EQUADIFF 1

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ON SOME STABILITY PROBLEMS

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The presented paper consists of two different parts. In the first part those problems of the Lyapunov theory will be dealt with which arise in the study of some notions of stability in which persistent perturbations are taken in account. In the second part an estimate of the probability that the solutions of a differential equation with random perturbations exceed a given bound in a special case will be given.

1

At the outset a well-known definition of stability under persistent perturbations will be mentioned.

The solution $x \equiv 0$ of the vector differential equation

$$x' = F(t, x) \tag{1}$$

will be called stable under persistent perturbations, if to every positive $\varepsilon > 0$ there exist positive numbers $\delta > 0$, $\eta > 0$ such that for every solution $y(t)$ of the differential equation

$$y' = F(t, y) + S(t, y) \tag{2}$$

$\|y(t)\| < \varepsilon$ for $t \geqq t_0$ whenever $\|y(t_0)\| < \delta$ and

$$\|S(t, y)\| < \eta\,. \tag{3}$$

If the magnitude of the disturbing terms $S(t, y)$ will be measured in a different way, i.e. if (3) is replaced by

$$\int_0^\infty \sup_y \|S(t, y)\| \,\mathrm{d}t < \eta\,, \tag{3'}$$

one obtains the definition of integral stability. In the case of integral stability the perturbations may be large in a small interval, whereas in the case of stability under persistent perturbations they have to be small, but they may be persistent. The properties of both these types are possessed by stability under persistent perturbations bounded in the mean value; for the sake of brevity, in what follows, this will be called stability in the mean. Stability in the mean was studied by C. Corduneanu, V. E. Germaidze, N. N. Krasovskiĭ [1].

This type of stability is obtained, if instead of (3) one puts

$$\int_t^{t+T} \sup_y \|S(t, y)\| \,\mathrm{d}t < \eta(T) \tag{3''}$$

δ and η depending in general on T.

In stability theory a great attention is paid to the second method of Lyapunov. In this method functions $V(t, x)$ are sought fulfilling some given conditions. The existence of such a function is sufficient for the stability of the solution $x \equiv 0$.

In a recently published paper [2] I showed that stability under persistent perturbations can be characterized by means of the Lyapunov functions, which is stated in the following theorem:

The solution $x \equiv 0$ is stable under persistent perturbations, if and only if there exists a function $V(t, x)$ with continuous partial derivatives fulfilling the following conditions:

i) *$V(t, x)$ is positive definite;*

ii) *$V(t, x)$ is bounded uniformly with respect to t;*

iii) *there exists a continuous function $U(x)$ which is positive except at the point $x = 0$, and the function $Q(t, x) = \mathrm{d}V/\mathrm{d}t + U(x)\sqrt{\sum_{i=1}^{n}(\partial V/\partial x_i)^2}$ is less than or equal to zero.*

This theorem is proved under the assumption that $F_i(t, x)$ are continuous. In [3] I proved that integral stability can be characterized by means of the Lyapunov functions:

The solution $x \equiv 0$ is integrally stable, if and only if there exists a function $V(t, x)$ with continuous partial derivatives fulfilling the following conditions:

i) *$V(t, x)$ is positive definite;*

ii) *$V(t, x)$ fulfils a Lipschitz condition with a constant independent of t;*

iii) *$\mathrm{d}V/\mathrm{d}t$ is less than or equal to zero.*

Also this theorem is proved under the assumption that the functions $F_i(t, x)$ are continuous.

In the case of stability in the mean only sufficient conditions are known at present.

In the theory of stability an important role is played by the notion of asymptotic stability which implies that the solution tends to a desired position after it was affected by a perturbation. For the above mentioned types of stability analogous concepts can be introduced.

Definition. *The solution $x \equiv 0$ is asymptotically stable under persistent perturbations, if it is stable under persistent perturbations and if to any sufficiently small numbers $\delta > 0$, $\eta > 0$ (i.e. $\delta \leqq K$, $\eta \leqq K$) there exist numbers $T(\delta, \eta) > 0$, $\gamma(\delta, \eta) > 0$ such that for every solution $y(t)$ of equation* (2)

$$\|y(t)\| < \eta \quad \textit{for} \quad t \geqq t_0 + T(\delta, \eta)$$

whenever

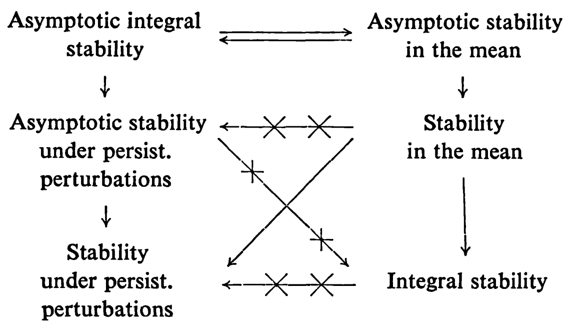
$$\|y(t_0)\| < \delta\,, \quad \|S(t, y)\| < \gamma(\delta, \eta)\,.$$

If one requires that the perturbations $S(t, y)$ are smaller than γ in the sense of inequalities (3′), (3″) one obtains the asymptotic analogues of the respective above mentioned concepts of stability.

Asymptotic stability under persistent perturbations can be characterized in the same manner as stability under persistent perturbations except that in condition iii) it is necessary to assume that $Q(t, x)$ is negative definite. An analogous theorem holds for asymptotic integral stability.

Also these theorems are proved under the hypothesis that the functions $F_i(t,x)$ are continuous.

It is of interest to compare the introduced concepts of stability:



In the presented scheme the arrows represent implications and crossed arrows mean that the implication does not hold.

In the autonomous case, for asymptotic stability under persistent perturbations and for asymptotic integral stability it is possible to construct the functions V independent of t.

For the non-asymptotic types of stability I succeeded only in constructing a function fulfilling locally a Lipschitz condition.

Comparing the above presented concepts of stability (in the autonomons case) one obtains a very simple scheme, as all three asymptotic types of stability are equivalent to each other. Furthermore, stability under persistent perturbations is equivalent to stability in the mean. Finally, each of the presented types of stability implies integral stability.

2

Deriving the sufficient conditions for stability, e.g. constructing a Lyapunov function, one has to respect the case when the perturbations in every point and in every moment of time tend to increase the distance of the solution from the origin at the greatest possible rate. Thus one is compelled to give too strict conditions for stability. Really, as usual at one moment the perturbations tend to increase the distance of the solution from the origin and at the other moment they tend to decrease it. To express exactly this property it is convenient to use the apparatus of random variables. The perturbations S are supposed to represent a stochastic process. Then the solution will also be a stochastic process and one can only speak about the probability $P(|x(t, \omega)| \geqq v)$.

The authors of papers [4], [5] defined by means of this expression the stability with respect to probability. These authors, however, are mostly interested in control problems.

Now a scalar differential equation with random perturbations will be examined, which will be written as

$$x' = -\lambda x + S(t, \omega), \quad \lambda > 0, \tag{4}$$

ω being an element of the probability space Ω where a σ-field of subsets of Ω and a probability measure P are defined. The function $S(t, \omega)$ and the initial value $x(0, \omega)$ are assumed to be P-measurable and $S(t, \omega)$ integrable with respect to t, ω being fixed. Further it is necessary that the perturbations S are small in some sense. It will be supposed that there exists a number $\delta > 0$ such that

$$E(|S(t, \omega)|) \leqq \delta,$$

E denoting the expect value.

Further assume that $S(t, \omega)$ is symmetrically distributed for every t. However, this condition does not imply that the perturbations behave in the way mentioned above. It is possible to construct $S(t, \omega)$ such that equation (4) is decomposed to two non-random equations both of them holding with a given probability. Consequently the following condition will be introduced:

There exists $d > 0$ such that random variables $S(t, \omega)$, $S(\tau, \omega)$ are independent, provided t, τ belong to different intervals of type $\langle kd, (k+1)\, d)$ (k integer).

Of course, $S(t, \omega)$ is supposed to be bounded in its absolute value, since in the opposite case one would obtain substantially weaker results. In the following, it will be assumed that

$$|S(t, \omega)| \leqq K,$$

where K may be large and δ very small. In fact it is the product δK that plays an important role.

Denote $P_1(v, d, t, \lambda) = \sup\limits_{S(t,\omega)} P(|x(t, \omega)| \geqq v)$, where $x(t, \omega)$ is a solution of equation (4) with the initial value $x(0, \omega) = 0$ and $S(t, \omega)$ represents any element of the set of stochastic processes fulfilling the above conditions.

For P_1 it is possible to prove an asymptotic formula.

Let the sequences of positive numbers d_n, t_n, v_n, λ_n fulfil the following conditions: $t_n/d_n \to \infty$, $\lambda_n d_n \to 0$, $\lambda_n t_n \to T > 0$, $v_n \sqrt{(\lambda_n/d_n)} \to a$, then

$$\lim_{n \to \infty} P_1(v_n, d_n, t_n, \lambda_n) = \sqrt{\left(\frac{2}{\pi}\right)} \int_{\frac{a}{\sqrt{(\delta K)}} \sqrt{\left(\frac{2}{1 - e^{-2T}}\right)}}^{\infty} e^{-\frac{1}{2}x^2} \, dx .$$

The case $\lambda = 0$ is to be examined especially. In this case it will be defined as above

$$P_2(v, d, t) = \sup_{S(t,\omega)} P(|x(t, \omega)| \geqq v),$$

where $x(t, \omega)$ is a solution of equation (4) where one puts $\lambda = 0$.

Let the sequence of positive numbers d_n, t_n, v_n fulfil the following conditions: $t_n/d_n \to \infty$, $v_n/\sqrt{(t_n d_n)} \to \alpha$, then

$$\lim_{n\to\infty} P_2(v_n, d_n, t_n) = \sqrt{\left(\frac{2}{\pi}\right)} \int_{\alpha/\sqrt{(\delta K)}}^{\infty} e^{-\frac{1}{2}x^2}\, dx .$$

In the case $\lambda = 0$ perturbations S may depend on x and it is not necessary to assume that $S(t, x, \omega)$ is symmetrically distributed. In this case it is necessary to add another condition to avoid systematic errors; this condition is a generalization of the equation

$$E(S(t, x, \omega)) = 0 .$$

Moreover the following equation is true:

$$\sup_{S(t,x,\omega)} P(\sup_{\tau \leqq t} |x(\tau, \omega)| \geqq v) = \sup_{S(t,x,\omega)} P(|x(t, \omega)| \geqq v) .$$

In much the same manner as above we will define

$$P_3(v, d, t) = \sup_{S(t,x,\omega)} P(\sup_{\tau \leqq t} |x(\tau, \omega)| \geqq v) .$$

If d_n, t_n, v_n fulfil the same conditions as for P_2, then

$$\lim_{n\to\infty} P_3(v_n, d_n, t_n) = 1 - \sum_{l=-\infty}^{\infty} \frac{(-1)^l}{\sqrt{(2\pi)}} \int_{l\alpha/\sqrt{(\delta K)}}^{(l+1)\alpha/\sqrt{(\delta K)}} e^{-\frac{1}{2}x^2}\, dx .$$

These formulae can be adapted even for the case that $x(0, \omega)$ is a random variable.

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