary conditions under the same suppositions about the functions A, C, E as before.

This problem being solved, it is now easy to get an existence theorem for the problem (1)–(3). This was the very reason for which to the solution of this problem the finite-difference method was chosen, for the conditions (2), (3) may be very simply put into account in finite differences. I got then an existence theorem for (1)–(3) at first supposing that the equation (1) is quasilinear, i.e. that it doesn't contain the term $B(x, t, u)(\partial u/\partial x)^2$. For details see [4].

What makes difficulties, that is the term $B(x, t, u)(\partial u/\partial x)^2$; the above mentioned balance method was derived for the case of a quasilinear equation and it cannot be immediately applied to the case of a nonquasilinear equation, as the equation (1) is. But, as I proved in [1], equation (1) may be easily transformed into a quasilinear equation by a substitution, containing only the functions A a B. By this, all questions, concerning the problem (1)–(3), are solved.

Finally, I have a small remark to add: Equation (1) was shown to have a character of a quasilinear equation. If its coefficients and the boundary functions are sufficiently smooth, then the solution and its derivatives are continuous up to the boundary of Q. It is a question whether this is also the case if equation (1) contains $\partial u/\partial x$ also in higher powers than in the second one. It is interesting, that it is no more the case. An example, showing it, may be found in [1].

REFERENCES

[1] K. Rektorys: Die Lösung des ersten Randwertproblems "im Ganzen" für eine nichtlineare parabolische Gleichung mit der Netzmethode. Czechoslovak Math. J. 12 (87), *1* (1962), 69–103.

[2] J. Kautský: Řešení quasilineární parabolické diferenciální rovnice s absolutním členem speciálního typu metodou sítí. Apl. Mat. *2* (1957), 327–341.

[3] O. A. Oleĭnik and T. D. Ventcel: The first boundary problem and the Cauchy problem for quasi-linear equations of parabolic type. Mat. Sb. N. S. 41 (*83*), (1957), 105–128 (Russian).

[4] K. Rektorys: Lösung der gemischten Randwertaufgabe und des Problems mit einer Integralbedingung "im Ganzen" für eine nichtlineare parabolische Differentialgleichung mit der Netzmethode. Czechoslovak Math. J. 13 (*88*), (1963), 189—208.